A logarithmic singularity for the end of bonded joints

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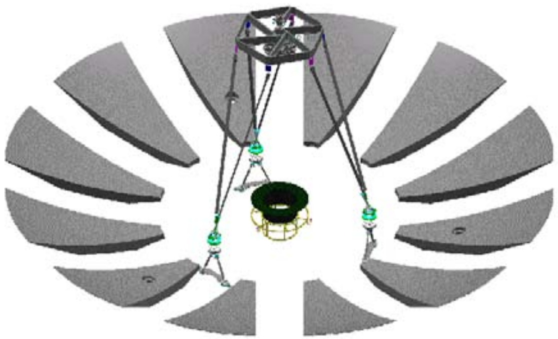
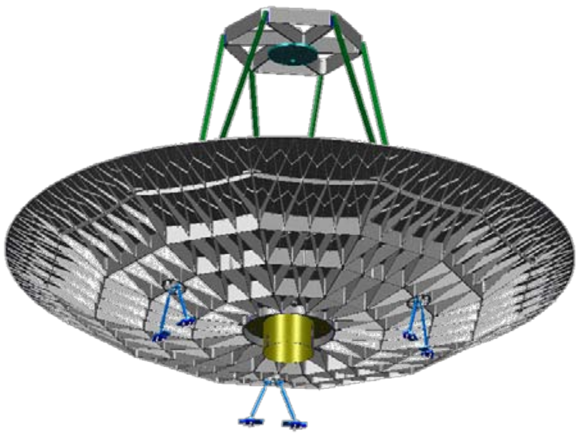
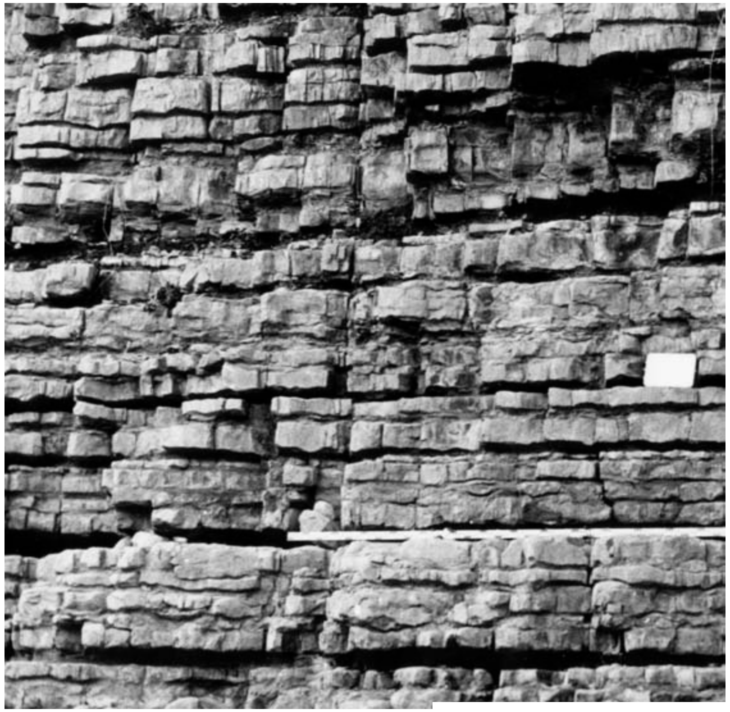
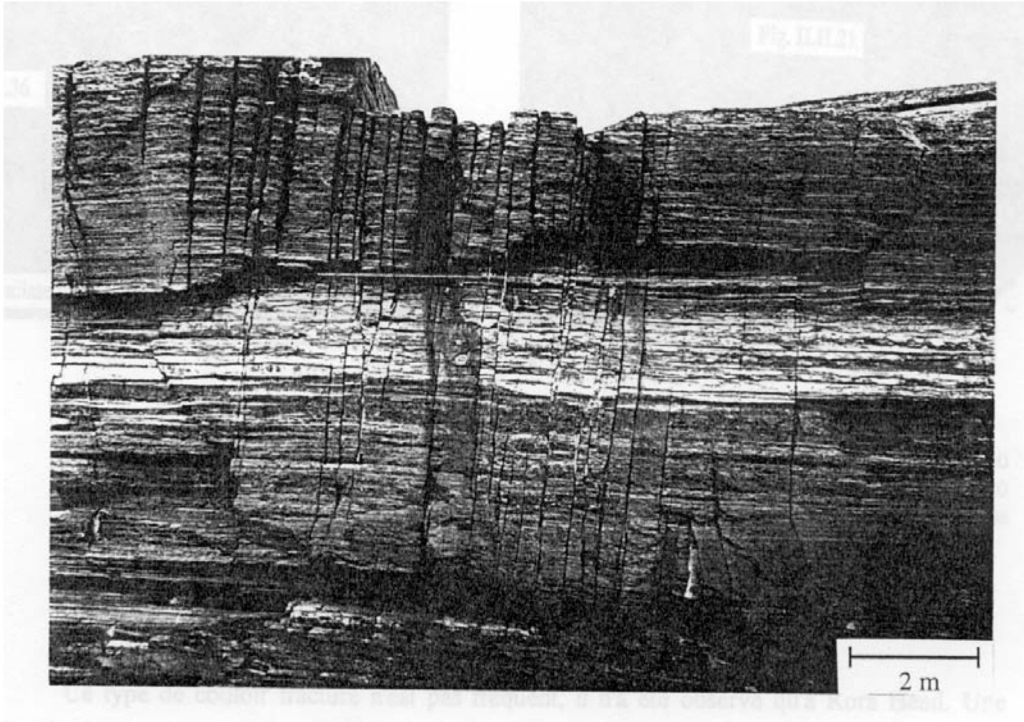
HERSCHEL MIRROR

