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Optimal elliptic regularity near 3-dimensional, heterogeneous Neumann vertices

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1 Introduction

Let $\Pi \subseteq \mathbb{R}^3$ be a domain, whose closure $\overline{\Pi}$ is simultaneously a polyhedron and a manifold with boundary. For a bounded, measurable coefficient function $\mu : \Pi \to \mathbb{R}^{3\times 3}$ we define the operator $-\nabla \cdot \mu\nabla : W^{1,2}(\Pi) \to (W^{1,2}(\Pi))'$ as usual by

$$\langle -\nabla \cdot \mu\nabla v, w\rangle := \int_\Omega \mu\nabla v \cdot \nabla\overline{w}\, d\mathrm{x}\,, \quad v, w \in W^{1,2}(\Pi), \tag{1}$$

in order to have (homogeneous) Neumann boundary conditions for the restriction of this operator to $L^2(\Pi)$.

THEOREM 1. *There is a $p > 3$, such that, for any $f \in (W^{1,p'}(\Pi))'$, every solution v of $-\nabla \cdot \mu\nabla v = f$ is in $W^{1,p}$ locally around a vertex a of Π, provided the following assumptions hold true:*

- *μ is elliptic and takes symmetric matrices as values.*
- *$\Pi = |K|$ for some finite, Euclidean complex K and μ is constant on the inner of every 3-cell belonging to K, i.e. μ is piecewise constant on a cellular subpartition of the polyhedron Π.*
- *Any edge from the boundary of Π that has one endpoint in a is a geometric edge or a bimaterial outer edge, such that both opening angles do not exceed π.*

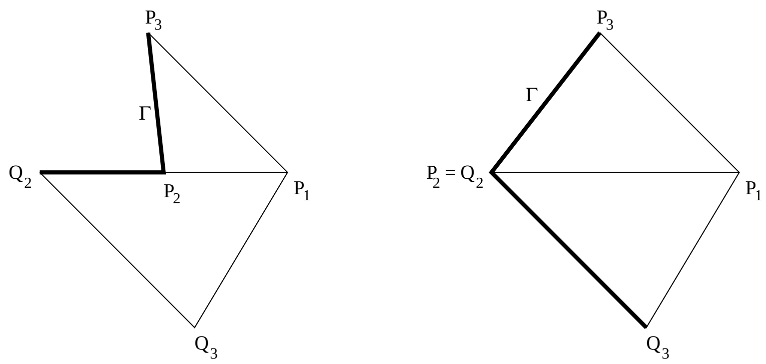

Abbildung 1:

- *Every inner edge with endpoint a is well-behaved, i.e. the singularity exponent associated to this edge, is larger than $1/3$.*

Strategy of proof

1) Deform a neighbourhood of a by a PL homeomorphism ϕ, such that $\phi(\mathrm{a}) = 0 \in \mathbb{R}^3$ and the corresponding boundary part becomes part of the $x - z$-plane

2) Diminish the neighbourhood such that the image under ϕ equals a suitable half cube and, additionally, the only occurring edges have either one of their endpoints in $0 \in \mathbb{R}^3$ or are situated on the boundary of the half cube

3) Reflect the problem across the $x - z$-plane and end up with a Dirichlet problem

4) Restrict the edge singularities and exploit a theorem on elliptic regularity in case of polyhedral Dirichlet problems

2 The PL flattening theorem

DEFINITION 2. *Let K be a complex in $\mathbb{R}^d$. A continuous mapping f from $|K|$ onto a subset of $\mathbb{R}^m$ is then called* piecewise linear*, if there is a subdivision K' of K, such that the restricted function $f|_\sigma$ is linear for every $\sigma \in K'$.*

DEFINITION 3. *If v is a vertex of the Euclidean complex K, then we call the set of all cells from K which contain v, together with all their faces, the star around v within K.*

LEMMA 4. *Let K be a finite simplicial complex in $\mathbb{R}^3$ whose polyhedron $|K|$ is a 3-dimensional manifold with boundary. Let $\mathrm{v} \in \partial|K|$ be any vertex of K. If we denote by $K_\mathrm{v}^\star$ the star around v within K, then the polyhedron $|K_\mathrm{v}^\star|$ is homeomorphic to the closed unit ball in $\mathbb{R}^3$. Moreover, the boundary of $|K_\mathrm{v}^\star|$ is topologically a 2-sphere and, additionally, a polyhedron.*

PROPOSITION 5. *Let S be a polyhedron in $\mathbb{R}^3$ which is topologically a 2-sphere, and let $\mathcal{W}$ be a convex, open set containing S. Then there is a PL homeomorphism*

$$\phi_S : \mathbb{R}^3 \leftrightarrow \mathbb{R}^3, \qquad S \leftrightarrow \partial\sigma^3,$$

where σ^3 is a tetrahedron, such that $\phi_S|_{\mathbb{R}^3\setminus\mathcal{W}}$ is the identity.

