How normal can the *t*-statistic possibly be?

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Outline

Prologue

Normalized sums are ubiquitous in statistics
(binomial models, location parameter tests, U-statistics, etc.)

Central Limit Theorem (plus Slutsky's theorem):

$(\zeta_n : n \in \mathbb{N})$ iid sequence, $\mathrm{Var}[\zeta_1] < \infty$, $\overline{\zeta} = \sum_{i=1}^n \zeta_i/n$, then

$$\mathcal{L}\left(\sqrt{n}\frac{\overline{\zeta} - \mathbb{E}[\zeta_1]}{s}\right) \underset{(n\to\infty)}{\to} \mathcal{N}(0,1), \quad s = \left(\frac{1}{n-1}\sum_{i=1}^n (\zeta_i - \overline{\zeta})^2\right)^{1/2}.$$

We call $\sqrt{n}(\overline{\zeta} - \mathbb{E}[\zeta_1])/s$ t-statistic of $(\zeta_j)_{j=1,\dots,n}$.

Berry-Esséen:

$\mathbb{E}[|\zeta_1|^3]$ finite, then rate of convergence is at least $O(1/\sqrt{n})$.

Questions in practice

1. Can convergence behavior be characterized more sharply?
2. What roles do higher moments play?
3. Are there means of speeding convergence up?
4. "**How valid**" is statistical inference based on the CLT?

Answers (at the end of this talk):

1. **YES!**
2. **A crucial role.**
3. **YES!**
4. **It depends.**

Edgeworth expansion for standardized sums

Let $(\zeta_n : n \in \mathbb{N})$ iid sequence, $\mathbb{E}[\zeta_1] = 0$ and Var$[\zeta_1] = 1$.

$$S_n = \sqrt{n}\frac{\overline{\zeta}}{s} \text{ with } \overline{\zeta} = \sum_{i=1}^{n} \zeta_i/n \text{ and } s = \left(\frac{1}{n}\sum_{i=1}^{n}(\zeta_i - \overline{\zeta})^2\right)^{1/2}.$$

Modern notation of an (Edgeworth) expansion for the cdf of S_n:

$$F_n(x) = \Phi(x) + \varphi(x)\sum_{i=1}^{r} n^{-i/2}q_i(x) + o(n^{-r/2}), \tag{1}$$

with Φ cdf and φ pdf of $\mathcal{N}(0,1)$. Each q_i is a polynomial of order $3i-1$ with coefficients depending on $\alpha_j = \mathbb{E}\zeta_1^j, j = 3, \ldots, i+2$.

Validity (cf. [5]): $\mathbb{E}|\zeta_1|^{r+2} < \infty$ and Cramér's condition holds.

The polynomials q_1 and q_2

The first two polynomials $q_1,\ q_2$ are computed for example in [5], [6] and can be found in various textbooks. They are given by

$$
\begin{aligned}
q_1(y) &= \left(\frac{1}{6}+\frac{1}{3}y^2\right)\alpha_3,\\
q_2(y) &= \left(\frac{1}{12}y^3-\frac{1}{4}y\right)\alpha_4+\left(\frac{1}{6}y-\frac{1}{18}y^5-\frac{1}{9}y^3\right)\alpha_3{}^2-\frac{1}{2}y^3.
\end{aligned}
$$

This representation shows that the rate of convergence is $O(n^{-1/2})$ in case of $\alpha_3 \neq 0$ and $O(n^{-1})$ in case of $\alpha_3 = 0$.

Obviously, the best possible rate of convergence is $O(n^{-1})$ and this may be the reason that usually only the first two polynomials are reported.

Impact of normalization

Lehmann and Romano (2005), Section 11.4.2:

Edgeworth expansion for the classical t-statistic with normalization $(n-1)^{-1}$ in the definition of s.

Approximation polynomials in this case differ from the q_i's in (1).

Hence, the norming sequence in the denominator of a self-normalized sum is of importance for its asymptotic behavior.

$\Rightarrow$ Question: Exist other norming sequences for specific values of the moments α_i, $i \geq 3$, such that the rate of convergence can be improved?

