

Singular behavior of the solution of the Helmholtz equation in weighted L^p -Sobolev spaces

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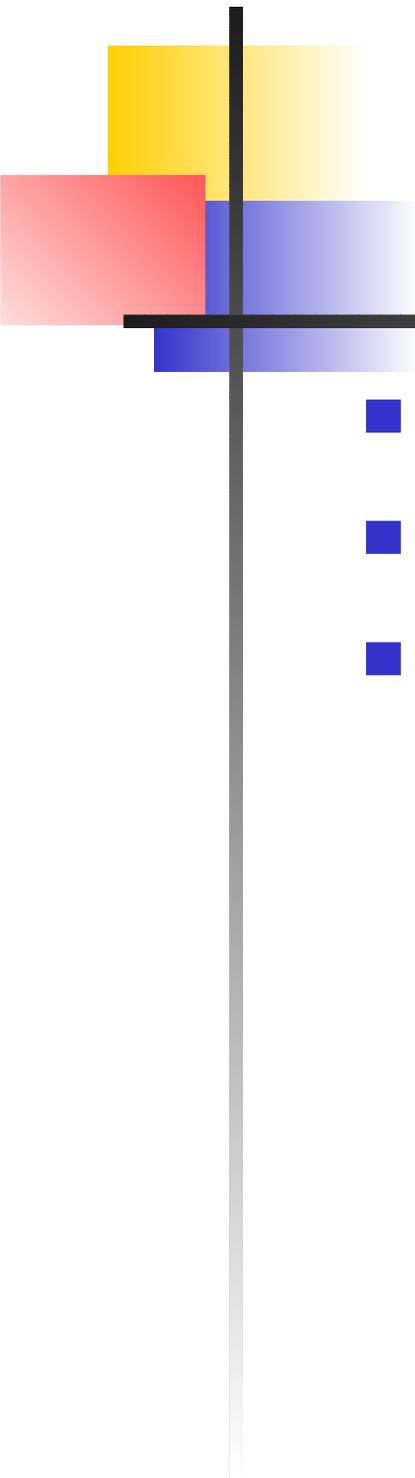
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Outline of the talk

- The Problem
- Some Embeddings and consequences
- The main result

The problem

Let Ω be a polygonal domain of \mathbb{R}^2 with a Lipschitz boundary $\partial\Omega$. On this domain, we consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & \text{in } \Omega \times (-\pi, \pi), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (-\pi, \pi), \\ u(\cdot, -\pi) = u(\cdot, \pi), & \text{in } \Omega, \end{cases}$$

where $f \in L^p((-\pi, \pi), L^p_\mu(\Omega))$ (described below) with $p \neq 2$.

Goal: Find regularity results for a large range of values of μ .

Tools: Uniform estimates for Helmholtz eq. and theory of sum of operators.

Some references

[Kozlov, 88]: $p = 2$, full asymptotic expansion.

Tool: Fourier analysis

[Grisvard, 95]: $p > 1, \mu = 0$, decomposition into regular and singular parts.

Tool: Theory of sum of operators.

[Nazarov, 01, 03], [Solonnikov, 01], [Pruss-Simonett, 07],

[Amann, 09]: $p > 1, \mu$ large enough to avoid the singularities.

Tools: Estimates of the Green function/Theory of sum of operators/blowing up.

Reduction

In order to give existence and regularity results for such a problem, we first localize the problem.

