A mathematical point of view in Electrowetting

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In collaboration with

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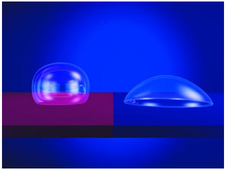
6^{th} singular days

30 April 2010

Electrowetting

Wetting?

Hydrophobic →



← Hydrophilic

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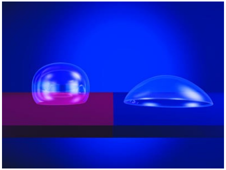

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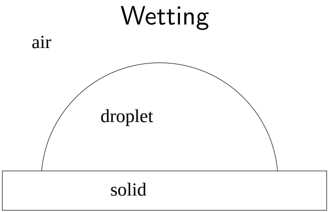
How to control wetting?

What is Electrowetting?



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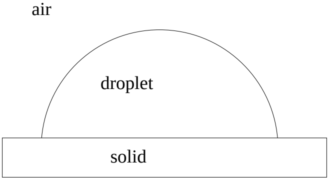
Wetting $\longrightarrow$



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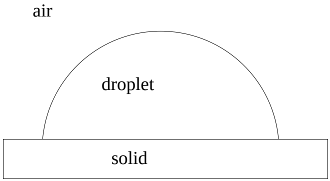
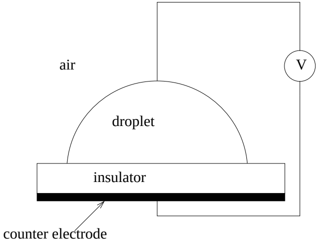
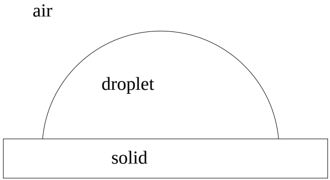
Wetting ⟶ Electrowetting

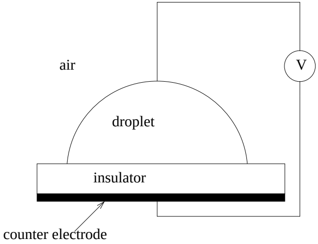




⟶

Wetting $\longrightarrow$ Electrowetting



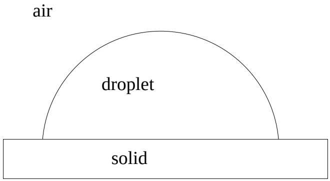


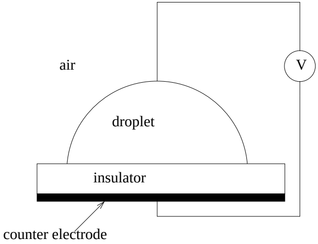
$\longrightarrow$

Modify the affinity between solid and liquid

What is Electrowetting?

Wetting $\longrightarrow$ Electrowetting





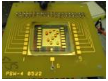
$\longrightarrow$

$\rightarrow$ Due to Bruno Berge : 1993

Many applications :

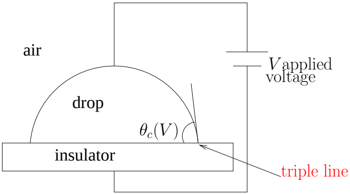
- Variable focal liquid lenses

- Microfluidics



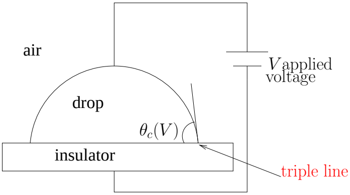
Outline

Observations and approximations



$\theta_c(V)$ contact angle at a given potential V

Observations and approximations



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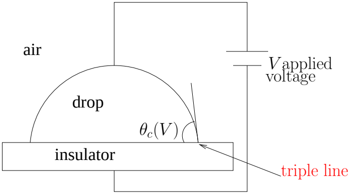

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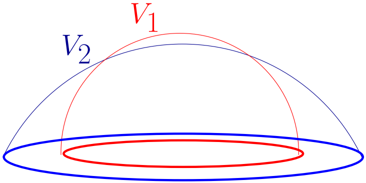

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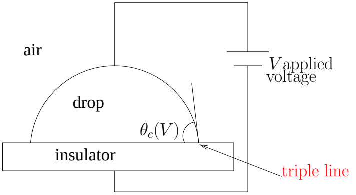

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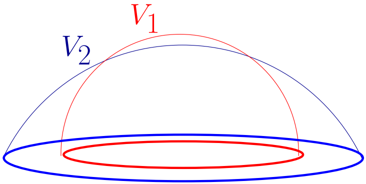

