From Damage to Delamination in Nonlinearly Elastic Materials

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joint work with Alexander Mielke and Tomáš Roubíček

6th Singular Days – Berlin, May, 1st, 2010

1. Modelling of damage and delamination
2. Setup
3. Mathematical tools
   - Energetic formulation of rate-independent processes
   - Existence result for partial damage
   - Γ-convergence of rate-independent processes
4. Delamination models as the limits of damage models
   - Partial damage
   - Gradient delamination ($\varepsilon \longrightarrow 0$, $\kappa > 0$ fixed)
   - Griffith-type delamination ($\kappa \longrightarrow 0$)
5. Conclusion

Damage, in its mechanical sense in solid materials is the creation and growth of microvoids or microcracks which are discontinuities in a medium considered as continuous at a larger scale. [Lemaitre/Desmorat05]

For all $t \in [0,T]$ and all $x \in \Omega \subset \mathbb{R}^d$ we define:

local damage variable $z(t,x) := \frac{\mathcal{L}^d((\Omega \cap B_r(x)) \setminus (\textbf{holes at time } t))}{\mathcal{L}^d(\Omega \cap B_r(x))}$ (r fixed).

$$\Rightarrow \quad \begin{aligned} z(t,x) &= 1 : && \text{no damage,} \\ z(t,x) &= 0 : && \text{complete damage,} \\ z(t,x) &\geq z_\star > 0 : && \text{partial damage} \end{aligned}$$

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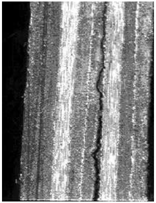


Delamination: Micro-cracking along interface $\Gamma_\mathrm{C} \subset \mathbb{R}^{d-1}$

local delamination variable $z : [0,T] \times \Gamma_\mathrm{C} \to [0,1]$

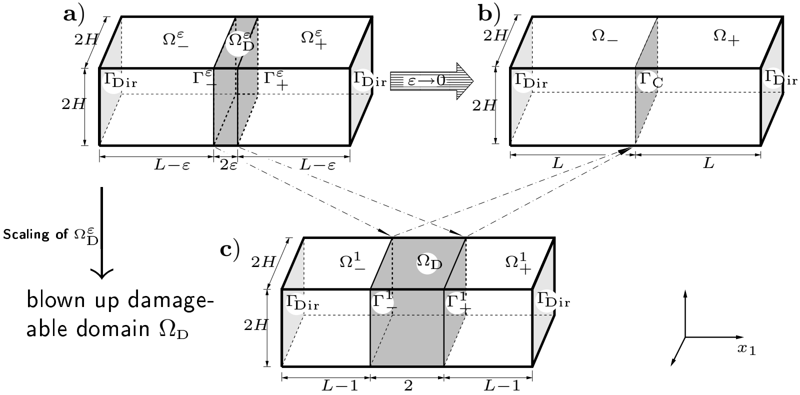
$$\left.\begin{aligned} z(t,x) &= 1: \text{bonding fully intact}\\ z(t,x) &> 0: \text{bonding} \end{aligned}\right\} \Rightarrow \text{transmission cond.s}$$

$$z(t,x) = 0: \text{bonding completely broken} = \text{crack}$$

Partial damage: $0 < \varepsilon^\gamma \leq z \leq 1$ in Ω_D^ε
No damage in $\Omega_-^\varepsilon \cup \Omega_+^\varepsilon$

$\xrightarrow{\varepsilon \to 0}$

Delamination along Γ_C
No damage in $\Omega_- \cup \Omega_+$



blown up damage-able domain Ω_D

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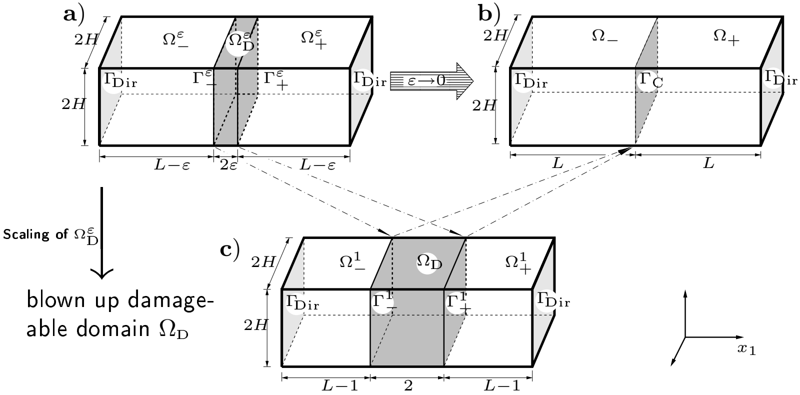