Silicon carbide components
Joining by brazing of 12 segments

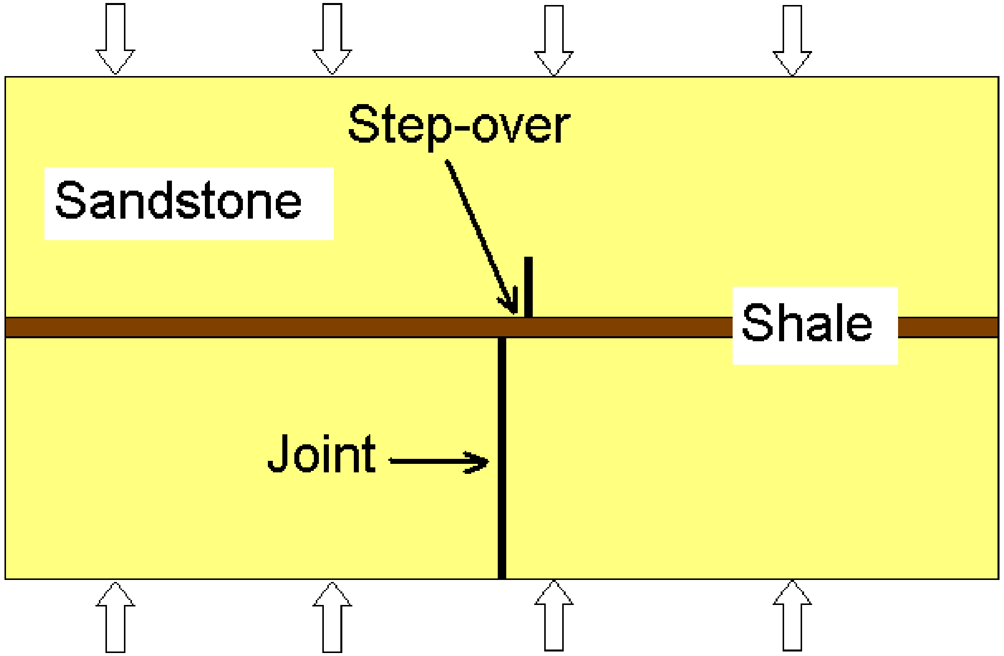






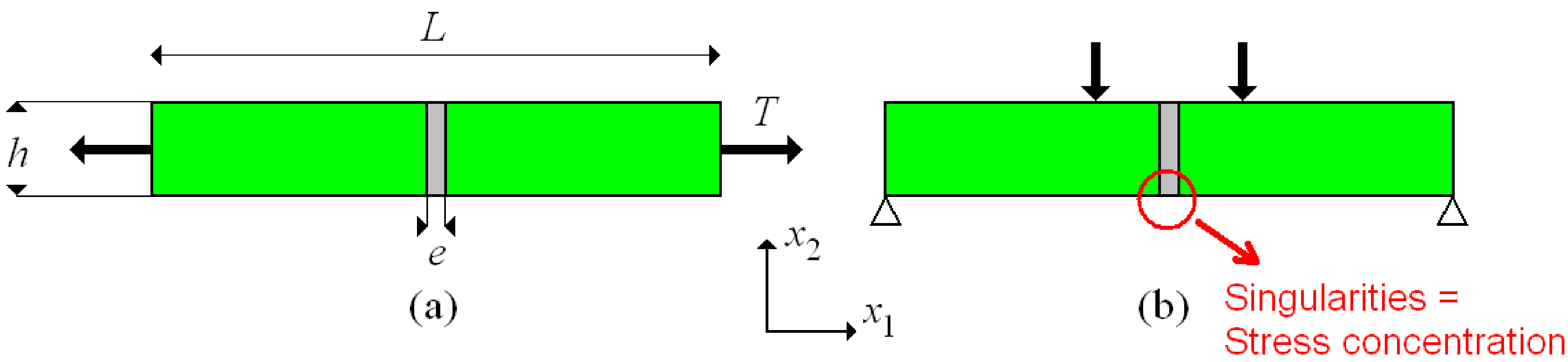


Picard, Putot, Leguillon (2005)
Quesada, Picard, Putot, Leguillon (2009)



Step over mechanism for bedded sediments in depth

The theoretical model and the matched asymptotic procedure



The bonded specimen and two kinds of loading, tension (a) and 4-point bending (b).

Lamé's coefficient λ^M and μ^M (the SiC substrates)
Lamé's coefficient λ^L and μ^L (the BraSiC® joint)

Assumption: $e << h$

The equations

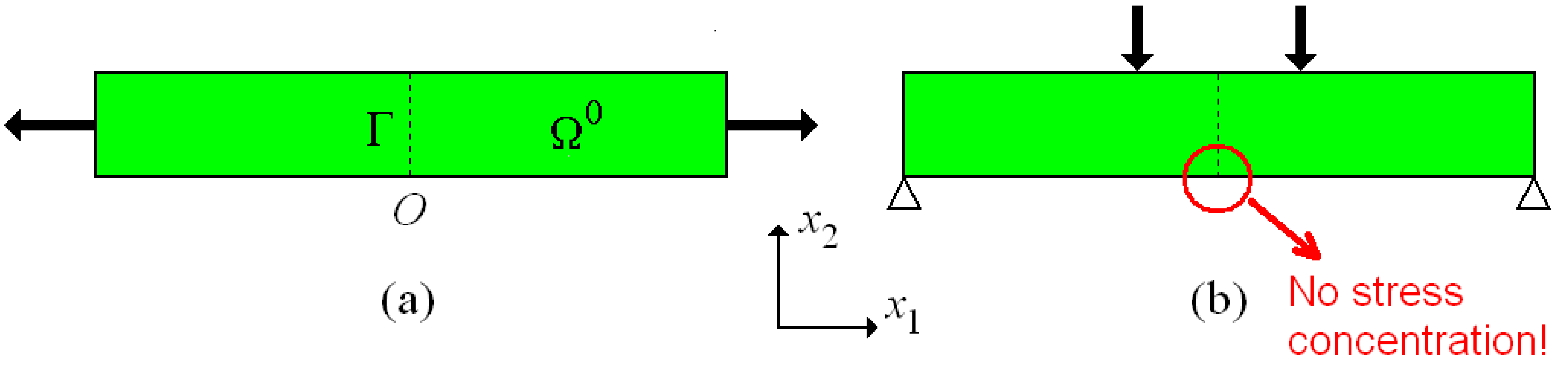
$$
\begin{cases}
-\sigma^e_{ij,j} = 0 & \text{in } \Omega^e \text{ (balance)} \\
\sigma^e_{11} = (\lambda + 2\mu)U^e_{1,1} + \lambda U^e_{2,2} & \text{in } \Omega^e \text{ (constitutive law)} \\
\sigma^e_{12} = \mu\left(U^e_{1,2} + U^e_{2,1}\right) & \text{in } \Omega^e \text{ (constitutive law)} \\
\sigma^e_{22} = (\lambda + 2\mu)U^e_{2,2} + \lambda U^e_{1,1} & \text{in } \Omega^e \text{ (constitutive law)} \\
\sigma^e_{11} = \pm T & \text{on the right (resp. left) face of the specimen} \\
\sigma^e_{12} = 0 & \text{on the right (resp. left) face of the specimen} \\
\sigma^e_{ij} n_j = 0 & \text{elsewhere on the boundary}
\end{cases}
$$

λ and μ stand for λ^L and μ^L in the layer and for λ^M and μ^M in the matrix.

Asymptotic expansions Nguetseng and Sanchez-Palencia (1985), Leguillon (1995), Leguillon and Abdelmoula (2000) and Haboussi et al. (2001)

$$\underline{U}^e(x_1,x_2)=\underline{U}^0(x_1,x_2)+e\,\underline{U}^1(x_1,x_2)+e^2\,\underline{U}^2(x_1,x_2).$$

The different terms are solution to problems settled on the simplified domain $\Omega^0=\lim_{e\to 0}\Omega^e$, the bonding layer is not visible



The limit domain Ω^0.

The simplified equations at the leading order

$\underline{U}^0(x_1, x_2)$ is continuous through the line Γ and fulfils

$$\begin{cases} -\sigma^0_{ij,j} = 0 & \text{in } \Omega^0 \text{ (balance)} \\ \sigma^0_{11} = (\lambda^M + 2\mu^M)U^0_{1,1} + \lambda^M U^0_{2,2} & \text{in } \Omega^0 \text{ (constitutive law)} \\ \sigma^0_{12} = \mu^M \left(U^0_{1,2} + U^0_{2,1}\right) & \text{in } \Omega^0 \text{ (constitutive law)} \\ \sigma^0_{22} = (\lambda^M + 2\mu^M)U^0_{2,2} + \lambda^M U^0_{1,1} & \text{in } \Omega^0 \text{ (constitutive law)} \\ \sigma^0_{11} = \pm T & \text{on the right (resp. left) face of the specimen} \\ \sigma^0_{12} = 0 & \text{on the right (resp. left) face of the specimen} \\ \sigma^0_{ij} n_j = 0 & \text{elsewhere on the boundary} \end{cases}$$

Then $\underline{U}^0(x_1, x_2) = T\ \underline{t}^1(x_1, x_2) + \ldots$

With (uniform tension)

$$\begin{cases} t_1^1(x_1, x_2) = \dfrac{\lambda^M + 2\mu^M}{4\mu^M(\lambda^M + \mu^M)}\, x_1 = \dfrac{\lambda^M + 2\mu^M}{4\mu^M(\lambda^M + \mu^M)}\, r\cos\theta \\ t_2^1(x_1, x_2) = -\dfrac{\lambda^M}{4\mu^M(\lambda^M + \mu^M)}\, x_2 = -\dfrac{\lambda^M}{4\mu^M(\lambda^M + \mu^M)}\, r\sin\theta \end{cases}$$

Note that power 1 has a multiplicity 2, the rigid rotation is the second mode.