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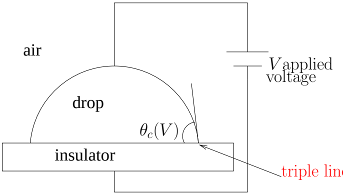

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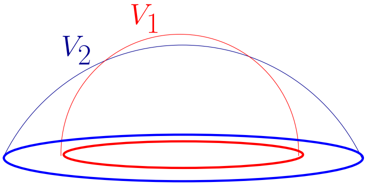

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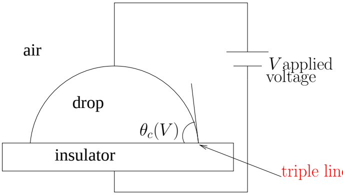

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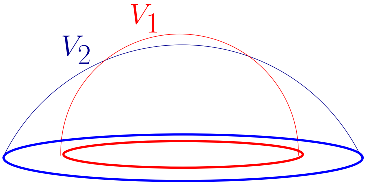

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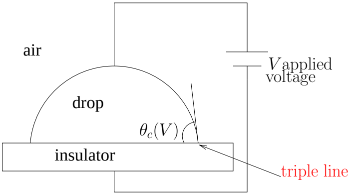

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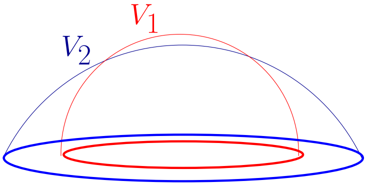

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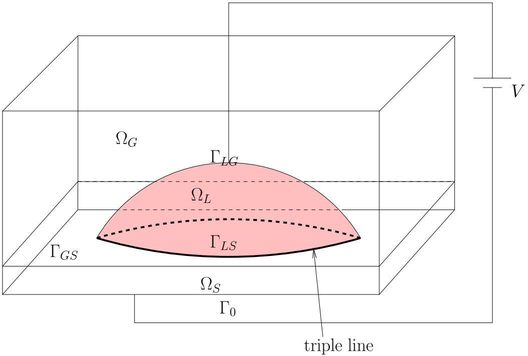
What can we add as mathematicians?

Outline

Notations



$\Omega = \Omega_G \cup \Omega_S \cup \Gamma_{GS}$, (white domain)

The energy of the drop

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- α, μ, δ : Physical parameters
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- $J(\Omega) := \mathcal{E}(\Omega_L, V) = J_{grav}(\Omega) + J_{LS}(\Omega) + J_{LG}(\Omega) + J_{elec}(\Omega)$
  cost function of the problem.

The potential: transmission problem

$$\begin{cases} \mathrm{div}(\varepsilon_i \nabla \phi^\Omega) = 0 & \text{in } \Omega_i \quad i = G, S \\ \phi^\Omega = V & \text{on } \Gamma_{LG} \cup \Gamma_{LS} \\ \phi^\Omega = 0 & \text{on } \Gamma_0 \\ \phi_G^\Omega = \phi_S^\Omega & \text{on } \Gamma_{GS} \\ \varepsilon_G \nabla \phi_G^\Omega . \overrightarrow{N_G} = -\varepsilon_S \nabla \phi_S^\Omega . \overrightarrow{N_S} & \text{on } \Gamma_{GS} \\ \varepsilon_i \nabla \phi_i^\Omega . \overrightarrow{N_i} = 0 & i = G, S \text{ on artificial boundaries} \end{cases}$$

- ϕ^Ω depends on Ω.
- Ω has a reentrant corner due to the triple line $\Rightarrow$ Loss of regularity

Optimal shape

To $V \geq 0$ and a given volume *vol*,

$$(P) \begin{cases} \text{Find } \Omega_L^* \text{ such that:} \\ \mathcal{E}(\Omega_L^*, V) = \min_{\{\Omega_L; \text{Vol}(\Omega_L) = vol\}} \mathcal{E}(\Omega_L, V) \end{cases}$$

- *Optimization under constraint* treated by a Lagrangian $\mathcal{L}(\Omega, \lambda) = J(\Omega) - \lambda C(\Omega)$, where $C(\Omega) = Vol(\Omega_L) - vol$, $\lambda \in \mathbb{R}$.
- Shape optimization gives a necessary condition for optimality:

$$\forall U \in \mathcal{U} \subset \mathcal{C}^1(\Omega^*, \mathbb{R}^3), DJ(\Omega^*).U = \lambda^* DC(\Omega^*).U$$

  if Ω^* saddle point and where $DJ(\Omega^*)$ is the shape derivative of J in Ω^*.
- Using the expression of the singularity at the triple line one obtains

The contact angle θ_c is independent of the applied potential $V \geq 0$ i.e.

$$\theta_c(V) = \theta_Y, \forall V \geq 0$$

Outline

Numerical approximation : axisymmetric case

V and physical constants are given.
Computation of the numerical shape, curvature and contact angle of the saddle point.