An early approach: Kai-Lai Chung (1946), cf. [2]

Back in 1946, Kai-Lai Chung derived an expansion for F_n.

Unfortunately, the explicit expansion given in equation (35) in [2] is incorrect as noted earlier by Wallace in [9] and to our knowledge there seems to be no published correction.

We corrected the main inaccuracy in [2] and extended the formulas where necessary.

⇒ Chung's method elementary, straightforward and efficient!

In principle, the q_i's are computable up to arbitrary order with an algebraic computer package.

Correction of Chung's error

In the derivations in [2], the function g defined by

$$g(\lambda) = z(1+\lambda^2 z^2)^{-1/2}\left[1+\sum_{j=1}^{\infty}\frac{\Gamma(3/2)}{\Gamma(3/2-j)\Gamma(j+1)}(\alpha_4-1)^{j/2}(\lambda x)^j\right]$$

and its derivatives $g^{(i)}$ play a crucial role.

The formulas given in [2], p. 458, struggle by abbreviating $z' = z(1+\lambda^2 z^2)^{-1/2}$ and ignoring that z' depends on λ.

Correct derivatives in $\lambda = 0$ are given by

$$\begin{aligned}
g^{(1)}(0) &= \frac{1}{2}z(\alpha_4-1)^{1/2}x, \quad g^{(2)}(0) = -z^3 - \frac{1}{4}z(\alpha_4-1)x^2, \\
g^{(3)}(0) &= -\frac{3}{2}z^3(\alpha_4-1)^{1/2}x + \frac{3}{8}z(\alpha_4-1)^{3/2}x^3, \\
g^{(4)}(0) &= 9z^5 + \frac{3}{2}z^3(\alpha_4-1)x^2 - \frac{15}{16}z(\alpha_4-1)^2x^4.
\end{aligned}$$

Chung's approximation technique

Formally, $F_n(z)$ is approximated in [2] by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w(x,y)\;dy\;dx + \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \gamma(x,y)\;dy\;dx,$$

with $w(x,y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right)$, $\rho = \alpha_3(\alpha_4 - 1)^{-1/2}$.

For the definition of $\gamma(x,y)$, we need some more notation:

$$w_{pq}(x,y) = \frac{\partial^{p+q}}{\partial x^p \partial y^q} w(x,y), \quad I_{pq}^r(z) = \int_{-\infty}^{\infty} x^r w_{pq}(x,z)\;dx,$$

$$f_{pq}(\lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w_{pq}(x,y)\;dy\;dx, \quad h(\zeta) = t_1 \frac{\zeta^2 - 1}{(\alpha_4 - 1)^{1/2}} + t_2\zeta,$$

where ζ has the same distribution as ζ_1.

Derivation of $\gamma(x, y)$

Let $U_j(t_1, t_2)$ denote the jth cumulant of $h(\zeta)$ and define

$$m_k(t_1, t_2) = \sum_{\ell=0}^{k-3} \frac{-i^{\ell+1} U_{\ell+3}(t_1, t_2)}{(\ell+3)!} \lambda^{\ell+1},$$

$$\Psi_k(it_1, it_2) = \sum_{j=1}^{k-3} \frac{m_j(t_1, t_2)^j}{j!}.$$

Expanding the U_i's in terms of t_1, t_2 and replacing $(it_1)^p (it_2)^q$ by $(-1)^{p+q} w_{pq}(x, y)$ in $\Psi_k(it_1, it_2)$ yields the representation

$$\Psi_k(it_1, it_2) \equiv \sum_{j=1}^{k-3} \gamma_j(x, y) = \gamma(x, y),$$

where $\gamma_j(x, y) = O(\lambda^j)$ and $w_{pq}(x, y)$ appears repeatedly in $\gamma_j(x, y)$ for various p, q with $p + q \leq 3r$.