The next term of the expansion

It results of a matched asymptotic procedure.

Change of variables $x_1 \to y_1 = x_1 / e \;\; x_2 \to x_2$

Stretched domain Ω^{in} unbounded in the y_1 direction as $e \to 0$

$$\underline{U}^e(x_1, x_2) = \underline{U}^e(ey_1, x_2) = \underline{V}^0(y_1, x_2) + e\,\underline{V}^1(y_1, x_2) + .$$

Derivation rules $*_{,1} = 1/e\,\partial * / \partial y_1 \;\; *_{,2} = \partial * / \partial x_2$ leads to a system of differential equations in the variable y_1 with matching conditions for $y_1 \to \pm\infty$ in the core (i.e. omitting the stress free boundary condition at $x_2 = 0$)

$$
\begin{cases}
V_1^0(y_1,x_2) &= U_1^0(0,x_2)\\
V_2^0(y_1,x_2) &= U_2^0(0,x_2)\\
V_1^{1\pm}(y_1,x_2) &= y_1\dfrac{\partial U_1^0}{\partial x_1}(0,x_2)+U_1^{1\pm}(0,x_2)\\
V_2^{1\pm}(y_1,x_2) &= y_1\dfrac{\partial U_2^0}{\partial x_1}(0,x_2)+U_2^{1\pm}(0,x_2)\\
V_1^{1\pm}(y_1,x_2) &= y_1\dfrac{\partial U_1^0}{\partial x_1}(0,x_2)+U_1^{1\pm}(0,x_2)+\left(2y_1\mp 1\right)\left[\dfrac{\lambda^M-\lambda^L}{2\left(\lambda^L+2\mu^L\right)}\dfrac{\partial U_2^0}{\partial x_2}(0,x_2)\right.\\
&\quad\left.+\dfrac{\left(\lambda^M+2\mu^M\right)-\left(\lambda^L+2\mu^L\right)}{2\left(\lambda^L+2\mu^L\right)}\dfrac{\partial U_1^0}{\partial x_1}(0,x_2)\right]\\
V_2^{1\pm}(y_1,x_2) &= y_1\dfrac{\partial U_2^0}{\partial x_1}(0,x_2)+U_2^{1\pm}(0,x_2)+\left(2y_1\mp 1\right)\dfrac{\mu^M-\mu^L}{2\mu^L}\left[\dfrac{\partial U_1^0}{\partial x_2}(0,x_2)+\dfrac{\partial U_2^0}{\partial x_1}(0,x_2)\right]
\end{cases}
$$

Jump at order 1

Continuity conditions on the previous term imply discontinuity conditions for $\underline{U}^1$ through the line Γ. They read as

$$\begin{cases} [\![U_1^1]\!] & = \sigma_{11}^0 \left(\dfrac{1}{\lambda^L + 2\mu^L} - \dfrac{1}{\lambda^M + 2\mu^M} \right) + U_{2,2}^0 \left(\dfrac{\lambda^M}{\lambda^M + 2\mu^M} - \dfrac{\lambda^L}{\lambda^L + 2\mu^L} \right) \\ [\![U_2^1]\!] & = \sigma_{12}^0 \left(\dfrac{1}{\mu^L} - \dfrac{1}{\mu^M} \right) \\ [\![\sigma_{11}^1]\!] & = 0 \\ [\![\sigma_{12}^1]\!] & = -\sigma_{12,1}^0 - \dfrac{\lambda^L}{\lambda^L + 2\mu^L} \sigma_{11,2}^0 - \dfrac{4\mu^L \left(\lambda^L + \mu^L \right)}{\lambda^L + 2\mu^L} U_{2,22}^0 \end{cases}$$

This is not surprising since a part of the stiff substrates is replaced by a more compliant material: the adhesive.

The stress free boundary condition out of the layer

$$
\begin{cases}
\sigma_{12}(y_1,0) & = \mu^M \left(\dfrac{\partial V_1^0}{\partial x_2}(y_1,0) + \dfrac{\partial V_2^{1\pm}}{\partial y_1}(y_1,0) \right) \\
 & = \mu^M \left(\dfrac{\partial U_1^0}{\partial x_2}(0,0) + \dfrac{\partial U_2^0}{\partial x_1}(0,0) \right) = \sigma_{12}^0(0,0) = 0 \\
\sigma_{22}(y_1,0) & = \left(\lambda^M + 2\mu^M\right) \dfrac{\partial V_2^0}{\partial x_2}(y_1,0) + \lambda^M \dfrac{\partial V_1^{1\pm}}{\partial y_1}(y_1,0) \\
 & = \left(\lambda^M + 2\mu^M\right) \dfrac{\partial U_2^0}{\partial x_2}(0,0) + \lambda^M \dfrac{\partial U_1^0}{\partial x_1}(0,0) = \sigma_{22}^0(0,0) = 0
\end{cases}
$$

The stress free boundary condition within the layer

$$
\begin{aligned}
\sigma_{12}(y_1,0) &= \mu^L\left(\frac{\partial V_1^0}{\partial x_2}(y_1,0)+\frac{\partial V_2^{1\pm}}{\partial y_1}(y_1,0)\right)\\
&= \mu^M\left(\frac{\partial U_1^0}{\partial x_2}(0,0)+\frac{\partial U_2^0}{\partial x_1}(0,0)\right)=\sigma_{12}^0(0,0)=0
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{22}(y_1,0) &= \left(\lambda^L+2\mu^L\right)\frac{\partial V_2^0}{\partial x_2}(y_1,0)+\lambda^L\frac{\partial V_1^{1\pm}}{\partial y_1}(y_1,0)\\
&= \left(\lambda^L+2\mu^L\right)\frac{\partial U_2^0}{\partial x_2}(0,0)+\lambda^L\frac{\lambda^M-\lambda^L}{\lambda^L+2\mu^L}\frac{\partial U_2^0}{\partial x_2}(0,0)+\lambda^L\frac{\lambda^M+2\mu^M}{\lambda^L+2\mu^L}\frac{\partial U_1^0}{\partial x_1}(0,0)
\end{aligned}
$$

Not surprisingly, this term vanishes only if (i.e. no necking)

$$
\frac{\lambda^M}{4\mu^M\left(\lambda^M+\mu^M\right)}=\frac{\lambda^L}{4\mu^L\left(\lambda^L+\mu^L\right)}
$$

The modified inner expansion

Let us replace $T\,\underline{t}^1(x_1, x_2)$ for $\underline{U}^0(x_1, x_2)$ in the jump conditions and in the set of equations