According to Lemma 4, we may apply Proposition 5 to the polyhedron $K_{\mathrm{a}}^{\bigstar}$. Clearly, $Int(K_{\mathrm{a}}^{\bigstar})$ is mapped onto $Int(\sigma^3)$ and $\partial(K_{\mathrm{a}}^{\bigstar})$ is mapped onto $\partial\sigma^3$. Modulo another PL homeomorphism $\phi_3 : \mathbb{R}^3 \to \mathbb{R}^3$ one may arrange that

- $\phi_S(\mathrm{a}) = 0$
- $\phi_S(\partial(K_{\mathrm{a}}^{\bigstar}))$ is an open neighbourhood of 0 in the plane $y = 0$.
- $\phi_S(Int(K_{\mathrm{a}}^{\bigstar}))$ is an open subset of $\{\mathrm{x} = (x, y, z) : x, z \in \mathbb{R}, y > 0\}$.

COROLLARY 6. *Let $\Lambda \subset \mathbb{R}^3$ be a polyhedron, which is the closure of its interior Ω, and suppose that Λ is a 3-manifold with boundary. Then Ω is a Lipschitz domain, even more: the local bi-Lipschitz charts around boundary points may be chosen as PL homeomorphisms.*

Consider now the image $\phi_S(K_{\mathrm{a}}^{\bigstar})$, which carries the Euclidean structure from the PL subdivision of $K_{\mathrm{a}}^{\bigstar}$. Denote the star around $\phi_S(\mathrm{a})$ within this complex by L. Finally, intersect this complex by a sufficiently small cube $\mathcal{C}$, such that all edges of $\mathcal{C} \cap L$ which intersect $intK$, have one endpoint in 0.

Bild We reflect the problem now symmetrically at the plane $y = 0$ and end up with a Dirichlet problem of the same type.

LEMMA 7.

$$-\nabla \cdot \hat{\mu} \nabla : W_0^{1,p}(\mathcal{C}) \to W^{-1,p}(\mathcal{C}) \tag{2}$$

is a topological isomorphism for a $p > 3$.

PROPOSITION 8. *Let $\{\Omega_k\}_k$ be a polyhedral partition of Ω, such that the coefficient function μ is constant on the inner of each Ω_k. If for every such edge the associated singularity exponent is larger than $\frac{1}{3}$, then there is a $p > 3$, such that*

$$-\nabla \cdot \mu \nabla : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega) \tag{3}$$

is a topological isomorphism.

Let us denote the upper half cube of $\mathcal{C}$ by $\mathcal{C}_+$ and the midplane of $\mathcal{C}$ by Σ.
Now we are going to identify the occurring edges E in $\overline{\mathcal{C}}$.

I	edges from $\partial\mathcal{C}$,	III	edges from Σ,
II	edges from $\mathcal{C}_+$,	IV	edges from $\mathcal{C}_-$.

DEFINITION 9. *Let E be an edge in $\overline{\Omega}$ that lies in $\partial\Omega$. Then we define:*

1. *E is a* geometric edge, *if all relative inner points of E possess a neighbourhood in $\overline{\Omega}$ on which μ is constant a.e. with respect to 3-dimensional Lebesgue measure.*
2. *E is a* bimaterial outer edge, *if it is adjacent to exactly two material sectors.*

PROPOSITION 10. *For any geometric edge E the kernels of the associated operators $\mathcal{A}_\lambda$ are trivial, if $\Re\lambda \in]0, 1/2]$. This same is true for bimaterial outer edges, if both sectors have an opening angle not larger than π.*

LEMMA 11. *The edges from $\partial\mathbb{C}$ are either geometrical edges or bimaterial outer edges with opening angles not larger than π. Hence, their singularity exponents are uncritical, due to Proposition 10.*

Edges from $\mathcal{C}_+$: By the definition of the cube K, all edges which intersect $\mathcal{C}_+$, have one endpoint in 0. Thus, their inverse image is either

I part of an original edge

or

II E lies in the inner of a tetrahedron from the original triangulation of $\overline{\Pi}$

or

III E does not intersect an edge from the original triangulation of $\overline{\Pi}$, but is contained in the intersection of two faces $\mathfrak{F}_1, \mathfrak{F}_2$ from two tetrahedra $\mathfrak{T}_1, \mathfrak{T}_2$.

By transforming back and exploiting known (but nontrivial) regularity theorems, one obtains

LEMMA 12. *The singularities associated to the edges from I, II, III are not critical.*

It remains to discuss the edges from Σ.