Taylor expansion for $f_{pq}(\lambda)$

We can write

$$F_n(z) = \sum_{j=0}^{r} \frac{f_{00}^{(j)}(0)}{j!} + \sum_{j=1}^{r} \int_{-\infty}^{\infty} \int_{-\infty}^{g(\lambda)} \gamma_j(x,y)\, dy\, dx + o(\lambda^r).$$

Now, $f_{pq}(\lambda)$ is approximated by the Taylor series in $\lambda = 0$ wherever it appears in $\int_{-\infty}^{\infty} \int_{-\infty}^{g(\lambda)} \gamma_j(x,y)\, dy\, dx$. More precisely, $f_{pq}(\lambda)$ is replaced by

$$\sum_{j=0}^{r} \frac{f_{pq}^{(j)}(0)}{j!} \lambda^j + o(\lambda^r).$$

This means, all we need to carry out Chung's method are tractable formulas for $f_{pq}^{(j)}(0)$!

Lemma:

For $q \geq 0$, non-vanishing I_{pq}'s are given by the following recursion.

$$\begin{aligned} I_{0q}^{0}(z) &= \varphi^{(q)}(z), \\ I_{0q}^{1}(z) &= -\rho\varphi^{(q+1)}(z), \\ I_{0q}^{r+1}(z) &= -\rho I_{0,q+1}^{r}(z) + rI_{0q}^{r-1}(z), r \geq 1, \\ I_{pq}^{r}(z) &= -rI_{p-1,q}^{r-1}(z),\ 1 \leq p \leq r. \end{aligned} \tag{2}$$

Remark:

Modified Hermite polynomials: $h_n(x) = -(I/\sqrt{2})^n H_n(Ix/\sqrt{2})$.
Interestingly, $I_{0q}^{r} = h_r(\rho D)(\varphi^{(q)})$, where D denotes the differential operator. Note that (2) corresponds to $h_{r+1}(x) = -xh_r(x) + rh_{r-1}(x)$.
(X, Z) bivariate normal, means 0, variances 1 and covariance ρ:

$$I_{0q}^{r}(z) = \frac{\partial^q}{\partial z^q}\mathbb{E}[X^r|Z = z].$$

Lemma: (Computation of $f_{pq}^{(j)}(0)$)

Setting $I_{pq}^r \equiv I_{pq}^r(z)$ for $r = 0, \ldots, 3$, we have for $p, q \geq 0$

$$
\begin{aligned}
f_{00}(0) &= \Phi(z), \\
f_{pq}(0) &= \begin{cases} -I_{p,q-1}^0, & q \geq 1, \\ 0, & q = 0, \end{cases} \\
f_{pq}^{(1)}(0) &= \frac{1}{2} z(\alpha_4 - 1)^{1/2} I_{pq}^1, \\
f_{pq}^{(2)}(0) &= \frac{1}{4}(\alpha_4 - 1)\left(-zI_{pq}^2 + z^2 I_{pq+1}^2\right) - z^3 I_{pq}^0, \\
f_{pq}^{(3)}(0) &= -\frac{3}{2}(\alpha_4 - 1)^{1/2}\left(z^3 I_{pq}^1 + z^4 I_{p,q+1}^1\right) \\
&\quad + \frac{1}{8}(\alpha_4 - 1)^{3/2}\left(3zI_{pq}^3 - 3z^2 I_{p,q+1}^3 + z^3 I_{p,q+2}^3\right).
\end{aligned}
$$

Computational remarks

Now all ingredients for the computation of the polynomials q_i are collected.

Computation by hand remains cumbersome. Therefore we prepared a Maple worksheet which allows the computation of the q_i's up to arbitrary order.

Due to the structure of the $f_{pq}^{(j)}$'s, the lemma can be extended by utilizing standard symbolic integration methods.

Clearly, resources limit the number of computable q_i's.

Remark:

We also computed the q_i's with Hall's 'smooth function' method described in [6] up to order 6 with complete coincidence. Hall's method involves the computation of moments of more complicated statistics and seems more time consuming.

Rates of convergence

Recall that $q_1(y) \equiv 0$ for $\alpha_3 = 0$.
Interpretation: Vanishing skewness of $\zeta_1 \Rightarrow$ On the $n^{-1/2}$ scale, the approximation of F_n cannot be distinguished from Φ.
However, the rate of convergence of F_n towards Φ can at most be $O(n^{-1})$, because q_2 never vanishes.

