Modeling of nonlinear effects at the tip zones for a crack onset

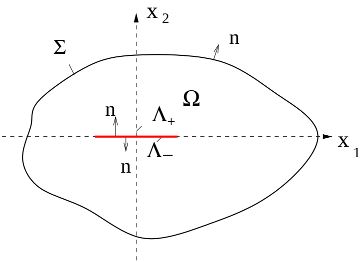
Maria Specovius

AAM, University of Kassel, Germany

Joint work with Sergej Nazarov, St. Petersburg

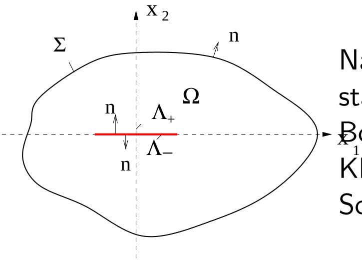
supported by the DFG Transregio 30

Singular Days, Berlin 2010



Griffith' energy criterion (1921)

The crack starts to propagate only if energy is released.
Engineering praxis: Lots of various criteria, the simplest: concept of critical SIF (Irwin 1957)



Nazarov (with various coauthors) starting 1988
Bourdin, Francfort, Marigo 2008
Khludnev, Kovtunenko 2000, K., Sokolowski 2000 Knees, Mielke 2008

Griffith' energy criterion (1921)

The crack starts to propagate only if energy is released.
Engineering praxis: Lots of various criteria, the simplest: concept of critical SIF (Irwin 1957)

$$\mathcal{L}u = f \text{ in } \Omega, \qquad \mathcal{N}u = g \text{ on } \Sigma,\ \mathcal{N}u = 0 \text{ on } \Lambda_\pm$$

Compatibility condition ($\mathbf{R}$ = space of rigid motions)

$$(f, r)_\Omega + (g, r)_\Sigma = 0, \forall\, r \in \mathbf{R} = \{D(\nabla)r = 0\}$$

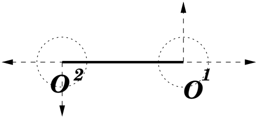
$\mathcal{L} = -D(\nabla)^\top A D(\nabla)$ second order strongly elliptic operator
$\mathcal{N} = n \cdot AD(\nabla)$ Neumann operator, $\mathcal{N}u$ normal stresses
u: displacement field,

Prop. $\exists$ solution $u_e \in H^1(\Omega)$, minimizer of the energy functional

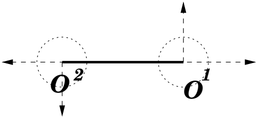
$$\mathcal{U}(u; f, g, \Omega) := \frac{1}{2}(D(\nabla)u, AD(\nabla)u)_\Omega - (f, u)_\Omega - (g, u)_\Sigma$$
$$= E_e(u) - A(u).$$

Normalization condition $(u_e, r)_\Omega = 0$ gives uniqueness.

u_e minimizer of $\mathcal{U}(u; f, g, \Omega) \Rightarrow u_e \in H^1(\Omega)$, but $u_e \notin H^2(\Omega)$
$u_e \in H^2_{loc}(\overline{\Omega} \setminus \{\mathcal{O}^1, \mathcal{O}^2\})$



u_e minimizer of $\mathcal{U}(u; f, g, \Omega) \Rightarrow u_e \in H^1(\Omega)$, but $u_e \notin H^2(\Omega)$
$u_e \in H^2_{loc}(\overline{\Omega} \setminus \{\mathcal{O}^1, \mathcal{O}^2\})$

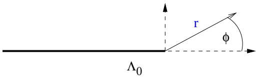


representation of u_e near the tips $\mathcal{O}^\nu$:

$$u_e(x) = u_e(\mathcal{O}^\nu) + K_1^\nu X^1(x^\nu) + K_2^\nu X^2(x^\nu) + \tilde{u}_e$$

K_j^ν: stress intensity factors (SIFs)

