Weierstraß's approach to analytic function theory

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Weierstraß's programme of lectures

Building a rigorous theory of analytic functions has been Weierstraß's standing concern for decades. In response to Riemann's achievements, since the early 1860's Weierstraß began to build his theory of analytic functions in a systematic way on arithmetical foundations, and to present it in his lectures. Following Poincaré their aim can be summarized as follows:

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- Eventually, to tackle Abelian functions themselves

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- This habit was coupled with a dislike of publishing his results in printed papers until they had reached the required level of rigour.

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- "The more I think about the principles of function theory – and I do it incessantly – the more I am convinced that this must be built on the foundations of *algebraic truths* [my emphasis], and that it is consequently not correct when the "transcendental", to express myself briefly, is taken as the basis of simple and fundamental algebraic propositions. This view seems so attractive at first sight, in that through it Riemann was able to discover so many of the most important properties of algebraic functions. (It is self-evident that, as long as he is working, the researcher must be allowed to follow every path he wishes; it is only a matter of systematic foundations" (Weierstraß to Schwarz on October 3, 1875)

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- At that time there was no way to deal with functions of several complex variables by resorting to "transcendental" methods as Cauchy and Riemann had done for functions of *one* variable.

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- His treatment of the natural numbers had "an almost mystic character" (Kopfermann 1965)

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- "By a numerical magnitude (*Zahlgröße*) we mean every complex number whose elements are the unit and its proper parts".
- In this way he introduced the rational numbers, and how to calculate with them. Then, by resorting to the concept of convergent, infinite series he introduced irrational numbers as numerical magnitudes with infinitely many "proper parts".

Complex and hypercomplex numbers

Weierstraß introduced complex numbers (in a modern sense) in two ways: first he introduced complex whole numbers in a geometric manner as Gauß had done.
Then he introduced complex numbers "in a purely analytical way, without any reference to their geometrical meaning". To this end he built an (abstract) algebraic structure on a 2-dimensional real vector space in great detail, and eventually showed that the (ordinary) field of complex numbers is obtained by giving the units the values $e = 1, e' = -1$ (and accordingly, $ii = -1, 1/i = -i$).
In the concluding remarks to the section on complex numbers of his 1878 lectures Weierstraß stated: "If one were to consider complex numbers with arbitrarily many units, then one would find that calculations with such numbers can always be reduced to calculations with numbers built by four units only". As Ullrich has pointed out, in modern terms this amounts to saying that that every finite dimensional, associative and commutative real algebra with a unit and no nilpotent elements is (isomorphic to) a ring-direct sum of copies of $\mathbb{R}$ and $\mathbb{C}$ (the Weierstraß–Dedekind theorem).

“Some theorems on magnitudes”

In Weierstraß’s lectures the development of analytic function theory proper was preceded by “some theorems on magnitudes in general’.
In Hurwitz’s 1878 lecture notes the historical development of the concept of function is followed by chapters on rational functions, power series and the differential calculus, and eventually by the study of what in modern terms are called the topological properties of $\mathbb{R}$ and $\mathbb{R}^n$. He introduced such concepts as the δ-neighborhood of a point of $\mathbb{R}^n$, the definition an open set – as we would denote what he called “a continuum” – and the definition of a path-connected domain as well. Then he stated and proved such fundamental theorems as the Bolzano–Weierstraß theorem in $\mathbb{R}$ and $\mathbb{R}^n$, the existence of the upper (resp. lower) bound and the existence of (at least one) accumulation point for an infinite, bounded set of real numbers (including their extension to sets of $\mathbb{R}^n$ and $\mathbb{C}^n$). As a consequence, he proved that a continuous function on a closed interval is uniformly continuous, and attains its upper and lower bounds there.

Analytic functions

After summarizing the principal properties of rational, entire functions of a single variable, including the Lagrange interpolation formula, Weierstraß turned to infinite series of rational functions by emphasizing the analogy of this procedure with what he had done in the introduction of numbers. Weierstraß then defined uniform convergence for series of functions, and proved that the sum of a uniformly convergent series of rational functions in a given domain is a continuous function there. Then he proved his famous theorems for power series (the existence of a disk of convergence, uniform convergence within the disk, inequalities for the coefficients of a series without resorting to Cauchy integral theorem, and so forth), and he remarked that "analogous propositions" hold for series of several variables.

