Wave-crack interaction in finite elastic bodies

Adriana Lalegname, Anna-Margarete Sändig

University Stuttgart, IANS

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The cracked elastic domains

The propagation of straight cracks by the influence of elastic waves will be considered as a moving boundary value problem:

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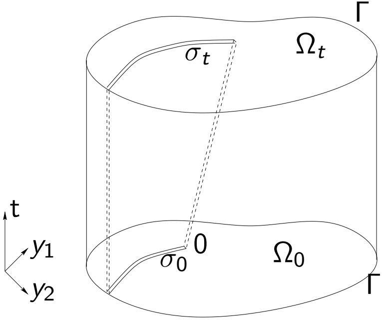
Reference config. $\Omega_0 = \Omega \diagdown \sigma_0 \longrightarrow$ Current config. $\Omega_t = \Omega \diagdown \sigma_t$,
where the motion of Ω_0 to Ω_t is given by a family of mappings

$$y = F_t(x) = x + h(t)\, \theta(x), \quad x \in \Omega_0, \quad y \in \Omega_t.$$

with unknown crack tip motion $h(t)$, $\theta = \eta(r)(1,0)^\top$.



The system of equations in the current configuration

$$(\lambda+\mu)\nabla(\nabla\cdot\vec{u})+\mu\nabla^2\vec{u}+\rho\vec{f}=\rho\vec{u}_{tt} \quad \text{in } Q:=\bigcup_{t=0}^{T}\Omega_t,$$

$$\sigma\vec{n}=0 \quad \text{on } \bigcup_{t=0}^{T}\sigma_t,$$

$$\sigma\vec{n}=\rho\vec{q} \quad \text{on } \Sigma_N:=\Gamma_N\times(0,T),$$

$$\vec{u}(t,y)=0 \quad \text{on } \Sigma_D:=\Gamma_D\times(0,T),$$

$$\vec{u}(0,y)=\vec{u_0},\ \partial_t\vec{u}(0,y)=\vec{u_1} \quad \text{in } \Omega_0.$$

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$(c_1^2-c_2^2)\nabla(\nabla\cdot\vec{u})+c_2^2\nabla^2\vec{u}-\ddot{\vec{u}}=\vec{f}$ Navier Lamé equations
$c_1^2=\frac{(\lambda+2\mu)}{\rho}=$ longitudinal or dilatational wave propagation speed,
$c_2^2=\frac{\mu}{\rho}=$ shear or rotational wave propagation speed.

Energy balance law

Assume that for every time t the rate of total energy $\hat{\Pi}$ is given by the rates of the dissipative energy D, the elastic energy E, the kinetic energy K and the external energy $\hat{A}$ for the wave displacement u, satisfying the above system:

$$0 = \hat{\Pi}(t) = \dot{D}(t) + \dot{E}(t) - \hat{A}(t) + \dot{K}(t)$$

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- $\dot{E}(t) = \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t} \sigma(\vec{u}) : \epsilon(\vec{u})\, dy$
- $\hat{A}(t) = \int\limits_{\Omega_t} \rho\vec{f} \cdot \vec{u}_t\, dy + \int\limits_{\Gamma_N} \rho\, \vec{q} \cdot \vec{u}_t\, ds,$
- $\dot{K}(t) = \frac{d}{dt}\int\limits_{\Omega_t} \frac{1}{2}\rho\, |\vec{u}_t|^2\, dy$ $\Longrightarrow \dot{D}(t).$
- D = energy, spent for irreversible processes (plastic deformations, voids, chemical reactions, noise,...)

Griffith criterion

Dynamic energy release rate in the plane strain case:

$$G(h,h') = \begin{cases} \frac{\dot{D}(t)}{h'(t)}, & \text{if } h'(t) \neq 0 \\ \frac{1-\nu}{2\mu}\,(k_1^2 + k_2^2), & \text{if } h'(t) \to 0. \end{cases}$$

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Griffith criterion: Let the fracture toughness $\Gamma(h,h')$ be known by experiments.

- If $G(h,h') < \Gamma(h,h') \Longrightarrow$ no crack propagation.
- If $G(h,h') = \Gamma(h,h') \Longrightarrow$ crack propagation, additional equation for calculation of the unknown crack position $h(t)$ for a running crack.

Challenges of the model

- Characterize the behaviour of the wave fields near the running crack tip (i.e. determine the dynamic crack singularities).