X^j: **solutions to the model problem:**



$$\mathcal{L}X = 0 \text{ in } \mathbb{R}^2 \setminus \Lambda_0,\ \mathcal{N}X = 0 \text{ on } \Lambda_0$$

power law solutions: $X = r^\lambda \Phi(\phi)$, $Y = r^{-\lambda}\Psi(\phi)$, $\lambda = \frac{k}{2}$, $k \in \mathbb{N}$
2 for each k, ($+$ 4 solutions corresponding to $\lambda = 0$)

$$X^{1,2} = r^{1/2}\Phi^{1,2}(\phi)$$

near the tips:

$$u(x) = u(\mathcal{O}^\nu) + K_1^\nu X^1(x^\nu) + K_2^\nu X^2(x^\nu) + \tilde{u}$$

$$\begin{aligned} &\|\widetilde{u}; H^2(\Omega)\| + \sum_{\nu=1}^{2} \left\{ |u(\mathcal{O}^\nu)| + \sum_{j=1}^{2} |K_j^\nu| \right\} \\ &\leq \quad c \left(\|f; L^2(\Omega)\| + \|g; H^{1/2}(\Sigma)\| + \|u; L^2(\Omega)\| \right). \end{aligned}$$

Remark

It is not enough to play with asymptotics in Kondratiev spaces V^l_β and embeddings here!

$$\text{Instead of } u \in H^2_{loc}(\overline{\Omega} \setminus \{\mathcal{O}^1, \mathcal{O}^2\}) \cap H^1(\Omega)$$
$$\text{require } u \in \mathfrak{D} =: H^2_{loc}(\overline{\Omega} \setminus \{\mathcal{O}^1, \mathcal{O}^2\}) \cap L^2(\Omega)$$

Again asymptotic representation, now with power-law solutions related to $\lambda = -\frac{1}{2}, 0, \frac{1}{2}$

$\lambda = \frac{1}{2}$: X^j, $\lambda = -\frac{1}{2}$: Y^j, $\lambda = 0$: e_j, T^j (logarithmic) $\Rightarrow$ near $\mathcal{O}^\nu$:

$$u(x) = a^\nu + \sum_{j=1}^{2} \left(d_j^\nu T^j(x^\nu) + c_j^\nu X^j(x^\nu) + b_j^\nu Y^j(x^\nu) \right) + \widetilde{u}(x),$$

$\widetilde{u}(x) \in H^2(\Omega)$, $\widetilde{u}(\mathcal{O}^\nu) = 0$ + estimate of $\widetilde{u}$ and coefficient vectors

$$\text{Instead of } u \in H^2_{loc}(\overline{\Omega} \setminus \{\mathcal{O}^1, \mathcal{O}^2\}) \cap H^1(\Omega)$$

$$\text{require } u \in \mathfrak{D} =: H^2_{loc}(\overline{\Omega} \setminus \{\mathcal{O}^1, \mathcal{O}^2\}) \cap L^2(\Omega)$$

Again asymptotic representation, now with power-law solutions related to $\lambda = -\frac{1}{2}, 0, \frac{1}{2}$

$\lambda = \frac{1}{2}$: X^j, $\lambda = -\frac{1}{2}$: Y^j, $\lambda = 0$: e_j, T^j (logarithmic) $\Rightarrow$ near $\mathcal{O}^\nu$:

$$u(x) = a^\nu + \sum_{j=1}^{2} \left(d_j^\nu T^j(x^\nu) + c_j^\nu X^j(x^\nu) + b_j^\nu Y^j(x^\nu) \right) + \widetilde{u}(x),$$

$\widetilde{u}(x) \in H^2(\Omega)$, $\widetilde{u}(\mathcal{O}^\nu) = 0$ + estimate of $\widetilde{u}$ and coefficient vectors

Various conditions for the coefficient vectors a, b, c, d are possible.
Kick out concentrated forces at the tips: $d = 0$

$$\mathfrak{E} = \left\{ u \in \mathfrak{D} : u = \sum_{j,\nu}^{2} \chi^\nu \left(c_j^\nu X^j + b_j^\nu Y^j \right) + \widetilde{u}, \mathcal{N} u = 0 \text{ on } \Lambda_\pm \right\},$$

$$\widetilde{u} \in H^2(\Omega), \quad \|u; \mathfrak{E}\|^2 = |b|^2 + |c|^2 + \|\widetilde{u}; H^2\|^2$$

and the **Generalized Green's formula**

$$(\mathcal{L}u, v)_\Omega + (\mathcal{N}u, v)_\Sigma - (u, \mathcal{L}v)_\Omega - (u, \mathcal{N}v)_\Sigma = \langle c_u, b_v \rangle - \langle b_u, c_v \rangle$$

Which linear and nonlinear conditions on b and c lead to well posed problems with a sensible physical interpretation?
$H^1(\Omega)$-solutions: $b_u = 0$ $(\rightarrow \mathfrak{E}_e)$
("Generalized Dirichlet condition" at the tips)