Difficulties arise at the triple point

- Need to adopt a microscopic view of the model at the triple point:
  $\longrightarrow$ "Macro-Micro" coupling model.
- Need to compute accurately the potential close to the triple point:
  $\longrightarrow$ Use of the Singular Complement Method (Ciarlet Jr. and al).

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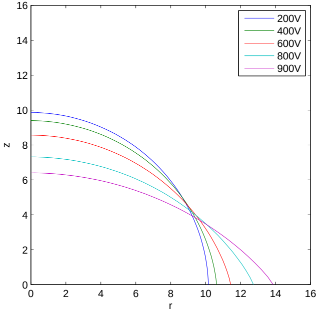
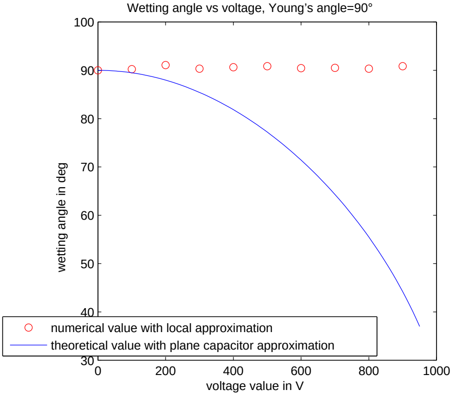
Shape of the drop



Contact angle



Outline

3D case: stakes

- Need of a good approximation of the electrostatic field and of its trace on the boundary of the drop
- Singular Complement Method less efficient in 3D than in 2D.

Method:

- Computation of the field, instead of the potential.
- Weighted weak formulation on the divergence of the field in order to solve the problem induced by the singularity (M. Costabel, M. Dauge, *Numer. Math.* 2002; P. Ciarlet Jr. et al., *M2AN*).

Point of view adopted:

- Numerical Analysis instead of computations.

Outline

Field formulation and space considered

The field E^Ω is solution of:

$$\begin{cases} \mathrm{curl} E_i^\Omega = 0 & \text{in } \Omega_i \quad i = G, S \\ \mathrm{div}(\varepsilon_i E_i^\Omega) = 0 & \text{in } \Omega_i \quad i = G, S \\ E_i^\Omega \times n = 0 & \text{on } \Gamma_{LG} \cup \Gamma_{LS} \cup \Gamma_0 \\ \varepsilon_G E_G^\Omega \cdot n = \varepsilon_S E_S^\Omega \cdot n \quad , & E_G^\Omega \times n = E_S^\Omega \times n \text{ on } \Gamma_{GS} \\ \varepsilon E^\Omega \cdot n = 0 & \text{on the artificial boundaries} \end{cases}$$

- Space considered: For $\alpha \in]0,1[$,

  $$\mathcal{X}_\alpha := \left\{ \mathcal{F} \in H(\mathrm{curl}, \Omega) | w_\alpha \mathrm{div} \varepsilon \mathcal{F} \in L^2(\Omega), \quad \mathcal{F} \times n_{/\Gamma_0 \cup \Gamma_L} = 0, \quad \varepsilon \mathcal{F}.n_{/\Gamma_{ext}} = 0 \right\}$$

  where $w_\alpha(.) \approx dist(., \text{triple ligne})^\alpha$.
- The boundary of Ω has two connected components.
  For $\alpha \in]0,1[$,

  $$\|\mathcal{F}\|_{\mathcal{X}_\alpha} := (\|\mathrm{curl}\mathcal{F}\|^2_{L^2} + \|w_\alpha \mathrm{div}(\varepsilon \mathcal{F})\|^2_{L^2} + |\int_{\Gamma_0} \varepsilon \mathcal{F} \cdot n|^2)^{\frac{1}{2}}$$

  is an equivalent norm to the graph norm.