Our approach to improve this rate of convergence:

$$T_n = \frac{\sqrt{n}\bar{\zeta}}{\sqrt{a_n \sum_{i=1}^n (\zeta_i - \bar{\zeta})^2}}, \quad \text{where} \quad a_n = \frac{1}{n(1 - \sum_{j=1}^M C_j n^{-j/2})}$$

Formal expansion for the generalized self-normalized sum T_n:

$$F_{T_n}(t) = \Phi(t) + \sum_{i=1}^{r} n^{-i/2} \tilde{q}_i(t) \varphi(t) + o(n^{-r/2}) \tag{3}$$

Coefficients of the $\tilde{q}_i$'s depend on cumulants of ζ_1 and on C_j's.

Derivation of the approximation for T_n

Notice that $T_n = S_n / b_n$ with $b_n = \sqrt{n a_n}$. Therefore,

$$F_{T_n}(t) = \mathbb{P}(T_n \leq t) = \mathbb{P}(S_n \leq b_n t) = F_n(b_n t).$$

From (1), we get under the necessary moment condition that

$$F_{T_n}(t) = \Phi(b_n t) + \sum_{i=1}^{r} n^{-i/2} q_i(b_n t) \varphi(b_n t) + o(n^{-r/2}).$$

In terms of $\Phi(t)$ and $\varphi(t)$, we can write

$$F_{T_n}(t) = \Phi(t) + \varphi(t) \left[h_n(t) + \sum_{i=1}^{r} n^{-i/2} q_i(b_n t) g_n(t) \right],$$

where the auxiliary functions h_n and g_n are defined by

$$h_n(t) = \left[\frac{\Phi(b_n t)}{\Phi(t)} - 1 \right] \frac{\Phi(t)}{\varphi(t)}, \qquad g_n(t) = \varphi(b_n t) / \varphi(t).$$

$\Rightarrow$ Expansions for b_n, $h_n(t)$, $g_n(t)$ needed!

Lemma:

Setting $\lambda = n^{-1/2}$, asymptotic expansions of b_n, $h_n(t)$ and $g_n(t)$ are given by

$$\begin{aligned} b_n &= 1 + \frac{C_1}{2}\lambda + \frac{C_2 + 3{C_1}^2/4}{2}\lambda^2 + O\left(\lambda^3\right), \\ h_n(t) &= \frac{C_1 t}{2}\lambda + \frac{t}{8}\left(4\,C_2 + 3\,{C_1}^2 - {C_1}^2 t^2\right)\lambda^2 + O\left(\lambda^3\right), \\ g_n(t) &= 1 - \frac{C_1 t^2}{2}\lambda - \frac{t^2}{2}\left({C_1}^2 + C_2 - \frac{{C_1}^2 t^2}{4}\right)\lambda^2 + O\left(\lambda^3\right). \end{aligned}$$

Proof:

The expansions for b_n and $g_n(t)$ are simple applications of the Taylor series of the square root and the exponential functions. For the expansion of $h_n(t)$, well-known asymptotic expansions for Mills' ratio are needed additionally.

Resulting approximation polynomials

Having expanded b_n, $h_n(t)$, and $g_n(t)$ in this manner, we finally obtain the first two $\tilde{q}_i$'s as

$$
\begin{aligned}
\tilde{q}_1(t) &= \frac{\alpha_3 t^2}{3} + \frac{\alpha_3}{6} + \frac{C_1 t}{2}, \\
\tilde{q}_2(t) &= \frac{3tC_1{}^2}{8} + \frac{\alpha_4 t^3}{12} + \frac{\alpha_3{}^2 t}{6} - \frac{t^3 C_1{}^2}{8} - \frac{\alpha_3{}^2 t^3}{9} - \frac{\alpha_3{}^2 t^5}{18} \\
&\quad + \frac{\alpha_3 C_1 t^2}{4} + \frac{tC_2}{2} - \frac{t^3}{2} - \frac{\alpha_3 C_1 t^4}{6} - \frac{\alpha_4 t}{4}.
\end{aligned}
$$