Hierarchy of conditions:

$$\mathfrak{R} := L^2(\Omega) \times H^{1/2}(\Sigma) \times \mathbb{R}^4.$$

$$\begin{aligned} \mathcal{L}u = f \text{ in } \Omega, \quad & \mathcal{N}u = g \text{ on } \Sigma \\ \mathcal{N}u = 0 \text{ on } \Lambda, \quad & \mathbf{H}_1 b + \mathbf{H}_2 c = h \in \mathbb{R}^4 \end{aligned} \qquad (*)$$

$(*)$ defines a Fredholm op. of index 0: $\mathbf{A} : \mathfrak{E} \to \mathfrak{R}$

Independent of $\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2)$: rigid motions $\subset \ker \mathbf{A}$, requires always

$$(f, r)_\Omega + (g, r)_\Sigma = 0.$$

$\mathbf{H} = (-T; \mathbb{I})$, T symmetric (and invertible f.s.),
$(*)$ has a solution as a stationary point of the
generalized energy functional:

$$\mathbf{U}(u; f, g, h) = \underbrace{\frac{1}{2}(\mathcal{L}u, u)_\Omega + \frac{1}{2}(\mathcal{N}u, u)_{\partial\Omega}}_{\text{elastic energy}} - \underbrace{((f, u)_\Omega + (g, u)_\Sigma)}_{\text{work of ext. forces}}$$

$$\underbrace{+\frac{1}{2}\langle T b_u - c_u, b_u \rangle}_{\text{el. energy stored at the tips}} \qquad - \underbrace{\langle h, b_u \rangle}_{\text{work at the tips}}$$

$$\mathfrak{E} = \mathfrak{E}_e \oplus \mathrm{span}\{\zeta^j, j = 1, \dots, 4\}$$

ζ^j *weight functions*:
solve $(*)$ with $f = 0$, $g = 0$, $b = \mathrm{e}_j \in \mathbb{R}^4$.

Z $:= (c_{\zeta^j})_j$ *Polarization matrix*, symmetric matrix, global integral characteristic of Ω.

Lemma If $T - Z$ is positive, then $(*)$ has a solution u, unique under the normalization $(u, r)_\Omega = 0$, and u is the minimizer of the generalized energy functional.

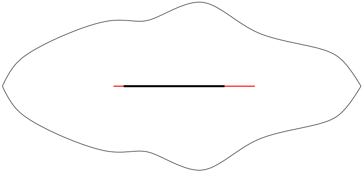
$$\mathcal{L}u = f \text{ in } \Omega, \quad \mathcal{N}u = g \text{ on } \Sigma, \quad \mathcal{N}u = 0 \text{ on } \Lambda,$$
$$\mathbf{T}(b_u) - c_u = 0, \quad \text{with } \mathbf{T}(0) = 0, \quad \mathbf{T} = \nabla \mathbf{E} \qquad (**)$$

Proposition 2

Z: polarization matrix,
$\mathbf{E}_\Omega(b) := \mathbf{E}(b) - \frac{1}{2}\langle Zb, b\rangle$ strictly convex and coercive
$(**)$ has a unique solution $u \in \mathfrak{E}^\perp$, u is the minimizer of the generalized energy functional

$$\mathbf{U}(u) = \frac{1}{2}(\mathcal{L}u, u)_\Omega + \frac{1}{2}(\mathcal{N}u, u)_\Sigma - (f, u)_\Omega - (g, u)_\Sigma + \mathbf{E}(b_u) - \frac{1}{2}\langle c_u, b_u\rangle.$$

Symmetric problem ("Mode I loading")



small one dimensional plastic zones at the tips: Find u with bounded stresses and d_ν with

$$\mathcal{L}u = 0 \quad \text{in } \Omega(d), \quad \mathcal{N}u = g \quad \text{on } \partial\Omega(d),$$
$$\text{with } g = 0 \quad \text{on } \Lambda_\pm, \qquad g = \mp\sigma_c \mathbf{e}_2 \text{ on } \Upsilon_{\nu,\pm}$$

Bounded stresses ⇒ determines the lengthes d_ν.