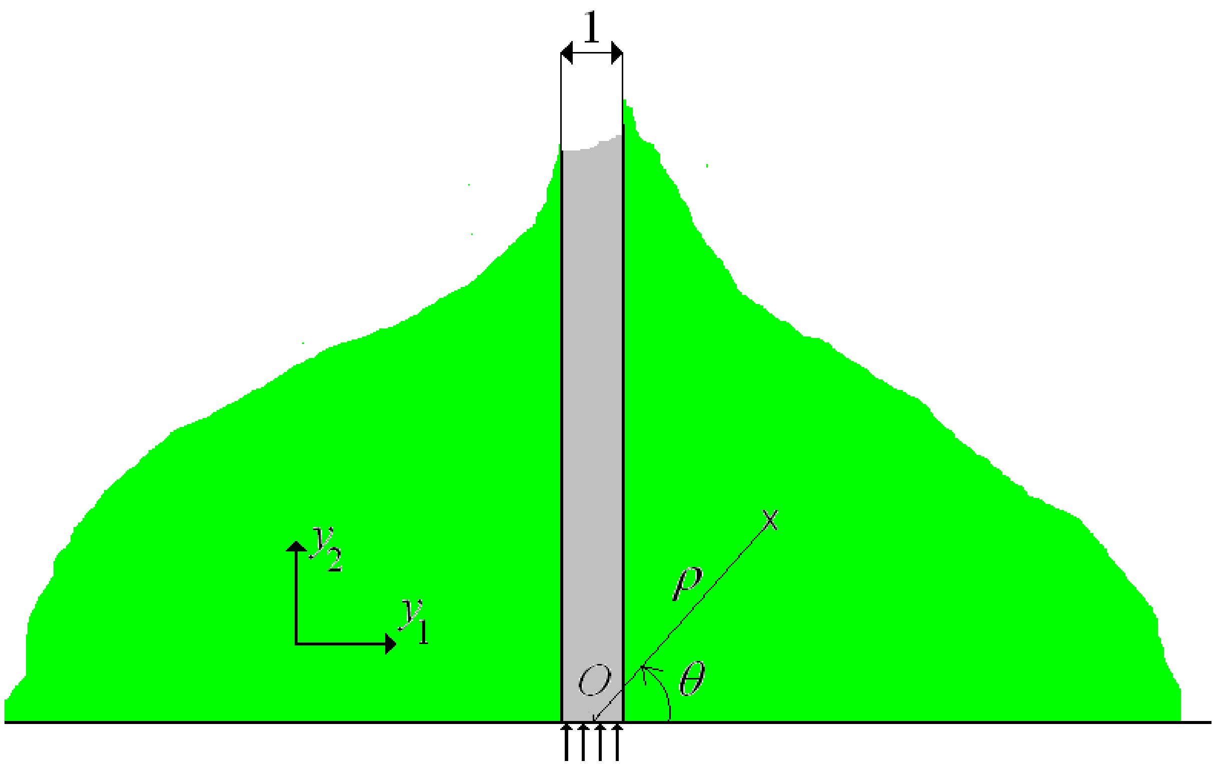
$$\begin{cases} [\![U_1^1]\!] = T\left[\left(\dfrac{1}{\lambda^L + 2\mu^L} - \dfrac{1}{\lambda^M + 2\mu^M}\right) - \dfrac{\lambda^M}{4\mu^M\left(\lambda^M + \mu^M\right)}\left(\dfrac{\lambda^M}{\lambda^M + 2\mu^M} - \dfrac{\lambda^L}{\lambda^L + 2\mu^L}\right)\right] = TC_1 \\ [\![U_2^1]\!] = 0\ ;\ [\![\sigma_{11}^1]\!] = 0\ ;\ [\![\sigma_{12}^1]\!] = 0 \end{cases}$$

We substitute also y_2 for ex_2 to have a domain spanned by y_1 and y_2 since we are only interested in the vicinity of the bonding end.

$$
\begin{cases}
V_1^0(y_1, ey_2) & = & 0 \\
V_2^0(y_1, ey_2) & = & T\dfrac{\lambda^M}{4\mu^M\left(\lambda^M + \mu^M\right)} ey_2 \\
V_1^{1\pm}(y_1, ey_2) & = & T\dfrac{\lambda^M + 2\mu^M}{4\mu^M\left(\lambda^M + \mu^M\right)} y_1 \pm T\dfrac{C_1}{2} \\
V_2^{1\pm}(y_1, ey_2) & = & TC_2 + \hat{U}_2^1(0, ey_2) \\
V_1^{1\pm}(y_1, ey_2) & = & T\dfrac{\lambda^M + 2\mu^M}{4\mu^M\left(\lambda^M + \mu^M\right)} y_1 \pm T\dfrac{C_1}{2} + T\left(2y_1 \mp 1\right)\left[-\dfrac{\lambda^M - \lambda^L}{2\left(\lambda^L + 2\mu^L\right)} \dfrac{\lambda^M}{4\mu^M\left(\lambda^M + \mu^M\right)}\right. \\
 & & \left. + \dfrac{\left(\lambda^M + 2\mu^M\right) - \left(\lambda^L + 2\mu^L\right)}{2\left(\lambda^L + 2\mu^L\right)} \dfrac{\lambda^M + 2\mu^M}{4\mu^M\left(\lambda^M + \mu^M\right)}\right] \\
V_2^{1\pm}(y_1, ey_2) & = & TC_2 + \hat{U}_2^1(0, ey_2)
\end{cases}
$$

with $y_2 = x_2 / e$

then $$\underline{U}^e(x_1,x_2)=\underline{U}^e(ey_1,ey_2)=\underline{W}^0(y_1,y_2)+e\,\underline{W}^1(y_1,y_2)+\ldots$$



The unbounded half space and the loading for $\hat{\underline{W}}^1(y_1,y_2)$.

Additional term $\underline{\hat{W}}^1(y_1, y_2)$ to compensate the boundary condition imbalance

$$\underline{U}^e(x_1, x_2) = \underline{U}^e(ey_1, ey_2) = \underline{W}^0(y_1, y_2) + e\left[\underline{W}^1(y_1, y_2) + \underline{\hat{W}}^1(y_1, y_2)\right] + .$$

New boundary conditions

$$\begin{cases} \hat{\sigma}_{12}(y_1,0) & = \mu^M \left(\dfrac{\partial \hat{W}_1^1}{\partial y_2}(y_1,0) + \dfrac{\partial \hat{W}_2^1}{\partial y_1}(y_1,0) \right) = 0 & \text{for } y_1 \le -1/2 \text{ and } y_1 \ge 1/2 \\ \hat{\sigma}_{22}(y_1,0) & = \left(\lambda^M + 2\mu^M\right)\dfrac{\partial \hat{W}_2^1}{\partial y_2}(y_1,0) + \lambda^M \dfrac{\partial \hat{W}_1^1}{\partial y_1}(y_1,0) = 0 & \text{for } y_1 \le -1/2 \text{ and } y_1 \ge 1/2 \\ \hat{\sigma}_{12}(y_1,0) & = \mu^L \left(\dfrac{\partial \hat{W}_1^1}{\partial y_2}(y_1,0) + \dfrac{\partial \hat{W}_2^1}{\partial y_1}(y_1,0) \right) = 0 & \text{for } -1/2 \le y_1 \le 1/2 \\ \hat{\sigma}_{22}(y_1,0) & = \left(\lambda^L + 2\mu^L\right)\dfrac{\partial \hat{W}_2^1}{\partial y_2}(y_1,0) + \lambda^L \dfrac{\partial \hat{W}_1^1}{\partial y_1}(y_1,0) = -P & \text{for } -1/2 \le y_1 \le 1/2 \end{cases}$$

Such a problem is unbalanced and thus ill-posed from a static viewpoint. It behaves at infinity like the point force solution $\underline{F}(y_1, y_2)$ (Timoshenko and Goodier 1951) involving a logarithmic term

$$\begin{cases} F_1(y_1, y_2) & = -\dfrac{P}{\pi}\left(\dfrac{1}{2\mu^M}\sin\theta\cos\theta + \dfrac{1}{2(\lambda^M + \mu^M)}\theta - \dfrac{\pi}{4(\lambda^M + \mu^M)} \right) \\ F_2(y_1, y_2) & = \dfrac{P}{\pi}\left(\dfrac{\lambda^M + 2\mu^M}{2\mu^M(\lambda^M + \mu^M)}\ln\rho + \dfrac{\lambda^M}{2\mu^M(\lambda^M + \mu^M)}\cos^2\theta - \dfrac{1}{2(\lambda^M + \mu^M)}\sin^2\theta \right) \end{cases}$$

$$\underline{\hat{W}}^1(y_1, y_2) = \underline{F}(y_1, y_2) + \underline{\hat{\hat{W}}}^1(y_1, y_2)$$