Then he stated and proved his double series theorem for one and several variables, established the identity theorem for series and defined term-by-term differentiation for series of one and several variables.

Next he turned to study the behaviour of a function on the boundary of its domain of definition. With a compactness argument he proved the existence of a singular point on the circumference of the disk of convergence of the series. This proof can be extended to series of several variables. Eventually, Weierstraß's expounded method of analytic continuation of a power series (a "function element", in Weierstrassian terminology) by means of chains of overlapping disks.

“Prime functions”

On the same day, December 16th, 1874, Weierstraß communicated to his closest students, Schwarz and Sonya Kovalevskaya, that he had finally succeeded in overcoming a major difficulty which for a long time had prevented him from building a satisfactory theory of single-valued functions of a complex variable. His starting point was the following question:

> *Given an infinite sequence of complex constants $\{a_n\}$ with $\lim |a_n| = \infty$ is there an entire, transcendental function $G(x)$ which vanishes at the points $\{a_n\}$ and only those, and in such a way that each of the constants a_j is a zero of order λ_j say, if a_j occurs λ_j times in the sequence?*

He had been able to find a positive answer by assuming that $a_n \neq 0$ for any n and by associating to the given sequence a sequence $\{\nu_n\}$ in such a way that $\sum\limits_{i=1} (\frac{x}{|a_i|})^{\nu_i+1} < \infty$. “This is always possible”, Weierstraß affirmed. Let $\nu_n = n - 1$ and consider the “prime functions”

$$E(x, 0) = 1 - x \qquad E(x, n) = (1 - x) \exp\left(\frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^n}{n}\right), \quad (1)$$

which he introduced here for the first time.

The infinite product

$$\prod_{n=1}^{\infty} E(x/a_n, n) \tag{2}$$

is convergent for finite values of x, and represents an analytic function that has "the character of an entire function" and vanishes in the prescribed way. The representation theorem followed easily: every single-valued analytical function $f(x)$ that "has the character of a rational function" for every finite value of x can be represented as the ratio of two convergent power series in such a way that the numerator and denominator never vanish for the same value of x. This theorem, "full of consequences", was until now "regarded as unproved in my theory of Abelian functions" Weierstraß admitted in his letter to Kovalevskaya.

This, and related theorems, constituted the core of what Weierstraß called "a very nice, small treatise" that he presented to the Akademie on December 10, 1874 and originally intended to publish in the *Monatsberichte* of that month. Instead, the material grew and Weierstraß presented an extended version of his "small treatise" to the Akademie on October 16, 1876. This time, this seminal paper on the "systematic foundations" of the theory of analytic functions was eventually published.

“A Promised Land”

In Poincaré’s opinion, the discovery of prime functions was Weierstraß’s main contribution to the development of function theory. They had a dramatic impact on Hermite when he first heard about them in 1877 on the occasion of the celebration of the 100th anniversary of Gauß’s birthday in Göttingen where Hermite met Weierstraß for the first time.
Reporting on the meeting in a letter to Mittag-Leffler on April 23, 1878 Hermite recorded:

> *I was talking Mr. Schwarz about elliptic functions, and I received from him the notion of prime factors, a notion of capital importance and completely new to me. But scarcely the most essential things were communicated to me. Only for an instant, as if the horizon was unveiled and then suddenly darkened, I glimpsed a new, rich and wonderful country in analysis, a Promised Land that I had not entered at all. I had this vision in my mind continuously during my entire journey back.*

Then Hermite suggested to Picard to provide the French translation of Weierstraß’s 1876 paper that appeared in 1879.

