Flux intensity functions along circular singular edges for the Laplace equation in 3-D domains

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Abstract

The solution of the Laplace equation in the vicinity of a circular singular edge is explicitly determined in a general 3-D domain, and the edge flux intensity functions are extracted by the quasi-dual function method (QDFM). It is shown that the solution is given in the form of an asymptotic series involving primal functions and two levels of shadow functions. Flux Intensity Functions along the circular singular edge are extracted by the QDFM from p-FE solutions.

Introduction

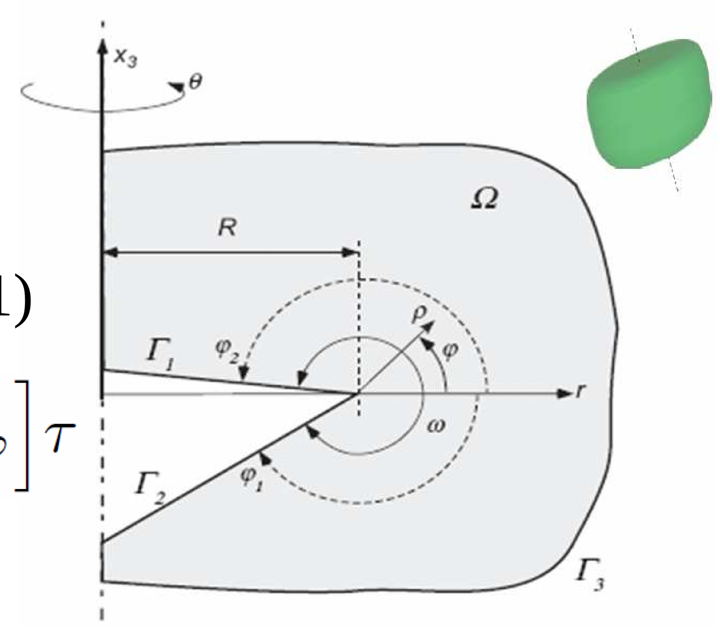
The Laplace equation in the vicinity of circular singular edges in (ρ,φ,θ) coordinates is given as:

$$\left(1+\frac{\rho}{R}\cos\varphi\right)^2\left[(\rho\partial_\rho)^2+\partial_{\varphi\varphi}\right]\tau \qquad (1)$$

$$+\frac{\rho}{R}\left(1+\frac{\rho}{R}\cos\varphi\right)\left[\cos\varphi(\rho\partial_\rho)-\sin\varphi\partial_\varphi\right]\tau$$

$$+\left(\frac{\rho}{R}\right)^2\partial_{\theta\theta}\tau=0$$

with $r=\rho\cos\varphi+R$



The solution of the Laplace equation can be represented as an infinite series involving primal functions and two levels of shadow functions as follows:

$$\tau=\sum_{\ell=0,2,4}\sum_{k=0}^{\infty}\partial_\theta^\ell A_k(\theta)\rho^{\alpha_k}\sum_{i=0}^{\infty}\left(\frac{\rho}{R}\right)^{\ell+i}\phi_{\ell ki}(\varphi) \qquad (2)$$

where R is the distance of the singular edge from the axisymmetric axis, ρ and φ are "polar" coordinates, and θ is the position along the edge.

α_k are the eigen values, $\phi_{\ell ki}(\varphi)$ are 1-D eigen functions, $A_k(\theta)$ are the Flux Intensity Function.

The primal eigen function and their shadows

Substituting (2) in (1) with the assumption that $\rho << R$, results in:

