Stationary Solutions to the Vlasov–Poisson System in Singular Geometries

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Outline

The Name of the Game:

Plasma near a conducting sharp end

Stationary Vlasov–Poisson system in $\Omega \subset \mathbb{R}^d$:

$$v \cdot \nabla_x f + E(x) \cdot \nabla_v f = 0, \qquad (x,v) \in \Omega \times \mathbb{R}^d, \tag{1}$$

$$E(x) = -\nabla_x \left(\phi[f] - \phi_e\right) \tag{2}$$

$$-\Delta\phi[f] = \int_{\mathbb{R}^d} f(x,v)\, dv := \rho[f], \qquad x \in \Omega, \quad + \text{ bdy cond'n}, \tag{3}$$

$$-\Delta\phi_e = \rho_e, \qquad x \in \Omega, \quad + \text{ bdy cond'n}. \tag{4}$$

- $f(x,v)$: distribution function of charged particles (electrons) $=$ density in phase space (x,v)
- $\rho[f](x)$: spatial density of particles.
- $\phi[f](x)$: self-consistent potential.
- $\phi_e(x)$: external (confining) potential.
- $\rho_e(x)$: density of "neutralising background" (ions).

Existence and Uniqueness

Any couple $(f, \phi[f])$ satisfying (Boltzmann problem)

$$\begin{cases} f(x,v) = \gamma\left(\frac{1}{2}|v|^2 + \phi[f](x) - \phi_e(x) - \beta\right) \\ -\Delta\phi[f] = \rho[f] \text{ (+ bdy cond'n)}, \quad \int_{\Omega\times\mathbb{R}^d} f\,dxdv = M, \end{cases} \tag{5}$$

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Proposition 1: Assume (inter alia) that γ is a positive, strictly decreasing function. There exists a unique solution $(f, \phi[f])$ to (5), for any mass M.

Proof: Consider the space:

$$L^1_M(\Omega \times \mathbb{R}^d) := \{f \in L^1(\Omega \times \mathbb{R}^d) : \int_{\Omega \times \mathbb{R}^d} f \, dxdv = M\}.$$

The function f is the minimum on $L^1_M(\Omega \times \mathbb{R}^d)$ of the functional:

$$J[f] = \int_{\Omega \times \mathbb{R}^d} \Big(\sigma(f) + (\frac{1}{2}\,|v|^2 - \phi_e)\, f\Big) dxdv + \frac{1}{2} \int_{\Omega} |\nabla \phi[f]|^2 dx,$$

where $\sigma' = -\gamma^{-1}$. The Euler–Lagrange equation reads:

$$-\gamma^{-1}(f) + \tfrac{1}{2}\,|v|^2 - \phi_e(x) + \phi[f] - \beta = 0 \qquad (\sigma' = -\gamma^{-1})$$

β: Lagrange multiplier of the constraint $\int_{\Omega \times \mathbb{R}^d} f \, dxdv = M$.

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Existence of the minimum: γ strictly $\searrow \implies \sigma$ strictly convex $\implies J$ strictly convex + technical assumptions.

Reinterpretation: non-linear elliptic problem

Problem (5) is equivalent to solving:

$$-\Delta\phi = \rho = G\left(\phi - \phi_e - \beta\right) \tag{6}$$

where $G(s) := C_d \int_0^{+\infty} \gamma(s+r)\, r^{d/2-1}\, dr$ and β is defined by:

$$\int_\Omega G\big(\phi - \phi_e - \beta\big) = M.$$

Boundary condition:

$$\phi = 0 \text{ on } \Gamma_C \cup \Gamma_D, \quad \partial_\nu \phi = 0 \text{ on } \Gamma_N. \tag{7}$$

The data ϕ_e is solution to the linear problem:

$$-\Delta\phi_e = \rho_e \in L^\infty(\Omega), \quad \partial_\nu\phi_e = 0 \text{ on } \Gamma_N,$$
$$\phi_e = 0 \text{ on } \Gamma_C, \quad \phi_e = \phi_{in} \text{ on } \Gamma_D, \quad \phi_{in} \in H^{1/2}(\Gamma_D) \cap L^\infty(\Gamma_D).$$

$\partial\Omega = \Gamma_C \cup \Gamma_D \cup \Gamma_N$, with Γ_D and Γ_N possibly empty.