Tools:

- **Energetic formulation** by functionals $\mathcal{E}_\varepsilon^\kappa$, $\mathcal{R}$ on common space $\mathcal{Q}$ wrt. $\Omega_- \cup \Omega_+$, Ω_D
- existence result for $(\mathcal{Q}, \mathcal{E}_\varepsilon^\kappa, \mathcal{R})$ for ε, κ fixed
- abstract **Γ-convergence result** [Mielke/Roubíček/Stefanelli08]

Rate-independence modelled by **positive 1-homogeneity** of $\mathcal{R} : \mathcal{Z} \to [0,\infty]$,
i.e. $\mathcal{R}(\alpha v) = \alpha \mathcal{R}(v)$ for all $\alpha \geq 0$ and all $v \in \mathcal{Z}$,

$\mathcal{Z}$: set of damage/delamination variables

Energetic formulation of the rate-independent damage/delamination processes,
i.e. $\widehat{\mathcal{E}} \in \{\mathcal{E}_\varepsilon^\kappa, \mathcal{E}^\kappa, \mathcal{E} \,|\, \varepsilon \in (0,\varepsilon_0], \kappa \in (0,\kappa_0]\}$

Definition: $q : [0,T] \to \mathcal{Q}$ is an **energetic solution** to $(\mathcal{Q}, \widehat{\mathcal{E}}, \mathcal{D})$, if for all $t \in [0,T]$ it holds $\partial_t \widehat{\mathcal{E}}(\cdot, q(\cdot)) \in L^1((0,T))$, $\widehat{\mathcal{E}}(t,q(t)) < \infty$ and:

$$\begin{cases} \text{(S) Stability: } & \text{for all } \tilde{q} \in \mathcal{Q}: \ \widehat{\mathcal{E}}(t,q(t)) \leq \widehat{\mathcal{E}}(t,\tilde{q}) + \mathcal{R}(\tilde{z} - z(t)), \\ \text{(E) Energy balance: } & \widehat{\mathcal{E}}(t,q(t)) + \mathrm{Diss}_{\mathcal{R}}(z,[0,t]) = \widehat{\mathcal{E}}(0,q(0)) + \int_0^t \partial_t \widehat{\mathcal{E}}(\xi, q(\xi))\mathrm{d}\xi, \end{cases}$$

where $\mathrm{Diss}_{\mathcal{R}}(z,[s,t]) := \sup_{\text{all part. of } [s,t]} \sum_{j=1}^N \mathcal{R}(z(\xi_j) - z(\xi_{j-1}))$.

- Existence result for partial damage (consequence of [Th.,Mielke09]):

Theorem: Under technical assumptions on the energy density and the given data there exists an energetic solution $q=(u,z):[0,T]\to\mathcal{Q}$ for $(\mathcal{Q},\mathcal{E}_\varepsilon^\kappa,\mathcal{R},q_0)$ for all fixed $\varepsilon>0$, $\kappa>0$ and all initial conditions $(t=0,q_0)$, which satisfy (S).

(S) $\mathcal{E}_\varepsilon^\kappa(t,q(t))\le\mathcal{E}_\varepsilon^\kappa(t,\tilde q(t))+\mathcal{R}(\tilde z(t)-z)$ for all $\tilde q=(u,z)\in\mathcal{Q}$

- Abstract Γ-convergence result for rate-independent processes $(\mathcal{Q},\mathcal{E}_j,\mathcal{R}_j)_{j\in\mathbb{N}}$ [Mielke, Roubíček, Stefanelli08]

$\mathcal{X}$: metric space, $\mathcal{G}_k, \mathcal{G} : \mathcal{X} \to \mathbb{R}_\infty = \mathbb{R} \cup \infty$,

Γ-convergence (De Giorgi): $\mathcal{G}_k \xrightarrow{\Gamma} \mathcal{G} \Leftrightarrow$ for every $w \in \mathcal{X}$ holds:

1. Γ-lim inf-inequality: $\forall\, (w_k)_{k\in\mathbb{N}},\, w_k \overset{\mathcal{X}}{\to} w :\ \mathcal{G}(w) \leq \liminf_{k\to\infty} \mathcal{G}_k(w_k)\,,$
2. Recovery sequence: $\exists\, (\hat{w}_k)_{k\in\mathbb{N}},\, \hat{w}_k \overset{\mathcal{X}}{\to} w :\ \mathcal{G}(w) \geq \limsup_{k\to\infty} \mathcal{G}_k(\hat{w}_k)\,.$

$$\mathcal{G}_k \xrightarrow{\Gamma} \mathcal{G} \ \Rightarrow\ \operatorname{argmin}_{v\in\mathcal{X}}\{\mathcal{G}_k(v)\} \ni \boldsymbol{w_k} \xrightarrow{\boldsymbol{\mathcal{X}}} \boldsymbol{w} \in \operatorname{argmin}_{v\in\mathcal{X}}\{\mathcal{G}(v)\}$$

For static minimization problems!

