Flux intensity functions along circular singular edges for the Laplace equation in 3-D domains

Samuel Shannon*, Zohar Yosibash*, Monique Dauge* & Martin Costabel*

* Computational Mechanics Lab, Dept Mech Eng, Ben-Gurion Univ, Beer-Sheva, Israel
* IRMAR, Univ of Rennes 1, Rennes, France

Abstract
The solution of the Laplace equation in the vicinity of a circular singular edge is explicitly determined in a general 3-D domain, and the edge flux intensity functions are extracted by the quasi-dual function method (QDFM). It is shown that the solution is given in the form of an asymptotic series involving primal functions and two levels of shadow functions. Flux Intensity Functions along the circular singular edge are extracted by the QDFM from p-FE solutions.

Introduction
The Laplace equation in the vicinity of circular edges in \((\rho, \phi, \eta)\) coordinates is given as:

\[
\begin{align*}
(1 + \frac{\rho}{R} \cos \phi)^2 \left( \frac{\partial \varphi}{\partial \rho} \right)^2 + \frac{\partial \varphi}{\partial \phi} + \frac{\partial \varphi}{\partial \eta} \tau &= \tau, \\
+ \frac{\rho}{R} (1 + \frac{\rho}{R} \cos \phi) \left( \cos \varphi \rho \frac{\partial \varphi}{\partial \rho} - \sin \varphi \frac{\partial \varphi}{\partial \phi} \right) + \left( \frac{\rho}{R} \right)^2 \partial_{\eta \eta} \tau &= 0
\end{align*}
\]

where \(\tau = \rho \cos \phi + R\).

The solution of the Laplace equation can be represented as an infinite series involving primal functions and two levels of shadow functions as follows:

\[
\tau = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \hat{A}_{\ell k}(\theta) \rho^\ell \left( \frac{\rho}{R} \right)^k \phi_{\ell k}(\varphi)
\]

where \(\ell, k, \rho \), and \(\varphi\) are the polar coordinates, and \(\dot{\phi}\) is the position along the edge.

The primal eigen function and their shadows
Substituting (2) in (1) with the assumption that \(\rho \ll R\), results in:

\[
\begin{align*}
0 &= A_k(\theta) \times \{ (\alpha_k + 1)^2 \phi_{k,0,0} + (\alpha_k - 1)^2 \phi_{k,0,0} + (\alpha_k + 1) \phi_{k,0,0} \cos \varphi - \phi_{k,0,0} \sin \varphi \} \\
+ \left( \frac{\rho}{R} \right)^2 \left[ (\alpha_k + 2)^2 \phi_{k,2,0} + (\alpha_k - 2)^2 \phi_{k,2,0} + (\alpha_k + 2) \phi_{k,2,0} \cos \varphi - \phi_{k,2,0} \sin \varphi \right] \\
+ \frac{\rho}{R} \left[ (\alpha_k + 1) \phi_{k,1,1} + (\alpha_k - 1) \phi_{k,1,1} \cos \varphi - \phi_{k,1,1} \sin \varphi \right] \\
&+ \left( \frac{\rho}{R} \right)^2 \left[ (\alpha_k + 3) \phi_{k,1,1} + (\alpha_k - 3) \phi_{k,1,1} \cos \varphi - \phi_{k,1,1} \sin \varphi \right]
\end{align*}
\]

Equation (3) has to hold for any \((\rho/R)^\ell\) and any \(\partial^\ell A_k\), resulting in the following recursive set of ordinary differential equations for the determination of the eigen values \(\alpha_k\), primal \(\phi_{k,0,0}(\varphi)\) and shadows \(\phi_{k,0,0}(\varphi)\): 

\[
\ell = 0, 2, 4, 6, \ldots, \quad \alpha_k < \varphi < \varphi_2
\]

\[
(\alpha_k + 1)^2 \phi_{k,0,0} + (\alpha_k - 1)^2 \phi_{k,0,0} - (\ell + 1 + \alpha_k + 1) \cos \varphi \phi_{k,0,0} - 2 \sin \varphi \phi_{k,0,0} + \phi_{k,0,0} = 0
\]

Extraction of FIFs by the QDFM (axisymmetric case)
Considering a torus surface around the circular edge we use the \(J[\tau, K_{n}^{\rho}]\) integral to extract the FIF \(A_{\tau}^{\rho}(\theta)\):

\[
J[\tau, K_{n}^{\rho}] = \int_{-\infty}^{\infty} \left( \frac{\rho}{R} \right)^{\alpha_k} \left( \frac{\rho}{R} \right)^{\alpha_k} K_{n}^{\rho} \phi_{\l,0,0}(\varphi) \phi_{\l,0,0}(\varphi) d\varphi
\]

Extracting \(A_{\tau}^{\rho}\) from FE solution

Summary and conclusion

- The solution of the Laplace equation in the vicinity of a circular edge can be explicitly represented in terms of eigen-pairs and 2 families of shadows.
- Having an explicit representation of the solution, the QDFM is being extended for the extraction of GFFs.
- The presented methods are being extended to the system of elasticity.