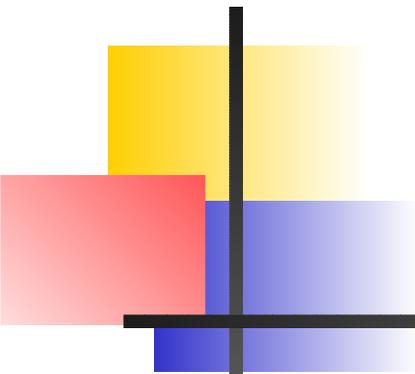
Reduce Ω to the **truncated sector**

$$\Omega = \{(r \cos \theta, r \sin \theta) \mid 0 < r < 1, 0 < \theta < \psi\}, \quad \psi \in (0, 2\pi].$$

Use the theory of the sum of operators, hence we need first to study the **Helmholtz** equation

$$-\Delta u + zu = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (1)$$

where $g \in L^p_\mu(\Omega)$ with $p \neq 2$ and $z \in \pi^+ \cup S_A$ where


$$\pi^+ = \{z \in \mathbb{C} \mid \Re(z) \geq 0\},$$

$$S_A = \{z \in \mathbb{C} \mid |z| \geq R \text{ and } |\arg z| \leq \theta_A\},$$

for $R > 0$ and $\theta_A \in]\frac{\pi}{2}, \pi[$ fixed.

Some definitions

For $p > 1, \mu \in \mathbb{R}$: weighted sp. with homogeneous norms:

$$L_{\mu}^p(\Omega) = \{f \in L_{loc}^p(\Omega) \mid r^{\mu} f \in L^p(\Omega)\}.$$

and

$$V_{\mu}^{k,p}(\Omega) = \{u \in L_{loc}^p(\Omega) \mid \|u\|_{V_{\mu}^{k,p}(\Omega)} < \infty\},$$

$$\|u\|_{V_{\mu}^{k,p}(\Omega)}^p := \sum_{|\gamma| \leq k} \int_{\Omega} |D^{\gamma} u(x)|^p r^{(\mu+|\gamma|-k)p}(x) dx.$$

In $H_0^1(\Omega)$ we will denote its semi-norm by

$$|u|_{H_0^1}^2 = \int_{\Omega} |\nabla u|^2.$$

Embeddings and consequences

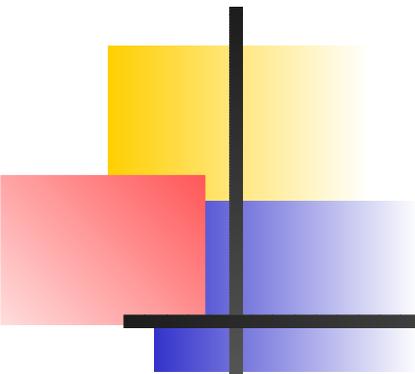
Le 1. Let $p \geq 2$ and μ satisfies,

$$\begin{aligned} \mu &< \frac{2p-2}{p}, & \text{if } p > 2, \\ \mu &\leq 1, & \text{if } p = 2. \end{aligned} \tag{2}$$

1. $L_{\mu}^p(\Omega) \hookrightarrow L_1^2(\Omega)$,
2. $L_{-1}^2(\Omega) \hookrightarrow (L_{\mu}^p(\Omega))' = L_{-\mu}^q(\Omega)$, $\frac{1}{q} + \frac{1}{p} = 1$,
3. $H_0^1(\Omega) \hookrightarrow L_{-\mu}^q(\Omega)$.

Proof

1. follows from Hölder's inequality and the fact that $r^{1-\mu} \in L^s(\Omega)$ if $1 - \mu > -\frac{2}{s}$.
2. consequence of the first one by using duality.
3.
 - a. $p = 2$, it is well known (see Thm 14.44 in [CDN book]) that $H_0^1(\Omega) \hookrightarrow L_{-1}^2(\Omega)$. We then conclude observing that, for $\mu \leq 1$: $r^{2(-\mu+1)} \in L^\infty(\Omega)$.
 - b. $p > 2$, we use the embedding $H_0^1(\Omega) \hookrightarrow L_{-1}^2(\Omega)$ and the second assertion.



Coro 1. Let $p \geq 2$ and μ satisfies (2). Then for all $g \in L^p_\mu(\Omega)$ and all $z \in \pi^+ \cup S_A$, the problem

$$\forall w \in H_0^1(\Omega), \quad \int_{\Omega} \nabla u \nabla \bar{w} + z \int_{\Omega} u \bar{w} = \int_{\Omega} g \bar{w}, \quad (3)$$

has a unique solution $u \in H_0^1(\Omega)$.