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Here: energetic solutions $q_j : [0,T] \to \mathcal{Q}$

Wanted:

energetic solutions of $(\mathcal{Q}, \mathcal{E}_j, \mathcal{R}_j)$ converge to energetic solutions of $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$

$\Rightarrow \mathcal{E}_j \overset{\Gamma}{\to} \mathcal{E}$ & $\mathcal{R}_j \overset{\Gamma}{\to} \mathcal{R}$ not sufficient

Important: Properties (S) & (E) have to be preserved under convergence!

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Sufficient conditions by abstract result in [Mielke/Roubíček/Stefanelli08]

Crucial properties:

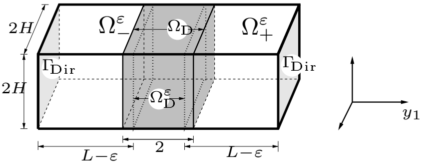
For all $(t_j, q_j) \to (t, q)$ in $[0,T] \times \mathcal{T}$ with $\mathcal{E}_j(t_j, q_j) \leq E$ and

$$(\mathrm{S_j}) \quad \mathcal{E}_j(t_j, q_j) \leq \mathcal{E}_j(t_j, \hat{q}) + \mathcal{R}_j(\hat{q} - q_j) \text{ for all } \hat{q} \in \mathcal{Q}$$

holds:

1. Γ-$\liminf$-inequalities for $\mathcal{E}_j$ and $\mathcal{R}_j$
2. $\partial_t \mathcal{E}_j(t_j, q_j) \to \partial_t \mathcal{E}_\infty(t, q)$
3. Upper semicontinuity of stable sets:

$$\mathcal{E}_\infty(t, q) \leq \mathcal{E}_\infty(t, \hat{q}) + \mathcal{R}_\infty(\hat{q} - q) \text{ for all } \hat{q} \in \mathcal{Q}$$

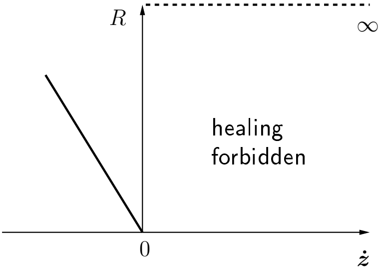


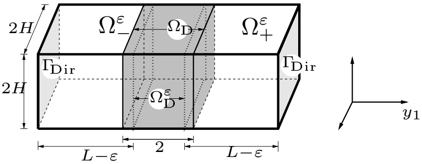
Dissipation potential for $v : \Omega_D^\varepsilon \to \mathbb{R}$:

$$\mathcal{R}_\varepsilon(v) = \int_{\Omega_D^\varepsilon} \frac{1}{\varepsilon} R(v(x))\mathrm{d}x$$

$$R(v) := \begin{cases} \rho|v| & \text{if } v \le 0 \\ \infty & \text{else} \end{cases}$$

for $\rho > 0$, $\varepsilon \in (0, \varepsilon_0]$, $v = \dot{z}$





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Scaling of Ω_D^ε ↓

Dissipation potential for $v : \Omega_D \to \mathbb{R}$:

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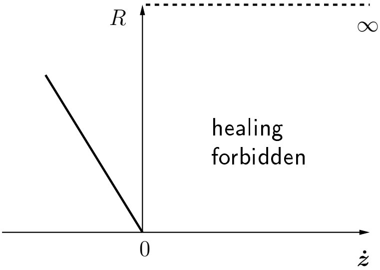
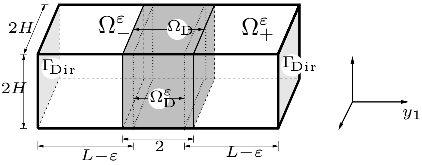