$\underline{\hat{\hat{W}}}^1(y_1, y_2)$ is now solution to a well-posed problem

(this is not trivial and more precisely a cut-off function must be used in the above relationship since $\underline{F}$ is singular at both ends in r i.e. at the origin and at infinity)

The modified outer expansion

$$\underline{U}^{e}(x_1,x_2)=\underline{U}^{0}(x_1,x_2)+e\ \underline{U}^{1}(x_1,x_2)+A^0Te^2\underline{U}^{2}(x_1,x_2)+.$$

with

$$\underline{U}^{1}(x_1,x_2)=T\underline{C}+\underline{F}(x_1,x_2)+\hat{\hat{\underline{U}}}^{1}(x_1,x_2)$$

i.e. jump + point force + complementary term

$$\underline{U}^{2}(x_1,x_2)=\underline{t}^{-1}(r,\theta)+\hat{\underline{U}}^{2}(x_1,x_2)$$

With

$$\begin{cases} t_{\rho}^{-1}(\rho,\theta)=\dfrac{1}{\rho}\dfrac{\cos 2\theta}{\mu^{M}} \\ t_{\theta}^{-1}(\rho,\theta)=-\dfrac{1}{\rho}\dfrac{\sin 2\theta}{\lambda^{M}+2\mu^{M}} \end{cases}$$

i.e. a kind of pinching (the dual mode to the uniform tension, note that for the same power -1, the torque is the dual mode to the rigid rotation)

Outline

Near the end of the bonding zone

Far field

Leading term → smooth solution, no stress concentration

Order 1 correction → jump in the core + point force acting on the boundary

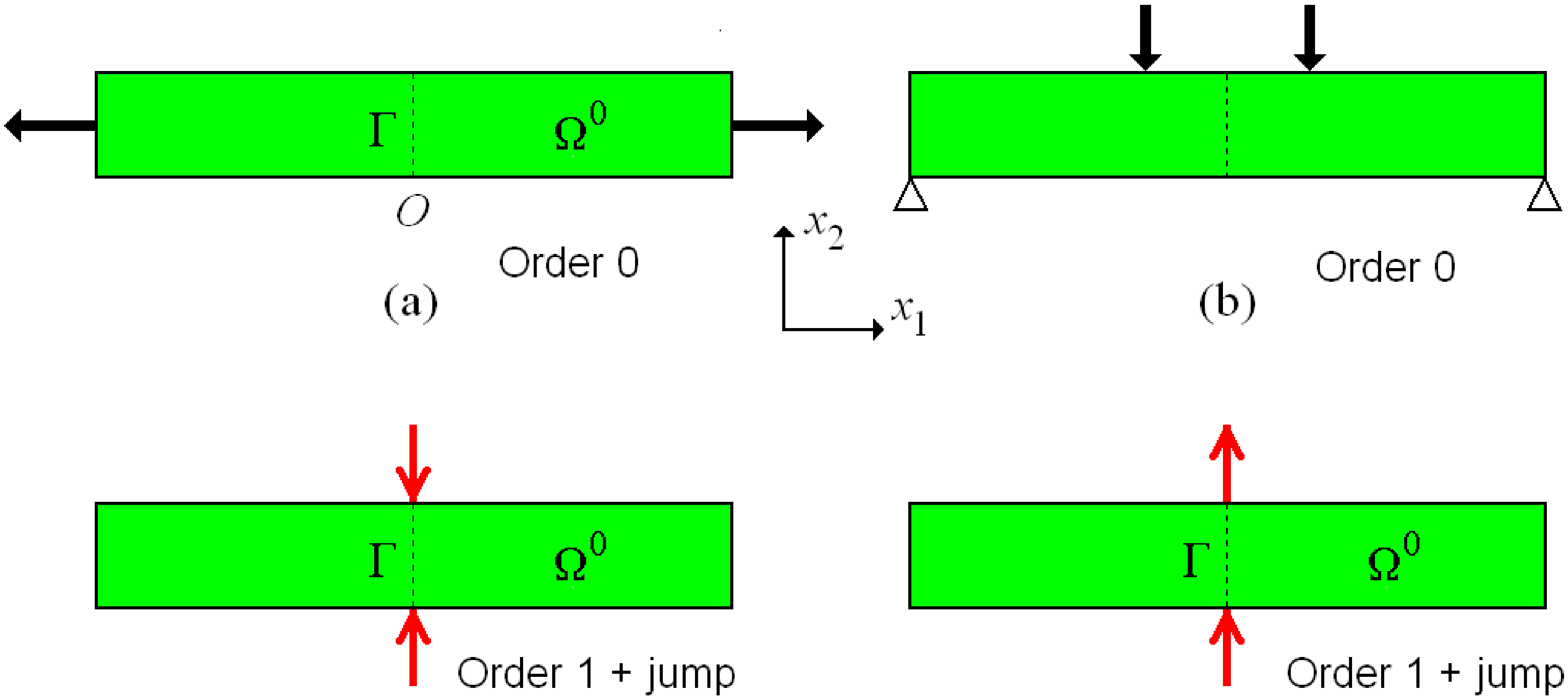
Order 2 correction → pinching acting on the boundary

Near field

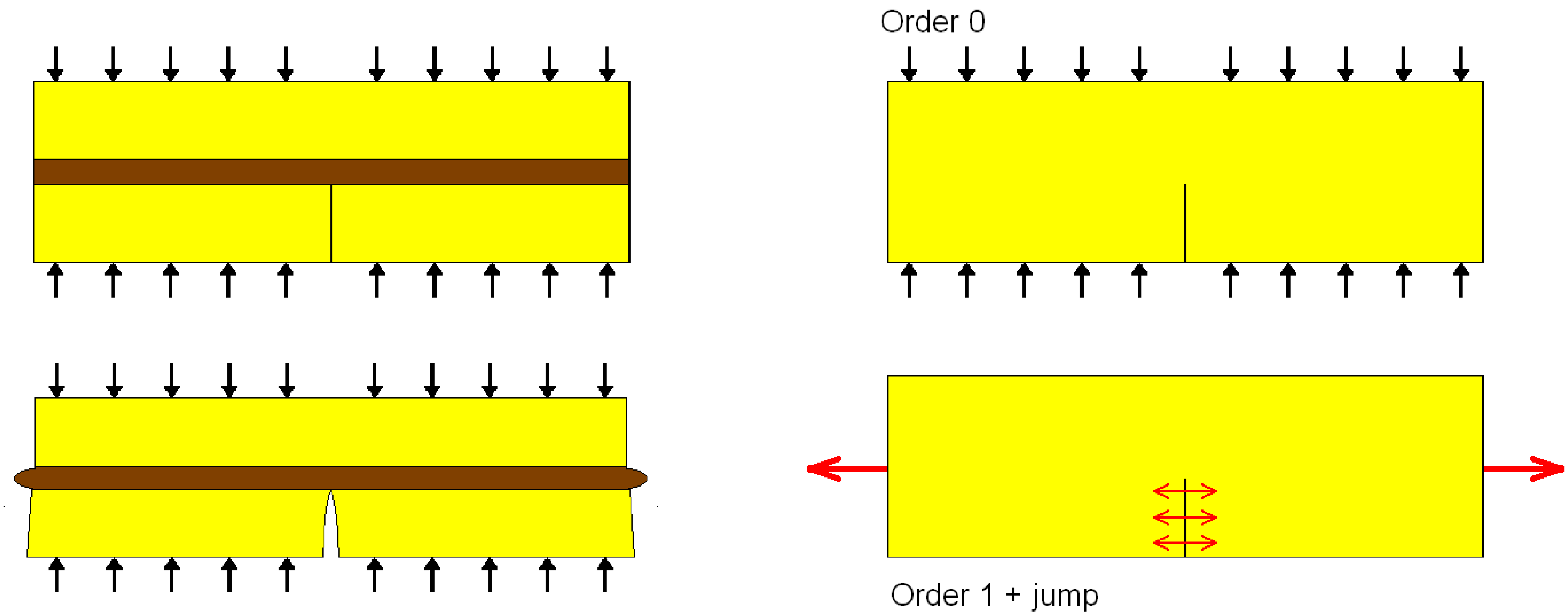
Leading term → smooth solution, no stress concentration