Sanity check

Setting $M = 2$, $C_1 = 0$ and $C_2 = 1$, we get the Studentized sum

$$\tilde{S}_n = \frac{\sqrt{n}\bar{\zeta}}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\zeta_i - \bar{\zeta})^2}}$$

with corresponding approximation polynomials

$$\begin{aligned}
\tilde{q}_1(t) &= \frac{\alpha_3\, t^2}{3} + \frac{\alpha_3}{6} = \frac{\alpha_3}{6}\left(2t^2 + 1\right), \\
\tilde{q}_2(t) &= \frac{\alpha_4 t^3}{12} + \frac{{\alpha_3}^2 t}{6} - \frac{{\alpha_3}^2 t^3}{9} - \frac{{\alpha_3}^2 t^5}{18} + \frac{t}{2} - \frac{t^3}{2} - \frac{\alpha_4 t}{4} \\
&= t\left[\frac{\kappa}{12}\left(t^2 - 3\right) - \frac{\alpha_3^2}{18}\left(t^4 + 2t^3 - 3\right) - \frac{1}{4}\left(t^2 + 1\right)\right],
\end{aligned}$$

where $\kappa = \alpha_4 - 3$ denotes the excess kurtosis of ζ_1. These are just the approximation polynomials given in Section 11.4.2 of the textbook [8] by Lehmann and Romano.

Rates of convergence for generalized self-normalized sums

Theorem:

Let $\Delta_n(x) = |F_{T_n}(x) - \Phi(x)|$.

(i) If $\alpha_3 \neq 0$ or $C_1 \neq 0$, then $\Delta_n(x) = O(n^{-1/2})$.

(ii) If $\alpha_3 = C_1 = 0$ and ($\alpha_4 \neq 6$ or $C_2 \neq 3$), then $\Delta_n(x) = O(n^{-1})$.

(iii) If $\alpha_3 = C_1 = 0$ and $\alpha_4 = 6$ and $C_2 = 3$ and ($\alpha_5 \neq 0$ or $C_3 \neq 0$), then $\Delta_n(x) = O(n^{-3/2})$.

(iv) If $\alpha_3 = C_1 = 0$ and $\alpha_4 = 6$ and $C_2 = 3$ and $\alpha_5 = C_3 = 0$, then $\Delta_n(x) = O(n^{-2})$.

Proof:

For parts (i)-(iii), we subsequently solve $\tilde{q}_i(t) \equiv 0$ for C_i and α_{i+2} for $i = 1, 2, 3$. For the proof of part (iv), we show that it is impossible to find values for (α_6, C_4) such that $\tilde{q}_4$ becomes the null polynomial.

Remark:

1. The studentized sum $\tilde{S}_n$ with $C_i = 0$ for all $i \neq 2$ and $C_2 = 1$ can only achieve a rate of convergence of $O(n^{-1})$.
2. Justification for the special role of $C_2 = 3$:
   Norming $a_n = (n-3)^{-1}$ leads to variance standardization of T_n, that is, $\mathrm{Var}[T_n] = 1$ if the ζ_i are iid normal as $\mathcal{N}(0,1)$.
3. The special role of $\alpha_4 = 6$ in parts (ii) - (iv) is not clear to us.

Example:

Let $\varphi(x|\sigma)$ denote the pdf of $\mathcal{N}(0,\sigma^2)$ and the density of ζ_1 given by

$$f_{\zeta_1}(x) = \alpha\varphi(x|\sigma_1) + (1-\alpha)\varphi(x|\sigma_2)$$

with $\sigma_1^2 = (2\alpha)^{-1}$, $\sigma_2^2 = (2(1-\alpha))^{-1}$ and $\alpha = (2+\sqrt{2})/4$.