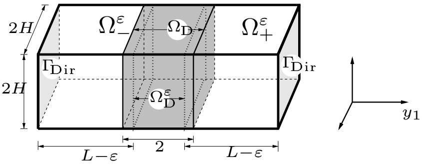


Ω : whole compound

$u : \Omega \to \mathbb{R}^d$ displacement

$$\mathcal{E}_\varepsilon^\kappa(t,q) := \begin{cases} \widetilde{\mathcal{E}}_\varepsilon^\kappa(t,q) & \text{if } q = (u,z) \in \mathcal{Q}_D = \mathcal{U}_D \times \mathcal{Z}_D \\ \infty & \text{otherwise} \end{cases}$$

$$\mathcal{U}_D := \{u \in W^{1,p}(\Omega,\mathbb{R}^d) \,|\, u = 0 \text{ on } \Gamma_{Dir}\}, \;\; \mathcal{Z}_D := W^{1,r}(\Omega_D)$$



Ω : whole compound

$u:\Omega\to\mathbb{R}^d$ displacement

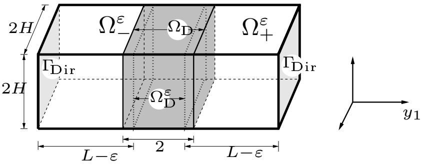
$e(u)=\frac{1}{2}(\nabla u+\nabla u^\top)$,

$\boldsymbol{\Pi^\varepsilon : W^{1,r}(\Omega_D)\to W^{1,r}(\Omega_D^\varepsilon)}$,

$\boldsymbol{\nabla}_\varepsilon=(\frac{1}{\varepsilon}\partial_{y_1},\partial_{y_2},\dots,\partial_{y_d})^\top$

$$\mathcal{E}_\varepsilon^\kappa(t,q):=\begin{cases}\widetilde{\mathcal{E}}_\varepsilon^\kappa(t,q) & \text{if } q=(u,z)\in\mathcal{Q}_D=\mathcal{U}_D\times\mathcal{Z}_D\\ \infty & \text{otherwise}\end{cases}$$

$$\widetilde{\mathcal{E}}_\varepsilon^\kappa(t,q):=\int_{\Omega_-^\varepsilon\cup\Omega_+^\varepsilon}W(e(u+g(t)))\,\mathrm{d}x+\int_{\Omega_D^\varepsilon}W_D(\boldsymbol{\Pi^\varepsilon z},e(u))\,\mathrm{d}x+\int_{\Omega_D}\Big(\frac{\kappa}{r}|\boldsymbol{\nabla}_\varepsilon z|^r+\underbrace{\delta_{[\varepsilon^\gamma,1]}(z)}_{\text{partial damage}}\Big)\,\mathrm{d}y$$



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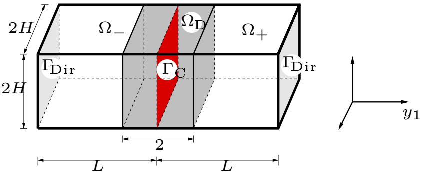
$$W_\mathrm{D}(\Pi^\varepsilon z, e) := \Pi^\varepsilon z W(e) + |(e_{11})^-|^p, \qquad (e_{11})^- = -\min\{e_{11}, 0\}$$



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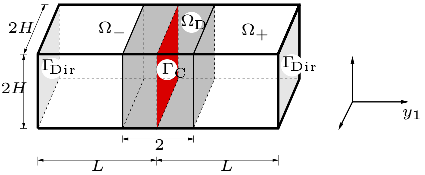
1. Γ-limit: $\varepsilon \longrightarrow 0$ gradient delamination
2. Γ-limit: $\kappa \longrightarrow 0$ Griffith-type delamination [Roubíček/Scardia/Zanini09]



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Crucial to describe delamination along Γ_C :

Transmission cond. $z[\![u]\!] = 0$ a.e. on Γ_C

Unilateral contact cond. $[\![u\cdot n_1]\!] \geq 0$ a.e. on Γ_C

(A1) Assumption: sequence $(t_\varepsilon, q_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ with $\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, q_\varepsilon) \le E$

Energy functional for the damage processes:

$$\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, q_\varepsilon) = \int_{\Omega_-^\varepsilon \cup \Omega_+^\varepsilon} W(e(u_\varepsilon + g(t_\varepsilon)))\,\mathrm{d}x + \int_{\Omega_\mathrm{D}^\varepsilon} W_\mathrm{D}(\Pi^\varepsilon z_\varepsilon, e(u_\varepsilon))\,\mathrm{d}x + \int_{\Omega_\mathrm{D}} \Big(\tfrac{\kappa}{r}|\nabla_\varepsilon z_\varepsilon|^r + \delta_{[\varepsilon^\gamma,1]}(z_\varepsilon)\Big)\,\mathrm{d}y$$

$$\nabla_\varepsilon = (\tfrac{1}{\varepsilon}\partial_{y_1}, \partial_{y_2}, \dots, \partial_{y_d})^\top$$