$$\begin{aligned}
0 \;=\; & A_k(\theta)\times\{[\alpha_k^2\phi_{0,k,0}+\phi''_{0,k,0}] \qquad (3)\\
& +\left(\frac{\rho}{R}\right)[(\alpha_k+1)^2\phi_{0,k,1}+\phi''_{0,k,1}+(\alpha_k\phi_{0,k,0}\cos\varphi-\phi'_{0,k,0}\sin\varphi)]\\
& +\left(\frac{\rho}{R}\right)^2[(\alpha_k+2)^2\phi_{0,k,2}+\phi''_{0,k,2}+((\alpha_k+1)\phi_{0,k,1}\cos\varphi-\phi'_{0,k,1}\sin\varphi)\\
& \quad -\cos\varphi\,(\alpha_k\phi_{0,k,0}\cos\varphi-\phi'_{0,k,0}\sin\varphi)]+\cdots\}\\
+\; & A''_k(\theta)\times\left\{\left(\frac{\rho}{R}\right)^2[(\alpha_k+2)^2\phi_{2,k,0}+\phi''_{2,k,0}+\phi_{0,k,0}]\right.\\
& +\left(\frac{\rho}{R}\right)^3[(\alpha_k+3)^2\phi_{2,k,1}+\phi''_{2,k,1}+((\alpha_k+2)\phi_{2,k,0}\cos\varphi-\phi'_{2,k,0}\sin\varphi)\\
& \quad +(\phi_{0,k,1}-2\cos\varphi\phi_{0,k,0})]\\
& +\left(\frac{\rho}{R}\right)^4[(\alpha_k+4)^2\phi_{2,k,2}+\phi''_{2,k,2}+((\alpha_k+3)\phi_{2,k,1}\cos\varphi-\phi'_{2,k,1}\sin\varphi)\\
& \quad -\cos\varphi((\alpha_k+2)\phi_{2,k,0}\cos\varphi-\phi'_{2,k,0}\sin\varphi)\\
& \quad \left.+(\phi_{0,k,2}-2\cos\varphi\phi_{0,k,1}+3\cos^2\varphi\phi_{0,k,0})]+\cdots\right\}\\
+\; & \cdots
\end{aligned}$$

Equation (3) has to hold for any $(\rho/R)^n$ and any $\partial_\theta^l A_k$, resulting in the following recursive set of ordinary differential equations for the determination of the eigen values α_k, primal $\phi_{\ell k0}(\varphi)$ and shadows $\phi_{\ell ki}(\varphi)$:

$$\begin{aligned}
\ell= \;& 0,2,4,6\cdots, \qquad i\geq 0, \qquad \varphi_1<\varphi<\varphi_2\\
& (\alpha_k+i+\ell)^2\phi_{\ell,k,i}+\phi''_{\ell,k,i}=-(\ell+i+\alpha_k-1)\left[2(\ell+i+\alpha_k)-1\right]\cos\varphi\phi_{\ell,k,(i-1)}\\
& +\sin\varphi\phi'_{\ell,k,(i-1)}-2\cos\varphi\phi''_{\ell,k,(i-1)}-(\ell+\alpha_k+i-2)(\ell+\alpha_k+i-1)\cos^2\varphi\phi_{\ell,k,(i-2)}\\
& +\cos\varphi\sin\varphi\phi'_{\ell,k,(i-2)}-\cos^2\varphi\phi''_{\ell,k,(i-2)}-\phi_{(\ell-2),k,i}
\end{aligned}$$

ℓ=0 is the case of axisymmetric load.

ℓ>0 is the general load case.

with $\phi_{\ell ki}=0\,\forall\ell,i<0$, and either homogeneous Dirichlet or Neumann BCs on the re-entrant surfaces intersecting at the circular edge.

Particular example problem: crack with homogeneous Neumann BCs

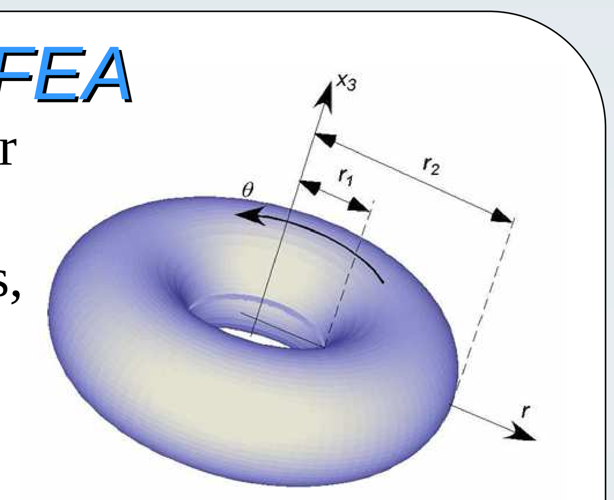
In the case of a crack $(\varphi_1=\varphi_2=\pi)$ with homogeneous Neumann boundary conditions on the crack face the asymptotic solution is given by:

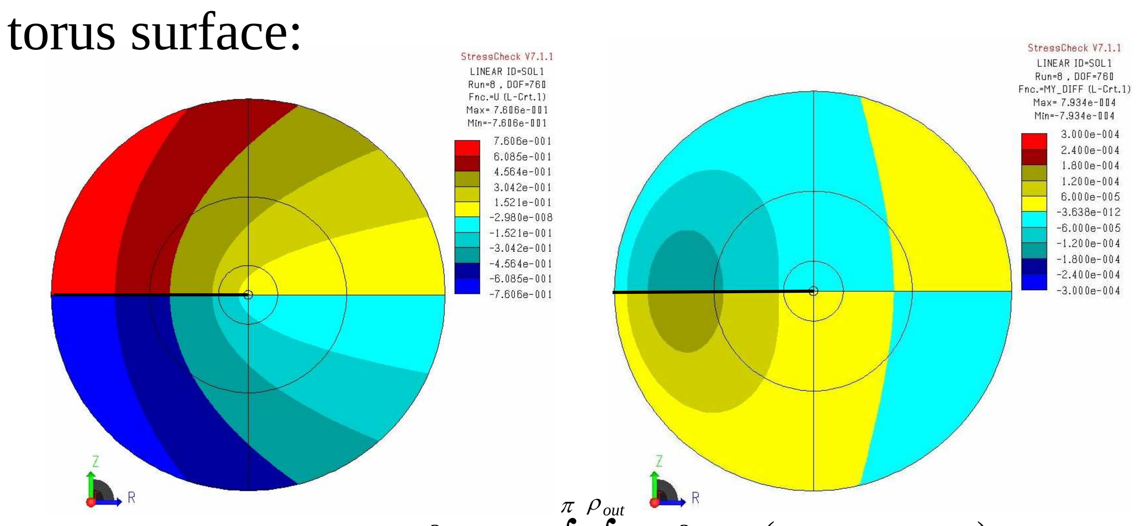
$$\begin{aligned}
\tau \;=\; & A_0(\theta)\\
+\; & A''_0(\theta)\left(\frac{\rho}{R}\right)^2\left[-\frac{1}{4}+\left(\frac{\rho}{R}\right)\frac{5}{16}\cos\varphi-\left(\frac{\rho}{R}\right)^2\left(\frac{19}{128}+\frac{11}{64}\cos2\varphi\right)+\cdots\right]+\cdots\\
+\; & A_1(\theta)\rho^{\frac{1}{2}}\left[\sin\frac{\varphi}{2}+\left(\frac{\rho}{R}\right)\frac{1}{4}\sin\frac{\varphi}{2}+\left(\frac{\rho}{R}\right)^2\left(\frac{1}{12}\sin\frac{\varphi}{2}-\frac{3}{32}\sin\frac{3\varphi}{2}\right)+\right.\\
& \left.\quad+\left(\frac{\rho}{R}\right)^3\left(\frac{1}{16}\sin\frac{\varphi}{2}-\frac{1}{30}\sin\frac{3\varphi}{2}+\frac{5}{128}\sin\frac{5\varphi}{2}\right)+\cdots\right]\\
+\; & A''_1(\theta)\rho^{\frac{1}{2}}\left(\frac{\rho}{R}\right)^2\left[-\frac{1}{6}\sin\frac{\varphi}{2}+\left(-\frac{1}{8}\sin\frac{\varphi}{2}+\frac{7}{60}\sin\frac{3\varphi}{2}\right)\left(\frac{\rho}{R}\right)+\cdots\right]+\cdots
\end{aligned}$$

Checking the Analytic solution by FEA

To verify the correctness of the solution, we consider a FE model of a torus with a circular crack.

Taking $A_1=1$, $A_k=0\,\forall k\neq 1$ with the 3 first terms, with $R=2$, $\rho_{out}=1/2$ we apply the solution on torus surface:





with

$$\|\tau\|_{L_2}^2=2\pi\int_{-\pi}^{\pi}\int_{0}^{\rho_{out}}|\tau|^2\times\rho(R+\rho\cos\varphi)d\rho d\varphi$$

$$\|e\|_{L_2}^2=2\pi\int_{-\pi}^{\pi}\int_{0}^{\rho_{out}}|\tau-\tau_{FE}|^2\times\rho(R+\rho\cos\varphi)d\rho d\varphi/\|\tau\|_{L_2}^2$$

Error between FE and analytical solution as the number of shadow functions is increased

Number of shadows	$\|e\|_{L_2}^2$	$\|\tau\|_{L_2}^2$
1	1.979E-01	1.554E-01
2	7.423E-03	1.275E-01
3	1.767E-03	1.318E-01

