Singular behavior of the solution of the Helmholtz equation in weighted L^p-Sobolev spaces

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6th Singular Days

WIAS Berlin, April 29, 2010

Outline of the talk

- The Problem
- Some Embeddings and consequences
- The main result

The problem

Let Ω be a polygonal domain of $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. On this domain, we consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f \text{ in } \Omega \times (-\pi, \pi), \\ u(x,\, t) = 0 \text{ on } \partial\Omega \times (-\pi, \pi), \\ u(\cdot, -\pi) = u(\cdot, \pi), \qquad \text{in } \Omega, \end{cases}$$

where $f \in L^p((-\pi, \pi), L^p_\mu(\Omega))$ (described below) with $p \neq 2$.
Goal: Find regularity results for a large range of values of μ.
Tools: Uniform estimates for Helmholtz eq. and theory of sum of operators.

Some references

[Kozlov, 88]: $p = 2$, full asymptotic expansion.

Tool: Fourier analysis

[Grisvard, 95]: $p > 1, \mu = 0$, decomposition into regular and singular parts.

Tool: Theory of sum of operators.

[Nazarov, 01, 03], [Solonnikov, 01], [Pruss-Simonett, 07], [Amann, 09]: $p > 1$, μ large enough to avoid the singularities.

Tools: Estimates of the Green function/Theory of sum of operators/blowing up.

Reduction

In order to give existence and regularity results for such a problem, we first localize the problem.
Reduce Ω to the truncated sector

$$\Omega = \{(r\cos\theta, r\sin\theta) \mid 0 < r < 1,\ 0 < \theta < \psi\}, \quad \psi \in (0, 2\pi].$$

Use the theory of the sum of operators, hence we need first to study the Helmholtz equation

$$-\Delta u + zu = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $g \in L^p_\mu(\Omega)$ with $p \neq 2$ and $z \in \pi^+ \cup S_A$ where

$$\pi^+ = \{z \in \mathbb{C} \mid \Re(z) \geq 0\},$$

$$S_A = \{z \in \mathbb{C} \mid |z| \geq R \text{ and } |\arg z| \leq \theta_A\},$$

for $R > 0$ and $\theta_A \in]\frac{\pi}{2}, \pi[$ fixed.

Some definitions

For $p > 1, \mu \in \mathbb{R}$: weighted sp. with homogeneous norms:

$$L^p_\mu(\Omega) = \{f \in L^p_{loc}(\Omega) \mid r^\mu f \in L^p(\Omega)\}.$$

and

$$V^{k,p}_\mu(\Omega) = \{u \in L^p_{loc}(\Omega) \mid \|u\|_{V^{k,p}_\mu(\Omega)} < \infty\},$$

$$\|u\|^p_{V^{k,p}_\mu(\Omega)} := \sum_{|\gamma| \leq k} \int_\Omega |D^\gamma u(x)|^p \, r^{(\mu+|\gamma|-k)p}(x)\, dx.$$

In $H^1_0(\Omega)$ we will denote its semi-norm by

$$|u|^2_{H^1_0} = \int_\Omega |\nabla u|^2.$$

Embeddings and consequences

Le 1. *Let $p \geq 2$ and μ satisfies,*

$$\begin{aligned} \mu &< \tfrac{2p-2}{p}, && \text{if } p > 2, \\ \mu &\leq 1, && \text{if } p = 2. \end{aligned} \tag{2}$$

1. $L^p_\mu(\Omega) \hookrightarrow L^2_1(\Omega)$,
2. $L^2_{-1}(\Omega) \hookrightarrow (L^p_\mu(\Omega))' = L^q_{-\mu}(\Omega)$, $\frac{1}{q} + \frac{1}{p} = 1$,
3. $H^1_0(\Omega) \hookrightarrow L^q_{-\mu}(\Omega)$.

Proof

1. follows from Hölder's inequality and the fact that $r^{1-\mu} \in L^s(\Omega)$ if $1-\mu > -\frac{2}{s}$.

2. consequence of the first one by using duality.

3.

a. $p = 2$, it is well known (see Thm 14.44 in [CDN book]) that $H_0^1(\Omega) \hookrightarrow L^2_{-1}(\Omega)$. We then conclude observing that, for $\mu \leq 1$: $r^{2(-\mu+1)} \in L^\infty(\Omega)$.

b. $p > 2$, we use the embedding $H_0^1(\Omega) \hookrightarrow L^2_{-1}(\Omega)$ and the second assertion.

Coro 1. *Let* $p \geq 2$ *and* μ *satisfies* (2). *Then for all* $g \in L^p_\mu(\Omega)$ *and all* $z \in \pi^+ \cup S_A$, *the problem*

$$\forall w \in H_0^1(\Omega), \qquad \int_\Omega \nabla u \nabla \overline{w} + z \int_\Omega u \overline{w} = \int_\Omega g \overline{w}, \tag{3}$$

has a unique solution $u \in H_0^1(\Omega)$.