(A2) Assumption: coercivity $c|e|^p \le W(e)$, **but**:

$$W_\mathrm{D}(\Pi^\varepsilon z_\varepsilon, e(u_\varepsilon)) := \underbrace{\Pi^\varepsilon z_\varepsilon}_{\ge \varepsilon^\gamma \to 0}\; W(e(u_\varepsilon)) + |(e_{11}(u_\varepsilon))^-|^p,$$

loss of coercivity!

$$(e_{11}(u_\varepsilon))^- = -\min\{\partial_{x_1} u_\varepsilon, 0\}$$

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$\geq\varepsilon^\gamma\to 0$ loss of coercivity!

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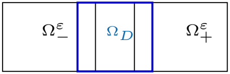
Theorem: Assume (A1). Let (A2) hold.

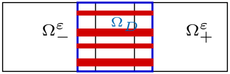
1. $r > d \Rightarrow$ transmission condition: $z|_{\Gamma_\mathrm{C}} [\![u]\!] = 0$ a.e. on Γ_C
2. $|(e_{11}(u_\varepsilon))^-|^p \Rightarrow$ unilateral contact condition: $[\![u\cdot \mathrm{n}_1]\!] \geq 0$ on Γ_C

$\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, q_\varepsilon) \le E$, thus subseq. $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega^\nu_- \cup \Omega^\nu_+, \mathbb{R}^d)$, $z_\varepsilon \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_\mathrm{D})$

1. $r > d$, W coerc. $\Rightarrow [\![u]\!] = 0$ a.e. on $\Gamma_\mathrm{C} \backslash N_z$ $\quad N_z = \{s \in \Gamma_\mathrm{C} \mid z|_{\Gamma_\mathrm{C}}(s) = 0\}$

- $\frac{\kappa}{r}\|\nabla_\varepsilon z_\varepsilon\|^r_{L^r(\Omega)} \le E$, hence $\frac{\kappa}{r}\|\partial_{y_1} z_\varepsilon\|^r_{L^r(\Omega)} \le E\varepsilon^r \rightsquigarrow \partial_{y_1} z = 0$





- smooth compact sets $K \subset \Omega_\mathrm{D} \backslash \{y \in \Omega_\mathrm{D} \mid z(y) = 0\}$
  s.th. $\mathcal{L}^{d-1}(K \cap \partial\Omega^{\varepsilon_0}_\pm) \neq \emptyset$

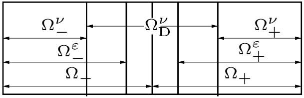
$\rightsquigarrow$ Korn's ineq. holds on $K \cup \partial\Omega^{\varepsilon_0}_\pm$

- $W^{1,r}(\Omega_\mathrm{D}) \Subset \mathrm{C}(\overline{\Omega_\mathrm{D}})$ for $r > d$
  $\Rightarrow \exists\, \delta_K > 0 : z, z_\varepsilon, \Pi^\varepsilon z_\varepsilon > \delta_K$ in K

  $\Rightarrow$ unif. coercivity on $K \cup \Omega^{\varepsilon_0}_\pm$
  $\Rightarrow u \in W^{1,p}(K \cup \Omega^{\varepsilon_0}_\pm)$ $\quad \rightsquigarrow [\![u]\!] = 0$ a.e. on $\Gamma_\mathrm{C} \backslash N_z$.

$\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, q_\varepsilon) \leq E$, thus subseq. $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$, $z_\varepsilon \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_\mathrm{D})$

2. W coerc., $|(e_{11}(u_\varepsilon))^-|^p \Rightarrow \int_{\Gamma_\mathrm{C}} (-[\![u \cdot \mathrm{n}_1]\!])^+ \,\mathrm{d}s = 0$



$$\int_{\Gamma_\mathrm{C}} (-[\![u \cdot \mathrm{n}_1]\!])^+ \,\mathrm{d}s = \lim_{\nu \to 0} \int_{\Gamma_\mathrm{C}} \big(u^1(-\nu, s) - u^1(+\nu, s)\big)^+ \,\mathrm{d}s$$