The criterion itself: deformation criterion crack propagates if

$$u_+(\mathcal{O}^\nu) - u_-(\mathcal{O}^\nu) > \delta_{crit}$$

σ_c large: the problem possesses a unique solution, $d_\nu = O(\sigma_c^{-2})$

Method of matched asymptotic expansions, model u^D by the first terms of the outer decomposition:

$$u^D \sim u^0 + a(K_1^3\zeta^1 + K_2^3\zeta^2) =: u^d \in \mathfrak{E}$$

u^d solves $(**)$ with nonlinear conditions at the tips

$$c^\nu = (\frac{1}{a} b^\nu)^{1/3} + (Z b)_\nu$$

Method of matched asymptotic expansions, model u^D by the first terms of the outer decomposition:

$$u^D \sim u^0 + a(K_1^3\zeta^1 + K_2^3\zeta^2) =: u^d \in \mathfrak{E}$$

u^d solves $(**)$ with nonlinear conditions at the tips

$$c^\nu = (\frac{1}{a}b^\nu)^{1/3} + (Zb)_\nu = \nabla \mathbf{E}(b)$$

u^d minimizes the generalized energy functional **U** with

$$\mathbf{E}(b) = \frac{3}{4\sqrt[3]{a}}(b_1^{4/3} + b_2^{4/3}) + \frac{1}{2}\langle Zb, b\rangle$$

Crack path is known a priori (only straight propagation).
$\mathbf{U} \to \mathbf{U}(h)$
Calculation of the energy release rate

$$\frac{d}{dh}\mathbf{U}|_{h=0}$$

- involves the geometry of the domain
- The condition $\frac{d}{dh}\mathbf{T}|_{h=0} < 0$ ($\mathbf{T}$ potential energy + surface energy) coincides with the original Dugdale criterion up to $O(|h|^2)$.

$$\mathscr{S}(t\sigma) = t\mathscr{S}(\sigma) \tag{vM}$$

Assume:

$$\sigma = \begin{cases} AD(\nabla)u & \text{for } \mathscr{S}(\sigma) \leq \mathscr{S}_0, \\ ?? & \text{for } \mathscr{S}(\sigma) \geq \mathscr{S}_0 \end{cases}$$

Examples for $\mathscr{S}$: conditions of von Mises or Tresca
$\mathscr{S}_0$ large, solutions to the nonlinear model problem in $\mathbb{R}^2 \setminus \Lambda_\infty$

$$W(K;x) = \sum_{j=1}^{2} \left(K_j X^j(x) + \mathcal{M}_j(K) Y^j(x)\right) + O(|x|^{-1}), \text{ as } |x| \to \infty. \tag{NMP}$$

$$\mathcal{M}(K) = (\mathcal{M}_1(K), \mathcal{M}_2(K))^\top : \mathbb{R}^2 \to \mathbb{R}^2$$

is a certain non-linear mapping.

Proposition 3

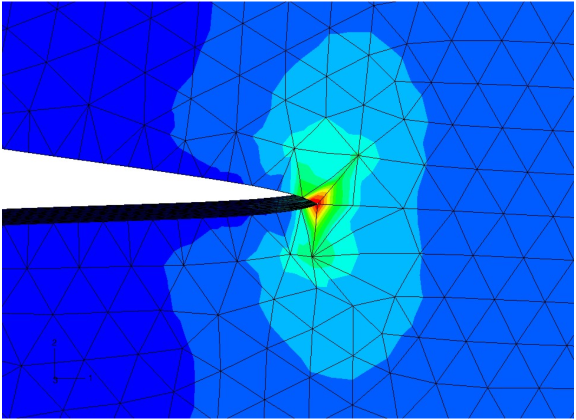
- For given $K \in \mathbb{R}^2$, assume the existence of a unique solution W to the homogeneous nonlinear model problem with

$$W(x) = K_1 X^1 + K_2 X^2 + O(1) \text{ as } |x| \to \infty \quad \Rightarrow$$
$$\mathcal{M}_j = |K|^3 \mathcal{M}_j \left(\frac{K}{|K|} \right)$$

- If $\mathcal{M}^{-1} = \nabla \mathbf{E}_\Omega$, $\mathbf{E}_\Omega$ convex $\Rightarrow$ solution to the nonlinear problem can be modeled by a minimizer u to the generalized energy functional, then

$$\mathbf{U}(u, g) = U(u_0) + U_p(K), K : \text{ SIFs of } u_0$$

Modelling of plastic effects: Adding a functional $U_p(K)$ homogeneous of order 4 to the classical potential energy.



calculated in the Institute of applied Mechanics, Paderborn

Thank You!