Order 1 correction → bimaterial interface singularities

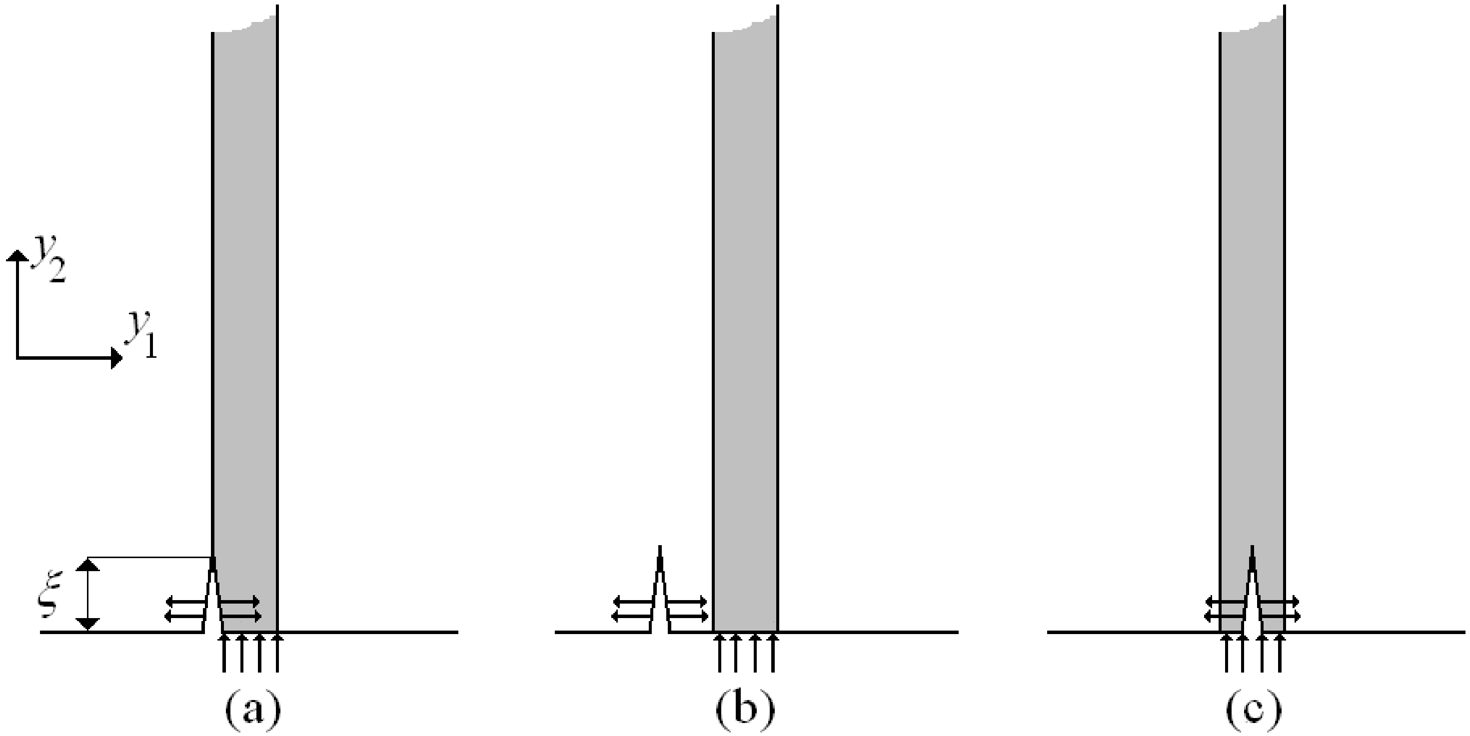
Brazing of ceramics



Step over mechanism



Nucleation of a crack at the end of the brazing zone



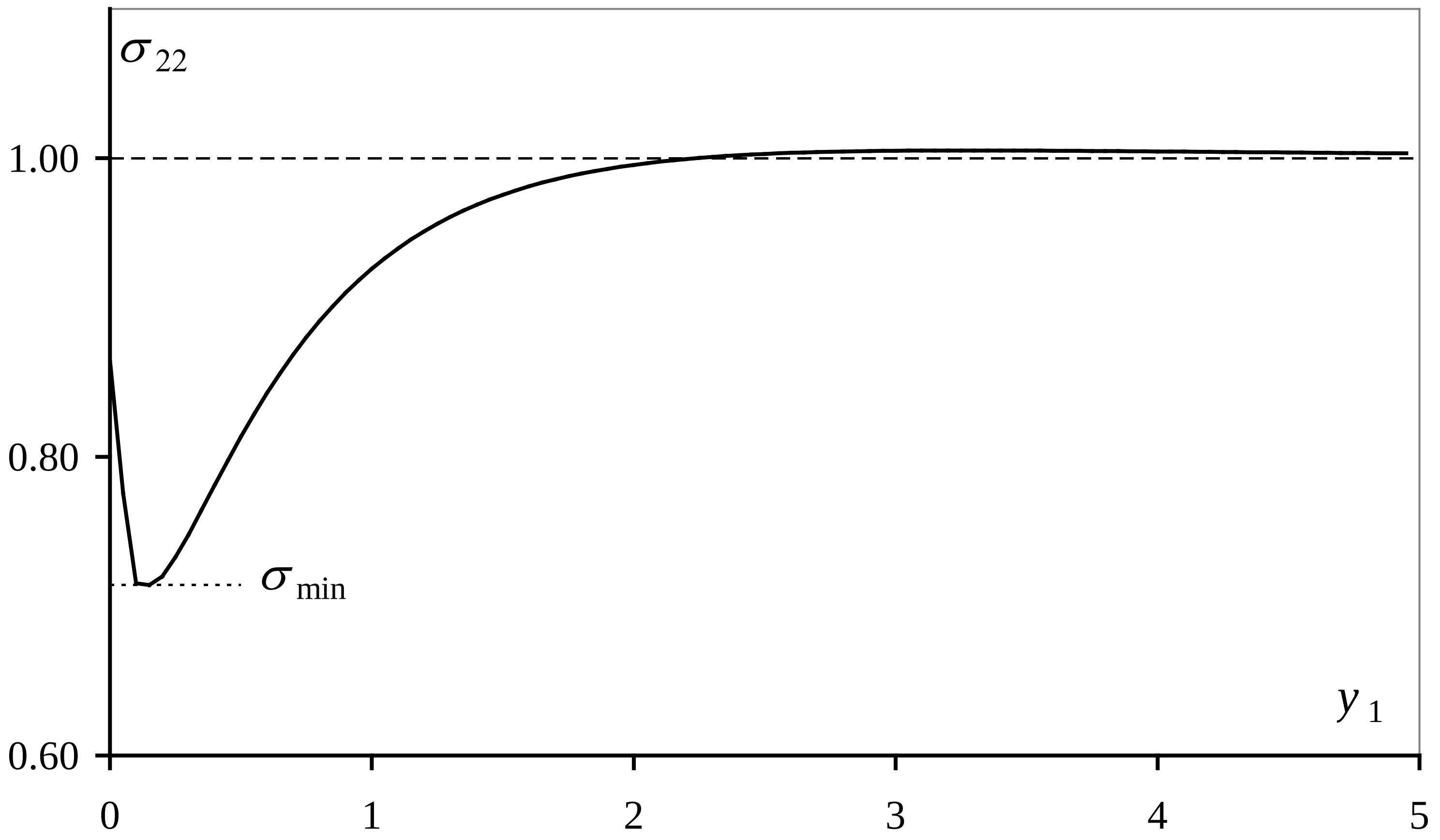
Adhesive (a) and cohesive (b and c) failures