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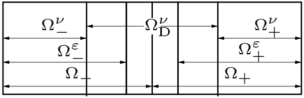

$$\int_{\Gamma_{\mathrm{C}}} (-[\![u \cdot \mathrm{n}_1]\!])^+ \, \mathrm{d}s = \lim_{\nu \to 0} \int_{\Gamma_{\mathrm{C}}} \big(u^1(-\nu, s) - u^1(+\nu, s)\big)^+ \, \mathrm{d}s$$

$$\text{Fatou} \leq \lim_{\nu \to 0} \liminf_{\varepsilon \to 0} \int_{\Gamma_{\mathrm{C}}} (u^1_\varepsilon(-\nu, s) - u^1_\varepsilon(+\nu, s))^+ \, \mathrm{d}s$$

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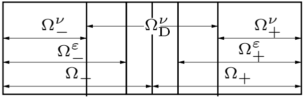

$$\int_{\Gamma_\mathrm{C}} (-[\![u\cdot \mathrm{n}_1]\!])^+\,\mathrm{d}s = \lim_{\nu\to 0}\int_{\Gamma_\mathrm{C}} \big(u^1(-\nu,s) - u^1(+\nu,s)\big)^+\,\mathrm{d}s$$

$$\text{Fatou} \le \lim_{\nu\to 0}\liminf_{\varepsilon\to 0}\int_{\Gamma_\mathrm{C}} (u_\varepsilon^1(-\nu,s) - u_\varepsilon^1(+\nu,s))^+\,\mathrm{d}s$$

$$\text{part. int.} = \lim_{\nu\to 0}\liminf_{\varepsilon\to 0}\int_{\Gamma_\mathrm{C}}\Big(\int_{-\nu}^{\nu} -\partial_{x_1} u_\varepsilon^1 \mathrm{d}x_1\Big)^+\mathrm{d}s$$

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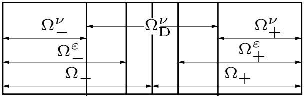

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$$\text{Jensen} \leq \liminf_{\nu\to 0} \liminf_{\varepsilon\to 0} \int_{\Omega_\mathrm{D}^\nu} \big(-\partial_{x_1} u_\varepsilon^1\big)^+ \,\mathrm{d}x$$

$\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, q_\varepsilon) \leq E$, thus subseq. $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$, $z_\varepsilon \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_{\mathrm{D}})$

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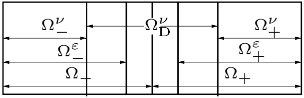

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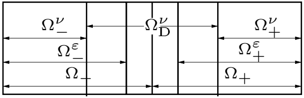
$$\text{Jensen} \leq \liminf_{\nu\to 0} \liminf_{\varepsilon\to 0} \int_{\Omega_{\mathrm{D}}^\nu} \big(-\partial_{x_1} u_\varepsilon^1\big)^+ \,\mathrm{d}x$$

- $W_{\mathrm{D}}(\Pi^\varepsilon z, e) = \Pi^\varepsilon z W(e) + |(e_{11})^-|^p$ & coercivity of W
  $\Rightarrow \|(\partial_{x_1} u_\varepsilon^1)^-\|_{L^p(\Omega)} \leq E \Rightarrow \exists\, b \in L^p(\Omega) : \ (\partial_{x_1} u_\varepsilon^1)^- \rightharpoonup b$ in $L^p(\Omega)$

$\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, q_\varepsilon) \le E$, thus subseq. $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$, $z_\varepsilon \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_D)$

2. W coerc., $|(e_{11}(u_\varepsilon))^-|^p \Rightarrow \int_{\Gamma_C} (-[\![u\cdot n_1]\!])^+ \,ds = 0$



$$\int_{\Gamma_C} (-[\![u\cdot n_1]\!])^+ \,ds = \lim_{\nu\to 0} \int_{\Gamma_C} \big(u^1(-\nu,s) - u^1(+\nu,s)\big)^+ \,ds$$

$$\text{Fatou} \le \lim_{\nu\to 0} \liminf_{\varepsilon\to 0} \int_{\Gamma_C} (u_\varepsilon^1(-\nu,s) - u_\varepsilon^1(+\nu,s))^+ \,ds$$

$$\text{part. int.} = \lim_{\nu\to 0} \liminf_{\varepsilon\to 0} \int_{\Gamma_C} \Big(\int_{-\nu}^{\nu} -\partial_{x_1} u_\varepsilon^1 dx_1\Big)^+ ds$$

$$\text{Jensen} \le \liminf_{\nu\to 0} \liminf_{\varepsilon\to 0} \int_{\Omega_D^\nu} (-\partial_{x_1} u_\varepsilon^1)^+ \,dx = \liminf_{\nu\to 0} \int_{\Omega_D^\nu} b \,dx = 0$$