- E^Ω is completely caracterized if one adds the equation:

$$\int_{\Gamma_0} \varepsilon E^\Omega \cdot n d\Gamma = -\mathbb{C} V$$

where $\mathbb{C} = \int_\Omega \varepsilon \nabla \chi_0^\Omega \cdot \nabla \chi_0^\Omega d\Omega$ is the capacitance matrix, with

$$\begin{cases} \mathrm{div}(\varepsilon \nabla \chi_0^\Omega) = 0 & \text{in } \Omega_i \quad i = G, S \\ \chi_0^\Omega = 0 & \text{on } \Gamma_{LG} \cup \Gamma_{LS} \\ \chi_0^\Omega = 1 & \text{on } \Gamma_0 \\ +\text{Transmission conditions} & \end{cases}$$

- Denote

$$PH^1(\Omega) := \left\{ v \in L^2(\Omega) | v \in H^1(\Omega_G) \text{ and } v \in H^1(\Omega_S) \right\}$$

There exists $\alpha_{min} \in]0, 1[$ such that

$$\mathcal{X}_\alpha \cap (PH^1(\Omega))^3 \text{ is dense in } \mathcal{X}_\alpha \text{ for all } \alpha \in]\alpha_{min}, 1[$$

$\Rightarrow$ Approximation by Lagrange Finite Elements envisageable.

Weak formulation and numerical approximation

Continuous weak formulation

$$a(E^{\Omega}, \mathcal{F}) = l(\mathcal{F}), \quad \forall \mathcal{F} \in \mathcal{X}_{\alpha}$$

$$a(\mathcal{E},\mathcal{F}) := \int_{\Omega} \mathrm{curl}\mathcal{E} \cdot \mathrm{curl}\mathcal{F} d\Omega + \sum_{i=G,S} \varepsilon_i^{-2} \int_{\Omega_i} w_{\alpha} \mathrm{div}(\varepsilon\mathcal{E}) w_{\alpha} \mathrm{div}(\varepsilon\mathcal{F}) d\Omega + \varepsilon_S^{-2} \int_{\Gamma_0} \varepsilon\mathcal{E} \cdot n \int_{\Gamma_0} \varepsilon\mathcal{F} \cdot n \quad (1)$$

and $l(\mathcal{F}) = -\mathbb{C} V \int_{\Gamma_0} \varepsilon\mathcal{F} \cdot n$.

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and $l(\mathcal{F}) = -\mathbb{C}V \int_{\Gamma_0} \varepsilon\mathcal{F} \cdot n$.

Approximation

$\mathcal{T}_h$ family of meshes of Ω.
$E_h^{\Omega} \in \mathcal{X}_{h,k} := \left\{ \mathcal{F}_h \in \mathcal{X}_{\alpha} \cap (PH^1(\Omega))^3 | (\mathcal{F}_h)_{K_l} \in (\mathbb{P}_k(K_l))^3, \quad \forall K_l \in \mathcal{T}_h \right\}$ solution of

$$a(E_h^{\Omega}, \mathcal{F}_h) = l_h(\mathcal{F}_h), \quad \forall \mathcal{F}_h \in \mathcal{X}_{h,k}$$

where l_h is an approximation of l.

- In our particular case, we know the value of α_{min} :

$$\alpha_{min} = 1 - \min \nu_Y(s),$$

and $\nu_Y(s)$ is the unique solution in $]0,1[$ of the equation:

$$\varepsilon_S \tan(\nu_Y(s)(\pi - \theta_Y(s))) = -\varepsilon_G \tan(\nu_Y(s)\pi).$$

- Error estimation obtained:

$$\forall \eta > 0, \quad \exists C_\eta, \quad \|E^\Omega - E_h^\Omega\|_{\mathcal{X}_\alpha} \leq C_\eta h^{\alpha - \alpha_{min} - \eta}$$

- Normal trace defined in $H^{-\frac{1}{2}}(\partial\Omega)$ and:

$$\forall \eta > 0, \quad \exists C_\eta, \quad \left\|\varepsilon E^\Omega \cdot n - \varepsilon E_h^\Omega \cdot n\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C_\eta h^{\alpha - \alpha_{min} - \eta}$$

Outline

Conclusions

- Modelling of Electrowetting phenomena
- Numerical simulation in the axisymmetric case.
- Numerical Analysis in 3D.

Taking into account the singularity is essential!!

Further works

- Saturation of the contact angle: Something is missing in the model! Corona discharge phenomenon.
- Non static case: Singularity to be taken into account.
- Existence of the optimal shape...
- Computations in the 3D case.