The change in the expansions occurs through a single parameter A^{ζ}

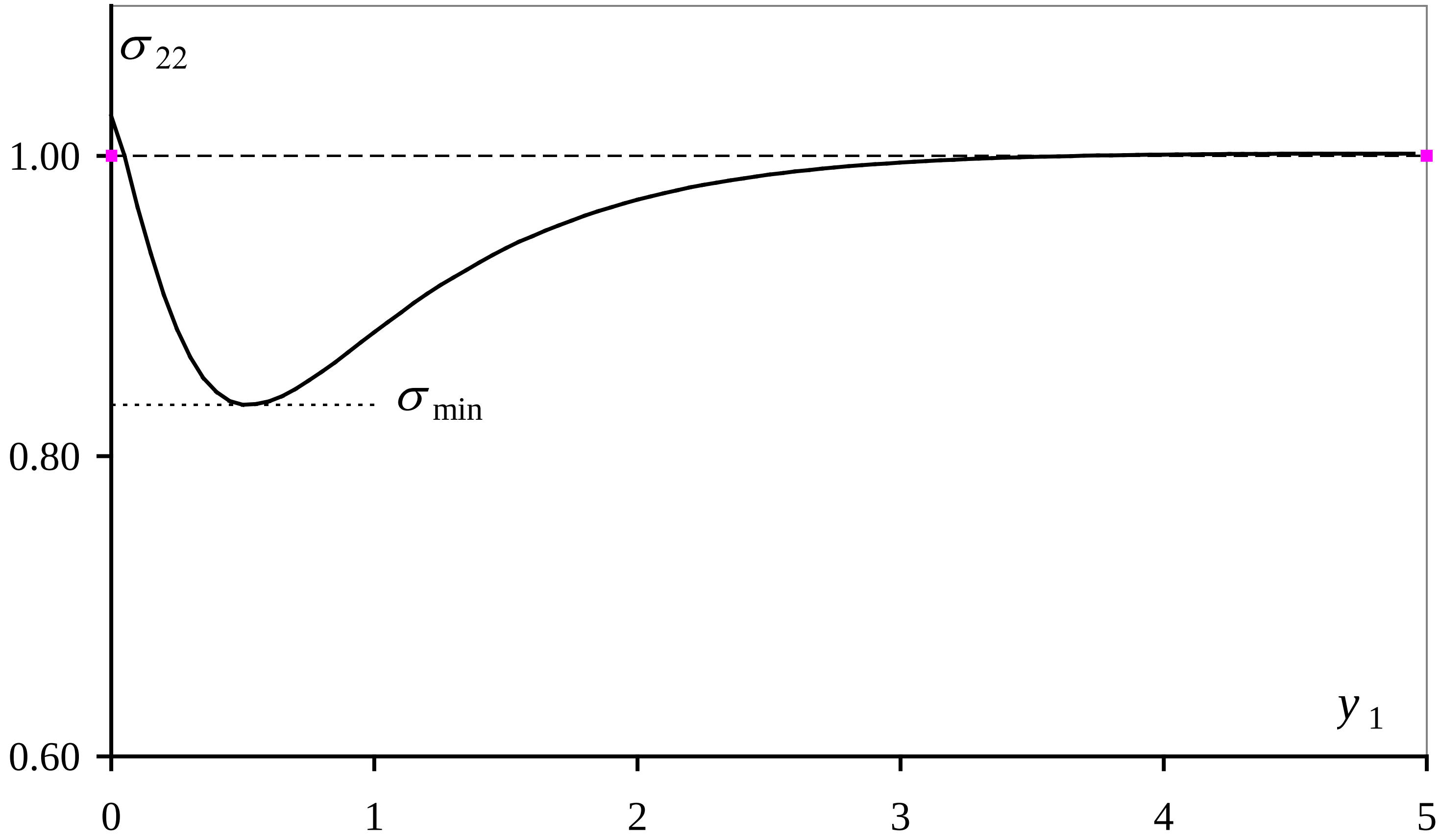
$$\underline{U}^{e}(x_1, x_2) = \underline{U}^{0}(x_1, x_2) + e\,\underline{U}^{1}(x_1, x_2) + A^{\xi} T e^{2} \underline{U}^{2}(x_1, x_2) + \ldots$$

The tensile stress prior to failure (first condition for fracture)