Rk. $u \in H_0^1(\Omega)$ is a weak solution of (1):

$$-\Delta u + zu = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Some Inequalities

Le 2. *Let* $p \geq 2$, μ *satisfies* (2), $z \in \pi^+ \cup S_A$, *and* $u \in H_0^1(\Omega)$ *be the solution of* (3). *Then*

$$|u|_{H_0^1(\Omega)} \lesssim \|g\|_{L^p_\mu(\Omega)}, \tag{4}$$

$$(1 + |z|^{1/2})|u|_{L^2(\Omega)} \lesssim \|g\|_{L^p_\mu(\Omega)}. \tag{5}$$

Proof

For $\Re z \geq 0$: Applying (3) with $w = u$ we have

$$|u|_{H_0^1}^2 + z \int_\Omega |u|^2 = \int_\Omega g\bar{u}. \tag{6}$$

By Lemma 1, taking the real part of (6), we obtain

$$|u|_{H_0^1}^2 + \Re z \int_\Omega |u|^2 \lesssim \|g\|_{L_\mu^p} |u|_{H_0^1}. \tag{7}$$

The result follows as $\Re z \geq 0$ and using Poincaré inequality.

Coro 2. *Let* $g \in L^2(\Omega)$, $z \in \mathbb{C}$ *with* $\Re z \geq 0$, *and* $u \in H_0^1(\Omega)$ *be the solution of* $($*3*$)$. *Then*

$$(1 + |z|)\|u\|_{L^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)}.$$

The domain

Def 1. *Let $p \geq 2$ and $\mu \in \mathbb{R}$. Then we define*

$$D(\Delta_{p,\mu}) = \{u \in H_0^1(\Omega) \mid \Delta u \in L_\mu^p(\Omega)\}.$$

Rk. If μ satisfies (2) and $2 - \frac{2}{p} - \mu \neq k\lambda, \forall k \in \mathbb{N}^*$. Then [Maz'ya-Plamenevskii, 78] $\Rightarrow$

$$D(\Delta_{p,\mu}) = V_\mu^{2,p}(\Omega) \cap H_0^1(\Omega)$$
$$+ \text{ Span } \{\eta(r) r^{\lambda'} \sin(\lambda'\theta) \mid 0 < \lambda' = k\lambda < 2 - \frac{2}{p} - \mu\},$$

η cut-off fct s. t. $\eta = 1$ near $r = 0$ and $\eta(1) = 0$.

An existence result

Le 3. *Let* $p \geq 2$*,* μ *satisfies* (2) *and* $\mu > -\lambda$*, with* $\lambda = \frac{\pi}{\psi}$*,* $z \in \pi^+ \cup S_A$*,* $u \in H_0^1(\Omega)$ *sol. of* (3) *with* $g \in L^p_\mu(\Omega)$*. Then* $u \in D(\Delta_{p,\mu})$*.*

Proof. As Le 1 $\Rightarrow H_0^1(\Omega) \hookrightarrow L^p_{\mu'}(\Omega)$ for all $\mu' > -\frac{2}{p}$, we distinguish different cases:

1. $\mu > -\frac{2}{p}$ and therefore $-\Delta u = g - zu \in L^p_\mu(\Omega)$.
2. $-2 - \frac{2}{p} < \mu < -\frac{2}{p}$. Take $\mu' = \mu + 2$. Since $\mu' > -\frac{2}{p}$, $u \in H_0^1(\Omega)$ is solution of

$$-\Delta u = g - zu \in L^p_{\mu'}(\Omega).$$

This implies that

$$u \in V^{2,p}_{\mu'}(\Omega),$$

because the set $\{\lambda' = k\frac{\pi}{\psi}, k \in \mathbb{Z} : 0 < \lambda' < 2 - \frac{2}{p} - \mu'\} = \emptyset$ (the assumption $\mu > -\lambda \Rightarrow \lambda > -\frac{2}{p} - \mu$). Accordingly

$$r^{\mu'-2}u \in L^p(\Omega) \Leftrightarrow u \in L^p_\mu(\Omega),$$

due to $\mu' - 2 = \mu$.

This guarantees $-\Delta u = g - zu \in L^p_\mu(\Omega)$.

The general case follows by induction.

An a priori estimate

Le 4. *Let* $\lambda = \frac{\pi}{\psi}$, $p \geq 2$, $\mu > -\lambda$ *satisfy* (2) *and*

$$\frac{4(p-1)\lambda^2}{p^2} + \frac{2\mu}{p} - \mu^2 > 0. \qquad (8)$$

Let $z \in \mathbb{C}$ *with* $\Re z \geq 0$, $u \in D(\Delta_{p,\mu})$ *sol. of* (3) *with* $g \in L^p_\mu(\Omega)$. *Then*

$$\Re z \, \|u\|_{L^p_\mu} \leq \|g\|_{L^p_\mu} \qquad \text{and} \qquad \Im z \, \|u\|_{L^p_\mu} \lesssim \|g\|_{L^p_\mu}.$$