Then $\mathbb{E}\zeta_1 = \mathbb{E}\zeta_1^3 = \mathbb{E}\zeta_1^5 = 0$, $\mathbb{E}\zeta_1^2 = 1$, $\mathbb{E}\zeta_1^4 = 6$, and $\mathbb{E}\zeta_1^6 = 90$.
Setting $C_1 = C_3 = 0$ and $C_2 = 3$, part (iv) applies in this case.

Expansion in terms of Student's t

Goal here: Derive an Edgeworth-type expansion for T_n of form

$$F_{T_n}(t) = F_{t_\nu}(t) + \sum_{i=1}^{r} n^{-i/2} Q_i(t) \varphi(t) + o(n^{-r/2}) \qquad (4)$$

in terms of Student's t with $\nu = n - 1$ degrees of freedom.

Note: T_n with norming sequence $a_n = (n-1)^{-1}$ and $\zeta_1 \sim \mathcal{N}(0,1)$ is exactly t_ν-distributed.

Questions:

1. Can an improved rate of convergence be obtained by changing the approximating distribution from $\mathcal{N}(0,1)$ to t_ν?
2. Can the norming constants C_j be employed to correct for higher-order moments of ζ_1 ?

Derivation of the Q_i's in (4)

Denote by $q_i^*, i = 1, \ldots, r$, the approximation polynomials for $T_n = t_\nu$, i.e., for special choices $M = 2$, $C_1 = 0$ and $C_2 = 1$ and $\alpha_j, j = 3, \ldots, (r+2)$, equal to the moments of $\mathcal{N}(0, 1)$.

By subtracting the resulting expansion from the general expansion for T_n, we immediately conclude that

$$Q_i(t) = \tilde{q}_i(t) - q_i^*(t), i = 1, \ldots, r.$$

Carrying out these calculations, we obtain the first four q_i^*'s as

$$\begin{aligned} q_1^*(t) = q_3^*(t) &\equiv 0, \\ q_2^*(t) &= -\frac{t}{4}\left(t^2+1\right), \\ q_4^*(t) &= -\frac{t}{96}\left(3\,t^6 - 7\,t^4 + 19\,t^2 + 21\right). \end{aligned}$$

Consequently, the first two Q_i's are given by

$$\begin{aligned} Q_1(t) &= \tilde{q}_1(t) = \frac{\alpha_3\, t^2}{3} + \frac{C_1 t}{2} + \frac{\alpha_3}{6}, \\ Q_2(t) &= -\frac{\alpha_3^2}{18} t^5 - \frac{\alpha_3 C_1}{6} t^4 - \left(\frac{1}{4} + \frac{\alpha_3^2}{9} - \frac{\alpha_4}{12} + \frac{C_1^2}{8} \right) t^3 + \frac{\alpha_3 C_1}{4} t^2 \\ &\quad + \left(\frac{3C_1^2}{8} + \frac{C_2}{2} + \frac{\alpha_3^2}{6} - \frac{\alpha_4}{4} + \frac{1}{4} \right) t. \end{aligned}$$

Rates of convergence:

- Q_1 only vanishes for $\alpha_3 = C_1 = 0$.
- Q_2 only vanishes if additionally $\alpha_4 = 3$ and $C_2 = 1$, i. e., in case of coincidence with the classical t-distribution case.
- This need for coincidence extends to the conditions for vanishing Q_3 and Q_4 (explicit formulas omitted here) .
- Conclusion: t-approximation instead of normal approximation does not help to increase convergence rates.

A link to Gayen's (1949) method

Substitute φ in (4) by the pdf f_{t_ν} of the t_ν-distribution:

$$F_{T_n}(t) = F_{t_\nu}(t) + \sum_{i=1}^{r} n^{-i/2}\, \tilde{Q}_i(t)\, f_{t_\nu}(t) + o(n^{-r/2}). \qquad (5)$$

Closely related expressions for F_{T_n} for fixed n have already been investigated in 1949 by A. K. Gayen based on M. S. Bartlett's famous paper [1].