Rk. $u \in H_0^1(\Omega)$ is a weak solution of (1):

$$-\Delta u + zu = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Some Inequalities

Le 2. *Let $p \geq 2$, μ satisfies (2), $z \in \pi^+ \cup S_A$, and $u \in H_0^1(\Omega)$ be the solution of (3). Then*

$$|u|_{H_0^1(\Omega)} \lesssim \|g\|_{L_\mu^p(\Omega)}, \quad (4)$$

$$(1 + |z|^{1/2})|u|_{L^2(\Omega)} \lesssim \|g\|_{L_\mu^p(\Omega)}. \quad (5)$$

Proof

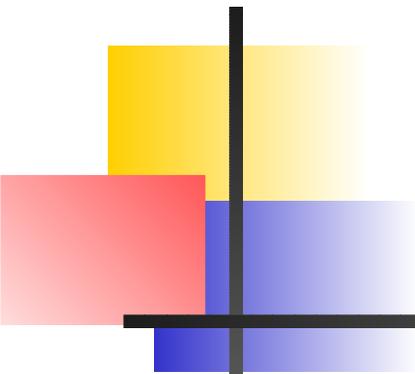
For $\Re z \geq 0$: Applying (3) with $w = u$ we have

$$|u|_{H_0^1}^2 + z \int_{\Omega} |u|^2 = \int_{\Omega} g \bar{u}. \quad (6)$$

By Lemma 1, taking the real part of (6), we obtain

$$|u|_{H_0^1}^2 + \Re z \int_{\Omega} |u|^2 \lesssim \|g\|_{L_{\mu}^p} |u|_{H_0^1}. \quad (7)$$

The result follows as $\Re z \geq 0$ and using **Poincaré** inequality.



Coro 2. Let $g \in L^2(\Omega)$, $z \in \mathbb{C}$ with $\Re z \geq 0$, and $u \in H_0^1(\Omega)$ be the solution of (3). Then

$$(1 + |z|) \|u\|_{L^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)}.$$

The domain

Def 1. Let $p \geq 2$ and $\mu \in \mathbb{R}$. Then we define

$$D(\Delta_{p,\mu}) = \{u \in H_0^1(\Omega) \mid \Delta u \in L_\mu^p(\Omega)\}.$$

Rk. If μ satisfies (2) and $2 - \frac{2}{p} - \mu \neq k\lambda, \forall k \in \mathbb{N}^*$. Then
[Maz'ya-Plamenevskii, 78] \Rightarrow

$$D(\Delta_{p,\mu}) = V_\mu^{2,p}(\Omega) \cap H_0^1(\Omega)$$

$$+ \text{Span} \{ \eta(r) r^{\lambda'} \sin(\lambda' \theta) \mid 0 < \lambda' = k\lambda < 2 - \frac{2}{p} - \mu \},$$

η cut-off fct s. t. $\eta = 1$ near $r = 0$ and $\eta(1) = 0$.

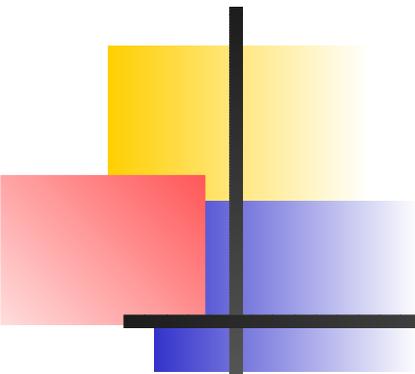
An existence result

Le 3. Let $p \geq 2$, μ satisfies (2) and $\mu > -\lambda$, with $\lambda = \frac{\pi}{\psi}$, $z \in \pi^+ \cup S_A$, $u \in H_0^1(\Omega)$ sol. of (3) with $g \in L_\mu^p(\Omega)$. Then $u \in D(\Delta_{p,\mu})$.

