Reduced-Order Models for Parameter-Dependent Partial Differential Equations

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We are interested in the efficient numerical solution of parameterized partial differential equations (PDEs) $\mathcal{L}_{\boldsymbol{\xi}} u(\cdot, \boldsymbol{\xi}) = f$ on a spatial domain $\mathcal{D}$, where the differential operator $\mathcal{L}$ depends on (a vector of) m parameters $\boldsymbol{\xi} \equiv (\xi_1, \ldots, \xi_m)^T$. A simple example is the diffusion operator $\mathcal{L}_{\boldsymbol{\xi}} = -\nabla a(\cdot, \boldsymbol{\xi}) \nabla$ where the diffusion coefficient is a random field, i.e., for each $x \in \mathcal{D}$, $a(x, \cdot)$ is a random variable depending on $\boldsymbol{\xi}$; other scenarios, for example with parameterized boundary conditions, wave numbers in acoustics or convection terms, are also possible. For any given $\boldsymbol{\xi}$, discretization of the PDE in space leads to a finite-dimensional algebraic system of equations $F_{\boldsymbol{\xi}}(\mathbf{u}_{\boldsymbol{\xi}}) = 0$ whose solution (which gives a discrete approximation $u_h(\cdot, \boldsymbol{\xi})$ to the PDE solution) also depends on $\boldsymbol{\xi}$. If high-resolution accuracy is desired, then the solution of the discrete models tends to be costly, and if in addition solutions are required for many parameters, then costs become prohibitive. We will discuss the use of *reduced-order models* based on *reduced-basis methods* to reduce these costs.

Let N denote the size (number of degrees of freedom) of the discrete system $F_{\boldsymbol{\xi}}(\mathbf{u}_{\boldsymbol{\xi}}) = 0$. The idea behind reduced-basis methods is to compute solutions $\mathbf{u}_{\boldsymbol{\xi}_1}, \ldots, \mathbf{u}_{\boldsymbol{\xi}_n}$ of this system corresponding to a specific set of parameters $\{\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}\}$, where $n \ll N$, and then, for other choices of $\boldsymbol{\xi}$, to compute approximate (or *surrogate*) solutions $\mathbf{u}_{\boldsymbol{\xi}}^{(s)}$ from the space spanned by the set of so-called "snapshot solutions" $\{\mathbf{u}_{\boldsymbol{\xi}_j}\}_{j=1}^n$. If Q is a matrix whose columns span the space of snapshot solutions, then the surrogate solution associated with parameter $\boldsymbol{\xi}$ has the form $\mathbf{u}_{\boldsymbol{\xi}}^{(s)} = Q\mathbf{y}_{\boldsymbol{\xi}}$. A typical strategy used to define the reduced problem is to impose a Galerkin condition, i.e., to require $Q^T F_{\boldsymbol{\xi}}(Q\mathbf{y}_{\boldsymbol{\xi}}) = 0$, so that the residual of the surrogate solution is orthogonal to the space spanned by the snapshots.

Two important aspects of this methodology are that the reduced solution should be an accurate approximation of the original (full) discrete system, and that the reduced problem should be much less expensive to solve. We will discuss various issues that enable these conditions to be valid as well as some things that are not fully resolved. Topics to be discussed include

- The separation of "offline" computations required to find the snap-

shot solutions and "online" computations used to find the surrogate solutions for multiple parameters.

- The costs of performing the online computations and the impact on the size of the parameter set on efficiency in the case of linear models.
- Issues that arise when the dependence of the model on the parameters is nonlinear, and ways to avoid some of the pitfalls that arise in this case.