The Preparation Theorem

In his lectures on analytic function theory Weierstraß regularly presented a series of theorems on single-valued functions of several variables that he needed in his lectures on Abelian functions, and in 1879 he collected this material in a lithographed paper that was eventually printed in 1886. The first theorem stated there was his *Vorbereitungssatz* (preparation theorem). In modern terms, this theorem can be stated in slightly different terms with respect to Weierstraß's original formulation as follows:

Let $F(x, x_1, \ldots, x_n)$ be a holomorphic function in the neighbourhood of the origin. Suppose $F(0, 0, \ldots, 0) = 0$, $F_0(x) = F(x, 0, \ldots, 0) \neq 0$ and let p be the integer such that $F_0(x) = x^p G(x)$, $G(0) \neq 0$. Then there exists both a "distinguished" polynomial

$$f(x, x_1, \ldots, x_n) = x^p + a_1 x^{p-1} + \ldots + a_p$$

(whose coefficients $a_j(x_1, \ldots, x_n)$ are holomorphic functions in the neighbourhood of the origin and vanish at the origin) and a function $g(x, x_1, \ldots, x_n)$ which is holomorphic and nonzero in the neighbourhood of the origin, such that $F = f \cdot g$ in the neighbourhood of the origin.

An "important question" in function theory

In 1880 Weierstraß proved that the domain of (uniform) convergence of a series may be built up of different, disjoint regions as shown by the series

$$F(x) = \sum_{\nu=0}^{\infty} \frac{1}{x^{\nu} + x^{-\nu}} \tag{3}$$

which is uniformly convergent for $|x| < 1$ and $|x| > 1$.

In such a case, an "important question" in function theory arises, namely whether the given series represents branches of the same monogenic function or not. The question had a negative answer, as Weierstraß proved. This would imply that "the concept of a monogenic function of one complex variable does not coincide completely with the concept of a dependence that can be expressed by means of (arithmetical) operations on magnitudes". Weierstraß was pleased to add in a footnote that "the contrary has been stated by Riemann" in §20 of his thesis, and also that "a function of one argument, as defined by Riemann, is always a monogenic function".

Natural boundaries

Weierstraß said he had found and presented in his lectures "for many years" the result that in either domain $|x| < 1$ and $|x| > 1$ the series (3) represents a monogenic function which cannot be analytically continued in the other one across their common boundary.

The proof was based on the identity

$$1 + 4F(x) = \left(1 + 2\sum_{n=1}^{\infty} x^{n^2}\right)^2, \ |x| < 1 \tag{4}$$

that Jacobi had established in the *Fundamenta nova*. ($|x| = 1$ is a natural boundary for $1 + 2\sum_{n=1}^{\infty} x^{n^2}$).

This was a particular example of the main theorem he proved in his paper, namely that a series of rational functions converging uniformly inside a disconnected domain may represent different monogenic functions on disjoint regions of the domain.

Theorems opposing "the standard view"

Weierstraß pointed out two theorems that "did not coincide with the standard view", namely

1. the continuity of a real function does not imply its differentiability
2. a complex function defined in a bounded domain cannot always be continued outside it.

The points where the function cannot be defined may be "not simply isolated points, but they can also make lines and surfaces". Weierstraß proved both theorems by resorting to his continuous nowhere differentiable function. Thus, the discovery of such a function seems to have to be related primarily to the problem of analytic continuation.

1884 Lecture at the Mathematical Seminar

Weierstraß summarised his approach to function theory, and his criticism of Cauchy's and Riemann's, in a lecture he gave at the Mathematical Seminar on May 28, 1884.

Weierstraß made a point of building analytic function theory without resorting to the Cauchy integral theorem whose proof, based on a double integration, he did not consider "to be completely methodical". Furthermore, his discovery of both continuous nowhere differentiable functions and series having natural boundaries convinced him to have nothing to do with "the old definitions" of function as given by Cauchy and Riemann involving the Cauchy-Riemann equations. But "all the difficulties vanish", he stated, when one follows his own power series approach. Indeed, if one starts from power series only "the first elements of arithmetic" are needed. Weierstraß himself was strengthened in this view by the remark that the very same approach could "more easily" be followed for functions of several variables as was the case of Abelian functions, whose theory was the ultimate goal of all his mathematical work.