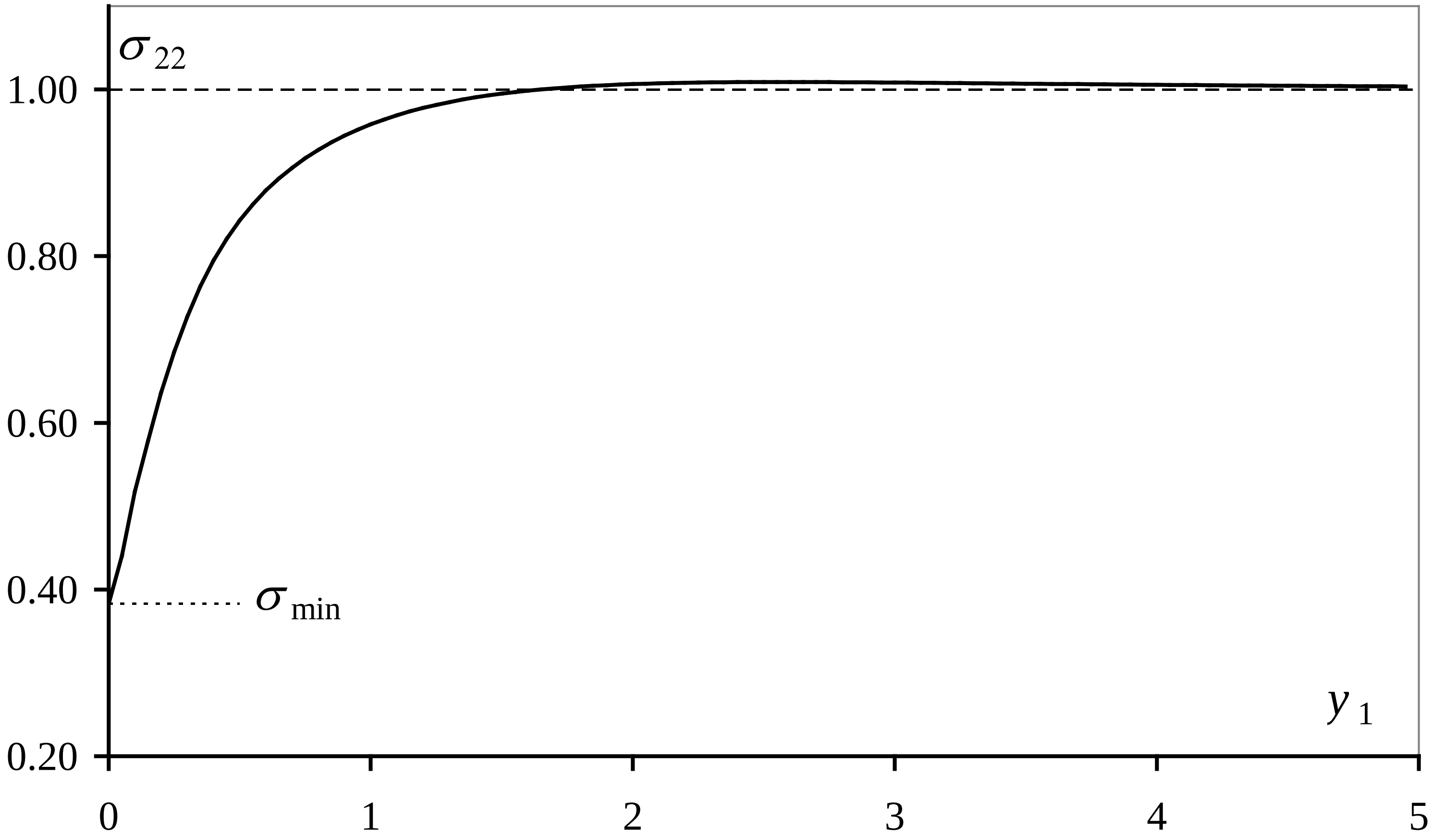
Normalized tensile stress



The normalized tensile stress along the interface.



The normalized tensile stress along the presupposed crack path in the matrix.



The normalized tensile stress along the middle line in the layer.

Stress condition $\sigma_{11} > \sigma_c$ (tensile strength)

The energy release rate (second condition for fracture)

Change in potential energy prior to and following the crack nucleation

$$\delta W^{p} = e^{2}\left(A^{\xi} - A^{0}\right) T\Psi\left(\underline{U}^{2}, \underline{U}^{0}\right) + \ldots$$

A^{ζ} is a priori the only coefficient to depend on the crack location

Energy release rate

$$G = -\frac{\delta W^{p}}{\ell} = e\frac{A^{\xi} - A^{0}}{\xi} T\Psi\left(\underline{U}^{0}, \underline{U}^{2}\right) + \ldots$$

$$G = -\frac{\delta W^{p}}{\ell} = e\frac{A^{\xi} - A^{0}}{\xi} T^{2}\Psi\left(\underline{t}^{1}, \underline{t}^{-1}\right) + \ldots = e\ g(\xi)\ T^{2} + \ldots \text{ with } g(\xi) = \frac{A^{\xi} - A^{0}}{\xi}\Psi\left(\underline{t}^{1}, \underline{t}^{-1}\right)$$

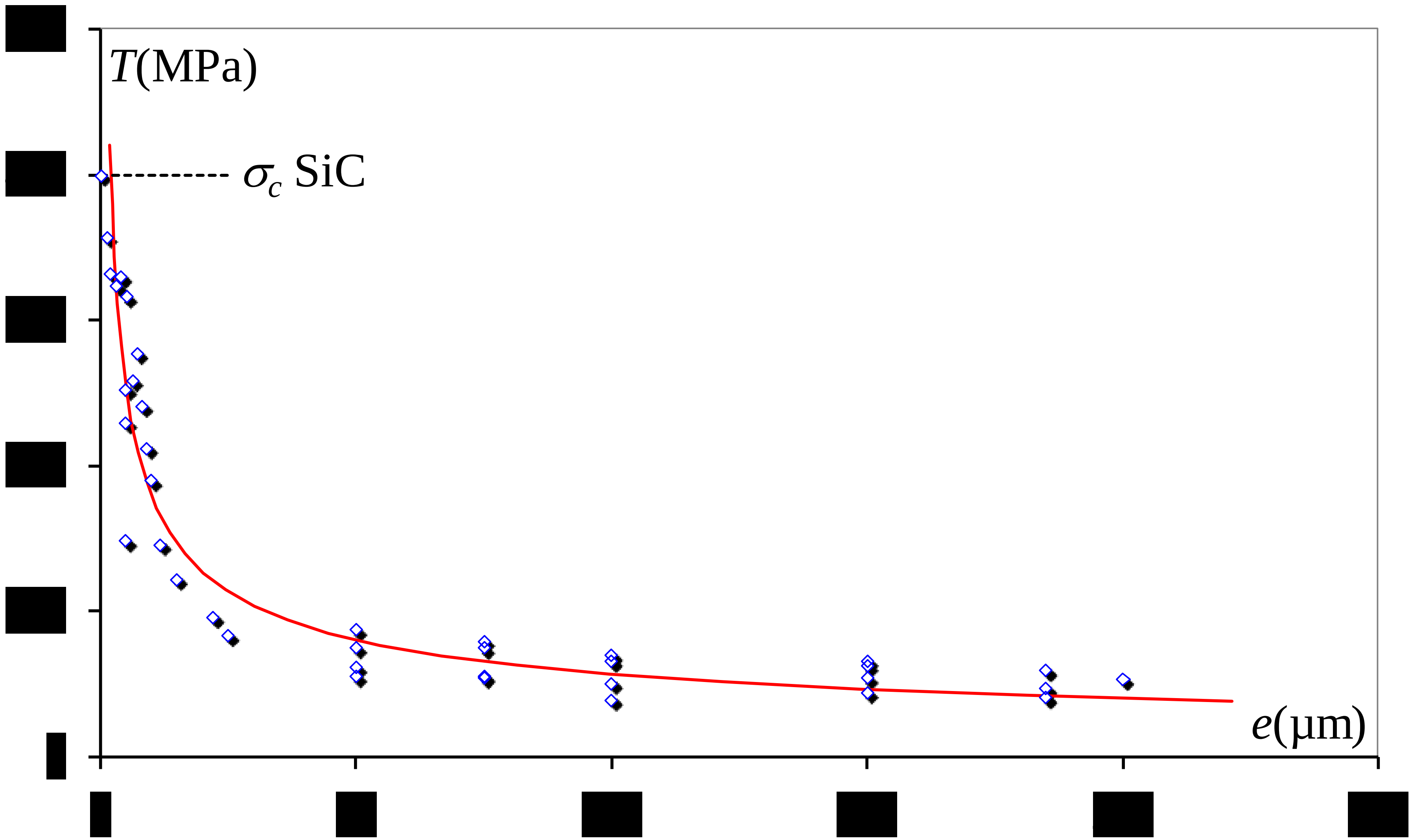
$$g(\xi) \simeq 0.0044 \times \xi \text{ almost independent of the crack location}$$

Energy criterion

$$e\, g(\xi)\, T^2 \geq G_c \quad \Rightarrow \quad T \geq \sqrt{\frac{G_c}{e\, g(\xi)}} \quad \text{(toughness)}$$

The tensile stress at failure depends on the layer thickness as $1/\sqrt{e}$.
The dimensionless crack length ξ is unknown.

The experimental points come from a large number of tests conducted on various grades of brazing material and are recorded in the databases of Astrium Co. They are fitted by a least-square method.



The approximation of the tensile stress at failure.

Nevertheless, to be predictive, the law must be identified without going through the least square procedure.

$$T = T_0 \sqrt{\frac{e_0}{e}}$$

Here e_0 and T_0 correspond to an experimental data point.

Thank you for your attention