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- Characterize the behaviour of the wave fields near the running crack tip (i.e. determine the dynamic crack singularities).
- Calculate the dynamic energy release rate in terms of dynamic stress intensity factors.
- Solve the ordinary differential equation for $h(t)$ given by the Griffith criterion for the running crack.
- Compute numerically the wave fields and the resulting motion of the crack tip $h(t)$ by an iterative scheme.

For the out-of-plane case (Mode III) see: S.Nicaise/S. 2007(JMAA), L./S./Sewell 2008(IntJFrac)

Helmholtz's decomposition

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Theorem

Let $\vec{u}$ be a solution of the Navier-Lamé equation. Then there exists scalar and vector potentials ϕ(dilatational part) and $\vec{\psi}$ (rotational part) in the 3D-case such that:

$$\vec{u} = \nabla\phi + \nabla \times \vec{\psi}, \quad \nabla \cdot \vec{\psi} = 0.$$

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Also there exist a scalar function f *and a vector function* $\vec{B}$*, such that the density vector of the volume forces* $\vec{f}(y,t) = \vec{f} = (f_1, f_2, f_3)^T$ *can be decomposed as:*

$$\vec{f} = \nabla \mathrm{f} + \nabla \times \vec{B}, \quad \nabla \cdot \vec{B} = 0,$$

Helmholtz's decomposition

Corollary If $\vec{f} = \vec{0}$, then we get in the 2D case two uncoupled scalar wave equations in $Q := \bigcup_{t=0}^{T} \Omega_t$:

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The plane strain wave-field is given by

$$u_1 = \partial_1 \Phi + \partial_2 \Psi,$$
$$u_2 = \partial_2 \Phi - \partial_1 \Psi.$$

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Two $z^{(i)}$-configurations, $z^{(i)}$-coordinates, the space principal parts with frozen coefficients are Laplacians.

Crack fields

In the $z^{(i)}$-configurations we consider the following crack fields with time depending coefficients

$$w_{\text{sing}}^{(i)}(z^{(i)}, t) = A_0^{(i)}(t) r_{z^{(i)}}^{\frac{3}{2}} \cos\left(\frac{3}{2}\varphi_{z^{(i)}}\right) + B_0^{(i)}(t) r_{z^{(i)}}^{\frac{3}{2}} \sin\left(\frac{3}{2}\varphi_{z^{(i)}}\right)$$

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$$u_{1,\text{sing}} = \partial_1\phi_{\text{sing}} + \partial_2\psi_{\text{sing}},$$
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- regard the Neumann conditions on the crack face

Crack fields

Finally we get under some assumptions

$$\vec{u}(\vec{y},t) = \vec{u}_{reg}(\vec{y},t) + k_1(t,h,h')\vec{u}_{1,\mathrm{sing}}(\vec{y},t) + k_2(t,h,h')\vec{u}_{2,\mathrm{sing}}(\vec{y},t),$$

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$$\vec{u}_{1,\mathrm{sing}} = \frac{(1+\alpha_2(t)^2)}{\mu\, D_{\mathrm{Ra}}} \begin{pmatrix} s_1^1(R,\vartheta,h,h') \\ s_1^2(R,\vartheta,h,h') \end{pmatrix}$$

$$\vec{u}_{2,\mathrm{sing}} = -\frac{\alpha_2(t)}{\mu\, D_{\mathrm{Ra}}} \begin{pmatrix} s_2^1(R,\vartheta,h,h') \\ s_2^2(R,\vartheta,h,h') \end{pmatrix}$$

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(R,ϑ) are the current polar coordinates in the moving crack tip,
$\alpha_i = \sqrt{1 - \frac{h'(t)^2}{c_i^2}}$.
The condition $D_{Ra} := 4\,\alpha_1(t)\,\alpha_2(t) - (1+\alpha_2(t)^2)^2 \neq 0$ excludes the Rayleigh velocity $h'(t) = v_{\text{Ra}}$ and $h'(t) = 0$.

Dynamical crack fields

First dynamic singular function with the two components:

$$s_1^1(R,\vartheta,h,h') = \sqrt{\frac{\sqrt{(R\cos\vartheta - h)^2 + \alpha_1^2(t)R^2\sin^2\vartheta} + (R\cos\vartheta - h)}{(R\cos\vartheta - h)^2 + \alpha_1^2(t)R^2\sin^2\vartheta}}$$

$$-\frac{2\,\alpha_1(t)\,\alpha_2(t)}{(1+\alpha_2(t)^2)}\sqrt{\frac{\sqrt{(R\cos\vartheta - h)^2 + \alpha_2^2(t)R^2\sin^2\vartheta} + (R\cos\vartheta - h)}{(R\cos\vartheta - h)^2 + \alpha_2^2(t)R^2\sin^2\vartheta}}$$