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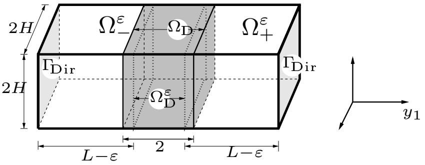


Ω : whole compound

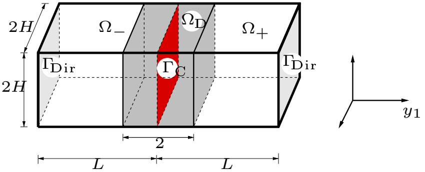
$u : \Omega \longrightarrow \mathbb{R}^d$ displacement

$\nabla_\varepsilon = (\frac{1}{\varepsilon}\partial_{y_1}, \partial_{y_2}, \dots, \partial_{y_d})^\top$

$$\mathcal{E}_\varepsilon^\kappa(t,u,z) := \begin{cases} \widetilde{\mathcal{E}}_\varepsilon^\kappa(t,u,z) & \text{if } (u,z) \in \mathcal{Q}_D = \mathcal{U}_D \times \mathcal{Z}_D \\ \infty & \text{otherwise} \end{cases}$$

$$\widetilde{\mathcal{E}}_\varepsilon^\kappa(t,u,z) := \int_{\Omega_-^\varepsilon \cup \Omega_+^\varepsilon} W(e(u+g(t)))\,\mathrm{d}x + \int_{\Omega_D^\varepsilon} W_D(\Pi^\varepsilon z, e(u))\,\mathrm{d}x + \int_{\Omega_D} \Big(\frac{\kappa}{r} |\nabla_\varepsilon z|^r + \underbrace{\delta_{[\varepsilon^\gamma,1]}(z)}_{\text{partial damage}} \Big)\,\mathrm{d}y$$

$$\mathcal{U}_D := \{u \in W^{1,p}(\Omega,\mathbb{R}^d) \,|\, u = 0 \text{ on } \Gamma_{Dir}\},$$
$$\mathcal{Z}_D := W^{1,r}(\Omega_D)$$
$$\mathcal{Q}_D := \mathcal{U}_D \times \mathcal{Z}_D$$



Ω : whole compound

$u : \Omega \to \mathbb{R}^d$ displacement

$[\![u]\!]$ jump across Γ_C

$$\mathcal{E}^\kappa(t,u,z) := \begin{cases} \widetilde{\mathcal{E}}^\kappa(t,u,z) & \text{if } (u,z) \in \mathcal{Q}_C\,, \\ \infty & \text{otherwise} \end{cases}$$

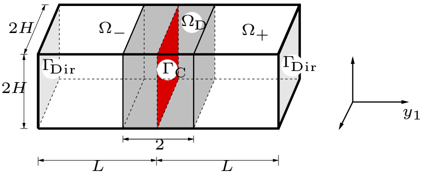
$$\widetilde{\mathcal{E}}^\kappa(t,u,z) := \int_{\Omega_- \cup \Omega_+} W(e(u+g(t)))\,\mathrm{d}x + \cancel{\int_{\Omega_D^\varepsilon} W_D(\Pi^\varepsilon z, e(u))\,\mathrm{d}x} + \int_{\Omega_D} \Big(\frac{\kappa}{r} |\nabla z|^r + \underbrace{\delta_{[0,1]}(z)}_{\text{delamination}} \Big)\,\mathrm{d}y$$

$$\mathcal{U}_C := \{ u \in W^{1,p}(\boldsymbol{\Omega_- \cup \Omega_+}, \mathbb{R}^d) \,|\, [\![u \cdot n_1]\!] \geq 0 \text{ a.e. on } \boldsymbol{\Gamma}_C,\ u = 0 \text{ on } \Gamma_{Dir} \},$$

$$\mathcal{Z}_C := \{ z \in W^{1,r}(\Omega_D) \,|\, \boldsymbol{\partial_{y_1} z = 0 \text{ in } \Omega_D} \},$$

$$\mathcal{Q}_C := \{ (u,z) \in \mathcal{U}_C \times \mathcal{Z}_C \,|\, z|_{\Gamma_C} [\![u]\!] = 0 \text{ a.e. on } \boldsymbol{\Gamma}_C \},$$

Griffith-type Delamination ($\kappa \longrightarrow 0$)



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$$\mathcal{Z}_C := \{z \in W^{1,r}(\Omega_D) \,|\, \partial_{y_1} z = 0 \text{ in } \Omega_D\},$$
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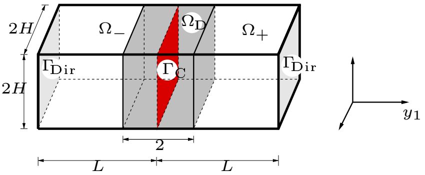