One can derive the first four $\tilde{Q}_i$'s in (5) by expanding

$$\varphi(t) = f_{t_\nu}(t)\left[1 + \frac{1+2t^2-t^4}{4n} + O(n^{-2})\right].$$

Plugging the latter expansion into (4) leads to

$$\begin{aligned}\tilde{Q}_i &\equiv Q_i,\ i = 1,2,\\ \tilde{Q}_i(t) &= Q_i(t) + \frac{1+2t^2-t^4}{4} Q_{i-2}(t),\ i = 3,4.\end{aligned}$$

Comparison with Gayen's results

Unfortunately, we could only reproduce Gayen's (1949) results up to order n^{-1}.

Taking limits ($n \to \infty$) in Gayen's paper also yields $\tilde{Q}_i \equiv Q_i,\ i = 1, 2$.

However, the expressions of order $O(n^{-3/2})$ associated with the factors α_3^3 and $\alpha_3\alpha_4$ seem to be in error in [4], p. 359, and also taking limits ($n \to \infty$) in these expressions does not coincide with our results.

Therefore, we also recomputed the original approximation method by Bartlett (cf. [1]) which underlies Gayen's (1949) calculations and finally reproduced "our" $\tilde{Q}_i$'s for $i = 1, \dots, 4$.

Asymptotic order of magnitude of $|F_{T_n} - F_{t_\nu}|$

Utilizing Chung's method and higher order expansions for $\varphi(t)/f_{t_\nu}(t)$, we calculated Q_i's and $\tilde{Q}_i$'s up to order 8.

$$\alpha_k^* : k-\text{th moment of } \mathcal{N}(0,1), \Delta\alpha_k = \alpha_k^* - \alpha_k$$

$$C_2^* = 1, C_k^* = 0 \text{ for } k \neq 2 \text{ and } \Delta C_k = C_k^* - C_k$$

Corollary:

Assume that the $(M+2)$-nd moment α_{M+2} of ζ_1 is finite for some integer $1 \leq M \leq 8$ and Cramér's condition holds. Then

$$|F_{T_n} - F_{t_\nu}| = O(n^{-k^*/2}),$$

where $k^* = \min\{k \in \{1, \ldots, M\} : \Delta\alpha_{k+2} \neq 0 \vee \Delta C_k \neq 0\}$.

If no such k^* exists, then $|F_{T_n} - F_{t_\nu}| = o(n^{-M/2})$.

What happens for $M > 8$?

Since each polynomial Q_i or $\tilde{Q}_i$, respectively, only depends on $\alpha_j,\ j = 3, \ldots, i+2$ and $C_j, j = 1, \ldots, i$, and equations (4), (5) are valid for $T_n = t_\nu$, it is clear that also for $M > 8$ the conditions

$$\Delta\alpha_{i+2} = 0 \wedge \Delta C_i = 0 \text{ for all } i = 1, \ldots, M \tag{6}$$

imply $Q_i(t) \equiv 0$ and $\tilde{Q}_i(t) \equiv 0$ for all $i = 1, \ldots, M$.

$\Rightarrow$ Conditions (6) are sufficient for vanishing polynomials up to the M-th for arbitrary $M \in \mathbb{N}$.

We conjecture that conditions (6) are also necessary conditions for any $M \geq 1$ as stated in the Corollary for $1 \leq M \leq 8$.

Conclusions

- Four different types of Edgeworth expansions for S_n, T_n. Once polynomials for one are obtained, they can be utilized to derive the polynomials for the others straightforwardly.
- At `http://www.helmut-finner.de`, find Maple sheets for Chung's, Hall's, and the Bartlett-Gayen method.
- Practical implications:
  - For skewed distributions, no convergence rate improvement upon $O(n^{-1/2})$ is possible with our approach.
  - If there is any evidence that α_4 is near 6, a normal approximation with $C_2 = 3$ is the best choice leading to $|F_{T_n}(x) - \Phi(x)| = O(n^{-3/2})$.
  - t-approximation works best for $a_n = (n-1)^{-1}$ and can achieve arbitrary rate of convergence for universes which are close to standard normal in terms of moments. This makes the t-approximation a more natural choice if we assume that the universe is "nearly normal".

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