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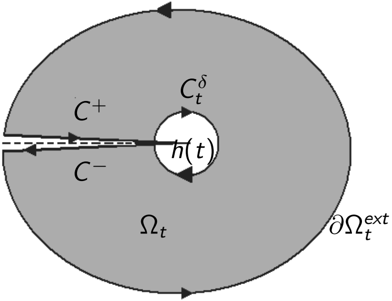
Theorem:

$$\dot{D}(t) = \frac{h'(t)}{2\mu}\left[\frac{\left(1-\alpha_2(t)^2\right)\,\left(\alpha_1(t)k_1^2(t,h,h') + \alpha_2(t)k_2^2(t,h,h')\right)}{4\alpha_1(t)\alpha_2(t) - \left(1+\alpha_2(t)^2\right)^2}\right].$$

Idea of the proof

Consider a family of annular domains cutting out the running crack tip.



In the annular domain, marked by the index δ, there holds:

$$\hat{A}^\delta(t) - \dot{E}^\delta(t) - \dot{K}^\delta(t) = -\frac{1}{2}\int_{\partial\Omega_t^\delta}\left[\left(\rho\,|\vec{u_t}|^2 + \sigma(\vec{u}) : \epsilon(\vec{u})\right)\frac{\partial y}{\partial t}\right]\cdot \vec{n}_y\, ds_y$$
$$-\int_{C_t^\delta}\Big((\lambda+\mu)(\nabla\cdot\vec{u})\vec{n} + \mu(\nabla\vec{u})\vec{n}\Big)\cdot\partial_t\vec{u}\, ds_y.$$

Limit procedure $\delta \to 0$ yields the statement.

Equation of motion for the running crack tip

If the crack growth resistance $\Gamma(h, h')$ is known by experiments, then we get the ordinary differential equation for $h(t)$:

$$\Gamma(h, h') = \frac{\dot{D}(t)}{h'(t)} = \frac{1}{2\mu} \left[\frac{\left(1 - \alpha_2(t)^2\right) \left(\alpha_1(t) k_1^2(t, h, h') + \alpha_2(t) k_2^2(t, h, h')\right)}{4\alpha_1(t)\alpha_2(t) - (1 + \alpha_2(t)^2)^2} \right],$$

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where the dynamical stress intensity factors can be extracted

$$\lim_{R(t)-h(t)\to 0} \frac{\mu\, D_{\mathrm{Ra}}\sqrt{2\pi(R(t)-h(t))}}{1+\alpha_2(t)^2 - 2\alpha_1(t)\alpha_2(t)}\partial_1 u_1(y,t)|_{y_2=0} = k_1(t,h,h'),$$

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Freund 1990 and other authors have proposed to consider a mode I crack, what leads to the following problem:
Find $h(t)$ such that

$$\Gamma(h, h') = \left(1 - \frac{h'(t)}{v_{\mathrm{Ra}}}\right) \frac{(1-\nu^2)}{E} k_{1\,static}^2 = \frac{h'(t)^2}{2\,\mu\, c_2^2}\ \frac{\alpha_1(t)\ k_1^2(t, h, h')}{D_{\mathrm{Ra}}}$$
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This equation will be used in our numerical experiments.

The complete formulation for the dynamic coupled problem for in-plane fracture case reads:

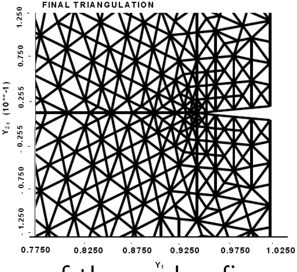
$$\left.\begin{aligned}
&(\lambda+\mu)\nabla(\nabla\cdot\vec{u})+\mu\nabla^2\vec{u}=\rho\vec{u}_{tt} \quad \text{in } Q:=\bigcup_{t=0}^{T}\Omega_t,\\
&\sigma\vec{n}=0 \quad \text{on } \bigcup_{t=0}^{T}\sigma_t,\\
&\sigma\vec{n}=\rho\vec{q} \quad \text{on } \Sigma_N:=\Gamma_N\times(0,T),\\
&\vec{u}(t,y)=0 \quad \text{on } \Sigma_D:=\Gamma_D\times(0,T),\\
&\vec{u}(0,y)=\vec{u_0},\ \partial_t\vec{u}(0,y)=\vec{u_1} \quad \text{in } \Omega_0.
\end{aligned}\right\}$$