Extracting FIFs by the QDFM (axisymmetric case)

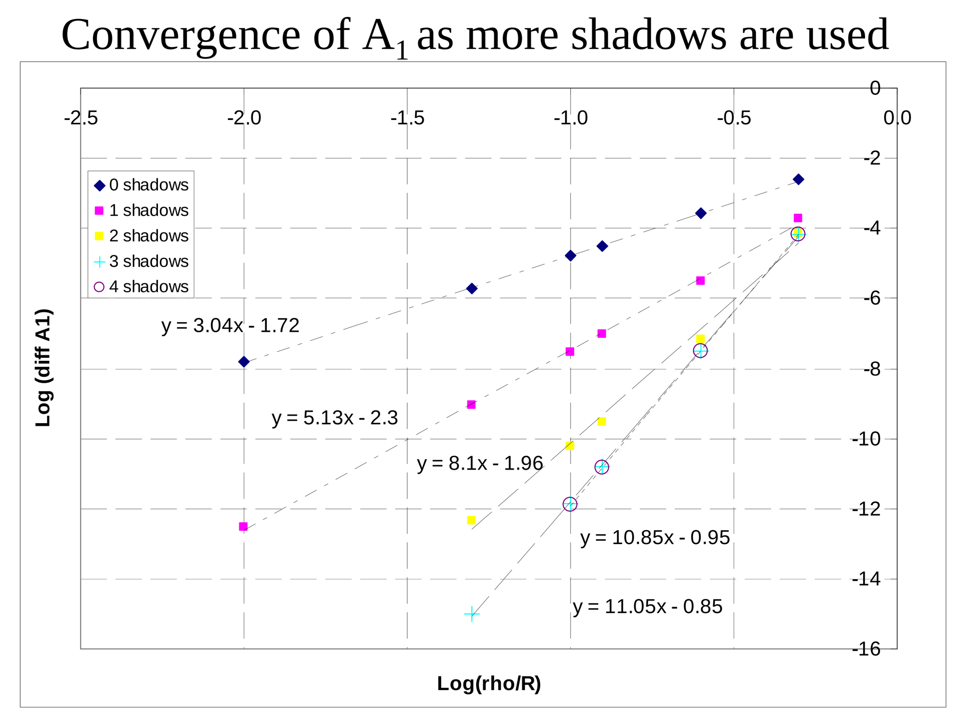
Considering a torus surface around the circular edge we use the $J[\tau,K_n^{\alpha_k}](\rho_o)$ integral to extract the FIF $A_k(\theta)$:

$$J[\tau,K_n^{\alpha_k}](\rho_o)\overset{\Delta}{=}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left((\partial_\rho\tau)K_n^{\alpha_k}-(\partial_\rho K_n^{\alpha_k})\tau\right)\rho(R+\rho\cos\varphi)d\varphi d\theta\big|_{\rho=\rho_0}=A_0+O\left(\frac{\rho}{R}\right)^m$$

with $K_n^{\alpha_k}=B_k\rho^{-\alpha_k}\sum_{i=0}^{n}\left(\frac{\rho}{R}\right)^i\psi_{k,i}(\alpha_k,\varphi)$, $\quad B_k=\frac{1}{2k\pi^2R}$ $\quad\psi_{i,k}(\varphi)$ are the dual primal and shadow eigen functions

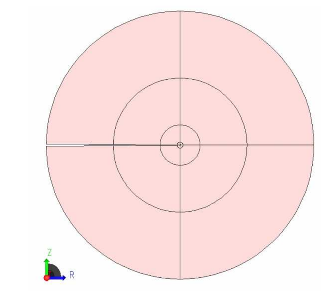
We determine *m* by numerical experiments:

Convergence of A_1 as more shadows are used



Extracting A_1 from FE solution

Error in A_1 (rho/R)		Number of shadows
$\rho/R=0.25$	$\rho/R=0.5$	
2.87E-04	2.39E-03	0
2.55E-05	1.25E-04	1
2.24E-05	1.20E-05	2
2.23E-05	6.18E-06	3



Summary and conclusion

- The solution of the Laplace equation in the vicinity of a circular edge can be explicitly represented in terms of eigen-pairs and 2 families of shadows.
- Having an explicit representation of the solution, the QDFM is being extended for the extraction of GFIFs.
- The presented methods are being extended to the system of elasticity.