Proof. As Le 1 $\Rightarrow H_0^1(\Omega) \hookrightarrow L_{\mu'}^p(\Omega)$ for all $\mu' > -\frac{2}{p}$, we distinguish different cases:

1. $\mu > -\frac{2}{p}$ and therefore $-\Delta u = g - zu \in L_\mu^p(\Omega)$.
2. $-2 - \frac{2}{p} < \mu < -\frac{2}{p}$. Take $\mu' = \mu + 2$. Since $\mu' > -\frac{2}{p}$, $u \in H_0^1(\Omega)$ is solution of

$$-\Delta u = g - zu \in L_{\mu'}^p(\Omega).$$



This implies that

$$u \in V_{\mu'}^{2,p}(\Omega),$$

because the set $\{\lambda' = k\frac{\pi}{\psi}, k \in \mathbb{Z} : 0 < \lambda' < 2 - \frac{2}{p} - \mu'\} = \emptyset$
(the assumption $\mu > -\lambda \Rightarrow \lambda > -\frac{2}{p} - \mu$). Accordingly

$$r^{\mu'-2}u \in L^p(\Omega) \Leftrightarrow u \in L_{\mu}^p(\Omega),$$

due to $\mu' - 2 = \mu$.

This guarantees $-\Delta u = g - zu \in L_{\mu}^p(\Omega)$.

The general case follows by induction.

An a priori estimate

Le 4. Let $\lambda = \frac{\pi}{\psi}$, $p \geq 2$, $\mu > -\lambda$ satisfy (2) and

$$\frac{4(p-1)\lambda^2}{p^2} + \frac{2\mu}{p} - \mu^2 > 0. \quad (8)$$

Let $z \in \mathbb{C}$ with $\Re z \geq 0$, $u \in D(\Delta_{p,\mu})$ sol. of (3) with $g \in L^p_\mu(\Omega)$.

Then

$$\Re z \|u\|_{L^p_\mu} \leq \|g\|_{L^p_\mu} \quad \text{and} \quad \Im z \|u\|_{L^p_\mu} \lesssim \|g\|_{L^p_\mu}.$$

Proof

Some integrations by parts $\Rightarrow v = r^\mu u$ satisfies

$$\begin{aligned} & \frac{p}{2} \int_{\Omega} |\nabla v|^2 |v|^{p-2} + \frac{p-2}{2} \int_{\Omega} |v|^{p-4} \bar{v}^2 (\nabla v)^2 - \mu^2 \int_{\Omega} r^{-2} |v|^p \\ & + 2\mu \int_{\Omega} r^{-1} \frac{\partial v}{\partial r} |v|^{p-2} \bar{v} + z \int_{\Omega} |v|^p = \int_{\Omega} r^\mu g |v|^{p-2} \bar{v}, \\ & p \Re \left(\int_{\Omega} r^{-1} \frac{\partial v}{\partial r} |v|^{p-2} \bar{v} \right) = \int_{\Omega} r^{-2} |v|^p. \end{aligned}$$

Setting $w = |v|^{p/2}$, these two identities lead to

$$\begin{aligned} \frac{4(p-1)}{p^2} \int_D |\nabla w|^2 + \left(\frac{2\mu}{p} - \mu^2\right) \int_D r^{-2} w^2 \\ + \Re z \int_D w^2 \leq \Re \left(\int_D g |v|^{p-2} \bar{v} \right). \end{aligned}$$

Poincaré \leq in $\theta \Rightarrow \int_{\Omega} |\nabla w|^2 \geq \lambda^2 \int_{\Omega} \frac{1}{r^2} w^2, \forall w \in H_0^1(\Omega) \Rightarrow$

$$\left(\frac{4(p-1)\lambda^2}{p^2} + \frac{2\mu}{p} - \mu^2\right) \int_D r^{-2} w^2 + \Re z \int_D w^2 \leq \Re \left(\int_D g |v|^{p-2} \bar{v} \right)$$