Proof

Some integrations by parts $\Rightarrow v = r^\mu u$ satisfies

$$\frac{p}{2}\int_\Omega |\nabla v|^2 |v|^{p-2} + \frac{p-2}{2}\int_\Omega |v|^{p-4}\overline{v}^2(\nabla v)^2 - \mu^2 \int_\Omega r^{-2}|v|^p$$
$$+2\mu \int_\Omega r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\overline{v} + z\int_\Omega |v|^p = \int_\Omega r^\mu g\,|v|^{p-2}\overline{v},$$

$$p\,\Re\left(\int_\Omega r^{-1}\frac{\partial v}{\partial r}|v|^{p-2}\overline{v}\right) = \int_\Omega r^{-2}|v|^p.$$

Setting $w = |v|^{p/2}$, these two identities lead to

$$\frac{4(p-1)}{p^2}\int_D |\nabla w|^2 + (\frac{2\mu}{p} - \mu^2)\int_D r^{-2}w^2 \\ + \Re z \int_D w^2 \leq \Re(\int_D g\,|v|^{p-2}\overline{v}).$$

Poincaré $\leq$ in $\theta \Rightarrow \int_\Omega |\nabla w|^2 \geq \lambda^2 \int_\Omega \frac{1}{r^2}w^2, \forall w \in H_0^1(\Omega) \Rightarrow$

$$(\frac{4(p-1)\lambda^2}{p^2} + \frac{2\mu}{p} - \mu^2)\int_D r^{-2}w^2 + \Re z \int_D w^2 \leq \Re(\int_D g\,|v|^{p-2}\overline{v})$$

The main result

Thm 1. *Let* $p \geq 2$*, and let* $\mu > -\lambda$ *satisfies* (2), (8) *and, for all* $k \in \mathbb{Z}^*\ 2 - \frac{2}{p} - \mu \neq k\lambda$*. Then, for all* $z \in \pi^+ \cup S_A$, $u \in D(\Delta_{p,\mu})$ *sol. of* (3) *with* $g \in L^p_\mu(\Omega)$*, i.e. weak sol. of (1):*

$$-\Delta u + zu = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

admits the decomposition

$$u = u_R + \sum_{0<\lambda'=k\lambda<2-\frac{2}{p}-\mu} c_{\lambda'}(z)\, P_{\lambda'}(r\sqrt{z})e^{-r\sqrt{z}}\, r^{\lambda'} \sin(\lambda'\theta), \quad (9)$$

with $u_R \in V^{2,p}_\mu(\Omega)$, $c_{\lambda'}(z) \in \mathbb{C}$ *and* $P_{\lambda'}(s) = \sum_{i=0}^{k_{\lambda'}-1} \frac{s^i}{i!}$,

$k_{\lambda'} > 2 - 2/p - \mu - \lambda'$.

The main result: uniform estimates

$$|u_R|_{V_\mu^{2,p}(\Omega)} + |z|^{1/2}\,|u_R|_{V_\mu^{1,p}(\Omega)} + |z|\,|u_R|_{L_\mu^p(\Omega)} \lesssim \|g\|_{L_\mu^p(\Omega)};$$

$$\sum_{0<\lambda'<2-\frac{2}{p}-\mu} |c_{\lambda'}(z)|\,(1+|z|^{1-\frac{1}{p}-\frac{\mu+\lambda'}{2}}) \lesssim \|g\|_{L_\mu^p(\Omega)}.$$

Sketch of the proof

Lemma 4 $\Rightarrow$

$$\|g - zu\|_{L^p_\mu(\Omega)} \lesssim \|g\|_{L^p_\mu(\Omega)},$$

hence u can be seen as a solution of

$$-\Delta u = g - zu \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and by standard regularity results:

$$u = u_{1R} + \sum_{0<\lambda'=k\lambda<2-\frac{2}{p}-\mu} c_{\lambda'}(z)\, r^{\lambda'} \sin(\lambda'\theta),$$

with $u_{1R} \in V^{2,p}_\mu(\Omega)$, $c_{\lambda'}(z) \in \mathbb{C}$.

Factors $P_{\lambda'}(r\sqrt{z})e^{-r\sqrt{z}}$: to have uniform estimates in z.

One application

Thm 2. *Let* $p \geq 2$*, and let* $\mu = 1 - \lambda$ *satisfies* (8) *and, for all* $k \in \mathbb{Z}^*$ $2 - \frac{2}{p} - \mu \neq k\lambda$*. Then* $\forall f \in L^p((0,\infty); L^p_\mu(\Omega))$*,*$\exists$ *a sol. of*

$$\partial_t u - \Delta u = f \text{ in } \Omega \times (0,\infty), \quad u = 0 \text{ on } \partial\Omega \cup \{t = 0\},$$

that admits the decomposition

$$u = u_R + \sum_{0<\lambda'=k\lambda<2-\frac{2}{p}-\mu} (E(r,\cdot) \star_t q_{\lambda'})\, r^{\lambda'} \sin(\lambda'\theta), \qquad (10)$$

with $u_R \in L^p((0,\infty); V^{2,p}_\mu(\Omega)) \cap W^{1,p}((0,\infty); L^p_\mu(\Omega))$*,*
$q_{\lambda'} \in W^{1-\frac{1}{p}-\frac{\mu+\lambda'}{2},p}(0,\infty)$ *and* $E(r,t) = r t_+^{-\frac{3}{2}} e^{-\frac{r^2}{4t}}$*.*