$$\begin{cases}
\Gamma(h,h')=\dfrac{1}{2\mu}\left[\dfrac{\left(1-\alpha_2(t)^2\right)\ \left(\alpha_1(t)k_1^2(t,h,h')+\alpha_2(t)k_2^2(t,h,h')\right)}{4\,\alpha_1(t)\ \alpha_2(t)-\left(1+\alpha_2(t)^2\right)^2}\right],\\[2ex]
h(0)=0,\ k_1(0,h,h')=k_1(0),\ k_2(0,h,h')=k_2(0).
\end{cases}$$

$$k_1(t,h,h')=\lim_{R(t)-h(t)\to 0}\ \frac{\mu\,D_{\mathrm{Ra}}\,\sqrt{2\,\pi\,(R(t)-h(t))}}{1+\alpha_2(t)^2-2\,\alpha_1(t)\,\alpha_2(t)}\,\frac{du_1(y,t)}{dy_1}\Big|_{y_2=0},$$

$$k_2(t,h,h')=\lim_{R(t)-h(t)\to 0}\ \frac{\mu\,D_{\mathrm{Ra}}\,\sqrt{2\,\pi\,(R(t)-h(t))}}{2\,\alpha_1(t)\ \alpha_2(t)-\left(1+\alpha_2(t)^2\right)}\,\frac{du_2(y,t)}{dy_1}\Big|_{y_2=0}.$$

Mode I crack propagation



zoom of the mesh-refinement near the crack tip

square in (y_1, y_2) coordinates

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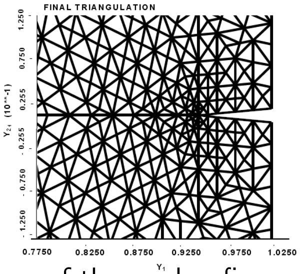

zoom of the mesh-refinement
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square in (y_1, y_2) coordinates

$$\vec{u}(y,t) = 0, \text{ at } \; y_1 = -1,$$

$$\sigma \cdot \vec{n} = 1000 N/m^2 \text{ on } y_2 = -1 \text{ and } \; y_2 = 1,$$

$$\sigma \cdot \vec{n} = 0 \text{ on the crack}, \quad \sigma \cdot \vec{n} = 0 \text{ on } y_1 = 1,$$

Note, that the crack is running from right with the starting position $h(0) = 0.9$.

Equation for the crack motion

$$\Gamma(h,h') = \left(1 - \frac{h'(t)}{v_{\mathrm{Ra}}}\right)\frac{(1-\nu^2)}{E}k^2_{1\ static} = G(h,h'),$$

$$k_{1\ static} = 0.01\,\mathrm{Pa}\ \cdot m$$

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$$\Gamma(h,h') = \left(1 - \frac{h'(t)}{v_{\mathrm{Ra}}}\right) \frac{(1-\nu^2)}{E} k_{1\ static}^2 = G(h,h'),$$
$$k_{1\ static} = 0.01\,\mathrm{Pa}\ \cdot m$$

Initial conditions fot $t = 0$ with $h(0) = 0.9, h'(0) = 0.5 v_{\mathrm{Ra}}$:

$$\vec{u}(y,0) = k_{1\ static} \frac{(1+\alpha_2(0)^2)}{\mu\, D_{\mathrm{Ra}}} \begin{pmatrix} s_1^1(R,\vartheta,h,h') \\ s_1^2(R,\vartheta,h,h') \end{pmatrix}$$
$$\partial_t \vec{u}(y,0) = 0.$$

The iterative procedure

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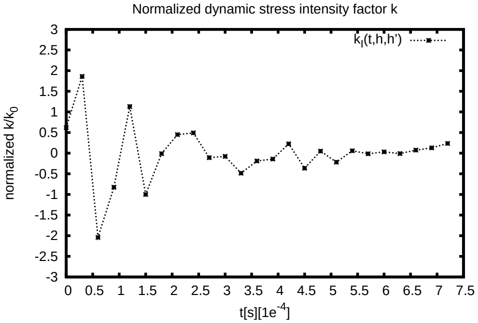
7. Interpolate previous mesh nodal data for finding corresponding data of present mesh.

The FEM-package PDE2D (Sewell, Univ.Texas) was used.

Numerical results

The relative stress intensity factor $\tilde{k}_1 = \frac{k_1}{k_{1\ static}}$ versus the time



There is an oscillatory behaviour as the initial crack length increases, but it tends to $k_{1\ static}$.

FEM-solutions for the first component u_1