The main result

Thm 1. Let $p \geq 2$, and let $\mu > -\lambda$ satisfies (2), (8) and, for all $k \in \mathbb{Z}^*$ $2 - \frac{2}{p} - \mu \neq k\lambda$. Then, for all $z \in \pi^+ \cup S_A$, $u \in D(\Delta_{p,\mu})$ sol. of (3) with $g \in L^p_\mu(\Omega)$, i.e. weak sol. of (1):

$$-\Delta u + zu = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

admits the decomposition

$$u = u_R + \sum_{0 < \lambda' = k\lambda < 2 - \frac{2}{p} - \mu} c_{\lambda'}(z) P_{\lambda'}(r\sqrt{z}) e^{-r\sqrt{z}} r^{\lambda'} \sin(\lambda'\theta), \quad (9)$$

with $u_R \in V_\mu^{2,p}(\Omega)$, $c_{\lambda'}(z) \in \mathbb{C}$ and $P_{\lambda'}(s) = \sum_{i=0}^{k_{\lambda'}-1} \frac{s^i}{i!}$,

$$k_{\lambda'} > 2 - 2/p - \mu - \lambda'.$$

The main result: uniform estimates

$$|u_R|_{V_\mu^{2,p}(\Omega)} + |z|^{1/2} |u_R|_{V_\mu^{1,p}(\Omega)} + |z| |u_R|_{L_\mu^p(\Omega)} \lesssim \|g\|_{L_\mu^p(\Omega)};$$

$$\sum_{0 < \lambda' < 2 - \frac{2}{p} - \mu} |c_{\lambda'}(z)| \left(1 + |z|^{1 - \frac{1}{p} - \frac{\mu + \lambda'}{2}}\right) \lesssim \|g\|_{L_\mu^p(\Omega)}.$$

Sketch of the proof

Lemma 4 \Rightarrow

$$\|g - zu\|_{L^p_\mu(\Omega)} \lesssim \|g\|_{L^p_\mu(\Omega)},$$

hence u can be seen as a solution of

$$-\Delta u = g - zu \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and by standard regularity results:

$$u = u_{1R} + \sum_{0 < \lambda' = k\lambda < 2 - \frac{2}{p} - \mu} c_{\lambda'}(z) r^{\lambda'} \sin(\lambda'\theta),$$

with $u_{1R} \in V_{\mu}^{2,p}(\Omega)$, $c_{\lambda'}(z) \in \mathbb{C}$.

Factors $P_{\lambda'}(r\sqrt{z})e^{-r\sqrt{z}}$: to have uniform estimates in z .

One application

Thm 2. Let $p \geq 2$, and let $\mu = 1 - \lambda$ satisfies (8) and, for all $k \in \mathbb{Z}^*$ $2 - \frac{2}{p} - \mu \neq k\lambda$. Then $\forall f \in L^p((0, \infty); L^p_\mu(\Omega)), \exists$ a sol. of

$$\partial_t u - \Delta u = f \text{ in } \Omega \times (0, \infty), \quad u = 0 \text{ on } \partial\Omega \cup \{t = 0\},$$

that admits the decomposition

$$u = u_R + \sum_{0 < \lambda' = k\lambda < 2 - \frac{2}{p} - \mu} (E(r, \cdot) \star_t q_{\lambda'}) r^{\lambda'} \sin(\lambda' \theta), \quad (10)$$

with $u_R \in L^p((0, \infty); V_\mu^{2,p}(\Omega)) \cap W^{1,p}((0, \infty); L^p_\mu(\Omega))$,

$q_{\lambda'} \in W^{1 - \frac{1}{p} - \frac{\mu + \lambda'}{2}, p}(0, \infty)$ and $E(r, t) = rt_+^{-\frac{3}{2}} e^{-\frac{r^2}{4t}}$.