Ω : whole compound

$u : \Omega \to \mathbb{R}^d$ displacement

$[\![u]\!]$ jump across Γ_C

$$\mathcal{E}(t,u,z) := \begin{cases} \widetilde{\mathcal{E}}(t,u,z) & \text{if } (u,z) \in \mathcal{Q}_G\,, \\ \infty & \text{otherwise} \end{cases}$$

$$\widetilde{\mathcal{E}}(t,u,z) := \int_{\Omega_- \cup \Omega_+} W(e(u+g(t)))\,\mathrm{d}x + \cancel{\int_{\Omega_D^\varepsilon} W_D(\Pi^\varepsilon z, e(u))\,\mathrm{d}x} + \int_{\Omega_D} \Big(\cancel{\tfrac{\kappa}{r}|\nabla z|^r} + \delta_{[0,1]}(z) \Big)\,\mathrm{d}y$$

$$\mathcal{U}_G := \{ u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d) \,|\, [\![u \cdot \mathrm{n}_1]\!] \geq 0 \text{ a.e. on } \mathbf{\Gamma}_C,\ u = 0 \text{ on } \Gamma_{Dir} \},$$

$$\mathcal{Z}_G := \{ z \in \boldsymbol{L^\infty(\Omega_D)} \,|\, \partial_{y_1} z = 0 \text{ in } \Omega_D \},$$

$$\mathcal{Q}_G := \{ (u,z) \in \mathcal{U}_C \times \mathcal{Z}_C \,|\, \boldsymbol{S_G z}[\![u]\!] = 0 \text{ a.e. on } \Gamma_C \},$$

$$S_G z(s) = \tfrac{1}{2} \int_{-1}^{1} z(y_1, s)\,\mathrm{d}y_1$$

Proposition (Griffith-crack property of $(\mathcal{Q},\mathcal{E},\mathcal{R})$): Let $p > d$ and $(u_0, z_0) \in \mathcal{Q}$ be a stable initial value. Then for all $t \in [0,T]$ and a.a. $y \in \Omega_{\mathrm{D}}$ an en. sol. $(u,z) : [0,T] \to \mathcal{Q}$ of $(\mathcal{Q},\mathcal{E},\mathcal{R})$ satisfies $z(t,y) \in \{0, z_0(y)\}$.

Proposition (Griffith-crack property of $(\mathcal{Q},\mathcal{E},\mathcal{R})$): Let $p>d$ and $(u_0,z_0)\in\mathcal{Q}$ be a stable initial value. Then for all $t\in[0,T]$ and a.a. $y\in\Omega_{\rm D}$ an en. sol. $(u,z):[0,T]\to\mathcal{Q}$ of $(\mathcal{Q},\mathcal{E},\mathcal{R})$ satisfies $z(t,y)\in\{0,z_0(y)\}$.

Hence:

If $z_0=1$ a.e. in $\Omega_{\rm D}$: $z(t,y)\in\{0,1\}$, then

$$\mathcal{R}(z(t)-z(0))=\rho\mathcal{L}^{d-1}\big(\Gamma_{z(t)}\big)\,,$$

where $\Gamma_{z(t)}=\{s\in\Gamma_{\rm C}\,|\,S_{\rm G}z(t,s)=0\}$.

Theorem: Under technical assumptions on W and g, with $p, r > d$ and $\gamma > (p-1)$ it holds:

- for fixed $\kappa \in (0, \kappa_0]$, as $\varepsilon \to 0$:
  energetic solutions of partial damage processes converge to energetic solutions of gradient delamination models,
- as $\kappa \to 0$:
  energetic solutions of gradient delamination models converge to energetic solutions of a Griffith-type delamination model,
- in particular:
  transmission conditions and unilateral contact conditions are satisfied.
- simultaneous convergence $\varepsilon \to 0$ and $\kappa \to 0$:
  There is $G : \mathbb{R}_+ \to \mathbb{R}_+$ so that any convergent subseq. $(q_\varepsilon^\kappa(t))_{\varepsilon\in(0,\varepsilon_0],\kappa\in(0,\kappa_0],\varepsilon\leq G(\kappa)}$ of energetic solutions of $(\mathcal{Q}, \mathcal{E}_\varepsilon^\kappa, \mathcal{R})_{\varepsilon\in(0,\varepsilon_0],\kappa\in(0,\kappa_0]}$ converges for all $t \in [0, T]$ to an energetic solution of the Griffith-type delamination model.