Proposition 2: For any fixed β, there exists a unique solution to Problem (6-7), and ϕ is the minimum of the functional:

$$F[\phi] = \int_\Omega \frac{1}{2} |\nabla \phi|^2 + \mathcal{G}(\phi - \phi_e - \beta) \, dx,$$

where $\mathcal{G}' = -G$, on the space V of functions defined by :

$$V = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_C \cup \Gamma_D\}.$$

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Proof: γ strictly $\searrow \implies G$ strictly $\searrow \implies \mathcal{G}$ strictly convex $\implies F$ strictly convex.

$$\text{V.F.:} \quad \int_\Omega \nabla\phi \cdot \nabla\xi = \int_\Omega G(\phi - \phi_e - \beta)\, \xi, \quad \forall \xi \in V.$$

A First Monotonicity Property

Non-linear elliptic comparison principle (**Lions**):
Let ϕ_1 and ϕ_2 be two solutions corresponding to $\beta = \beta_1$ and β_2.

$$\text{If } \; \beta_1 \geq \beta_2 \quad \text{then } \; \phi_1 \geq \phi_2 \quad \text{in } \; \Omega.$$

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Theorem 1: Define the mapping

$$\begin{array}{rccc} \mu : & \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ & \beta & \longmapsto & M = \displaystyle\int_\Omega G(\phi - \phi_e - \beta)\, dx, \end{array}$$

where ϕ is the solution to Problem (6–7).
Then, μ is a nondecreasing, one to one and onto mapping.

Regularity of solutions

$$f \in L^1(\Omega \times \mathbb{R}^d),\ \phi[f] \in H^1(\Omega),\ \rho[f] \in L^1(\Omega) \cap H^{-1}(\Omega).$$

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We assume γ such that $G \in L^\infty_{\mathrm{loc}}(\mathbb{R})$.

$$G(\bullet) \geq 0 \quad \Longrightarrow \quad \rho \geq 0 \quad \Longrightarrow \quad \phi \geq 0$$

$$\left.\begin{array}{l} \phi_e \in L^\infty(\Omega) \\ G \in L^\infty_{\mathrm{loc}}(\mathbb{R}),\ G \searrow \end{array}\right\} \quad \Longrightarrow \quad \rho \in L^\infty(\Omega)$$

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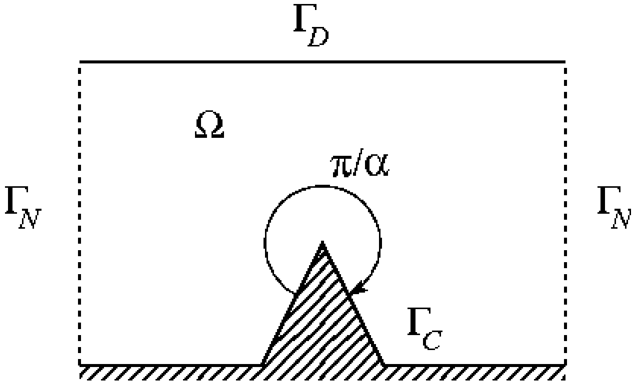
Thus, for all $p \in [1, \infty]$,

$$\phi \in \Phi_p := \{u \in W^{1,p}(\Omega) : \Delta u \in L^p(\Omega),\ u = 0 \text{ on } \Gamma_C \cup \Gamma_D\}.$$

p large enough: $\Phi_p \subset C(\overline{\Omega}) \quad \Longrightarrow \quad \rho \in C(\overline{\Omega})$ if $\phi_e \in C(\overline{\Omega})$.

Corner behaviour

We assume that Ω is a bounded polygonal domain in $\mathbb{R}^2$, with one re-entrant corner of opening π/α $(1/2 < \alpha < 1)$.



Corner singularities

[Grisvard 85, 92. . .]: for all $p \in \left(\frac{2}{2-\alpha}\,,\ \frac{1}{1-\alpha}\right)$ we have:

$$\phi = \phi_R + \lambda\,\chi(r)\,r^{\alpha}\,\sin(\alpha\theta)$$

where $\phi_R \in W^{2,p}(\Omega)$ is the regular part of ϕ, $\lambda = -\int_\Omega \Delta\phi\, P_s$ is the singularity coefficient and P_s is the dual singularity given by:

$$-\Delta P_s = 0 \quad \text{in } \Omega, \quad P_s = 0 \quad \text{on } \Gamma_D \cup \Gamma_C, \quad \frac{\partial P_s}{\partial \nu} = 0 \quad \text{on } \Gamma_N\,;$$

$P_s = \frac{1}{\pi}\, r^{-\alpha}\,\sin(\alpha\theta) + \text{l.s.t.}$ near the reentrant corner.

Theorem 2: Let ϕ be the solution to Problem (6) and $\phi_R \in W^{2,p}(\Omega)$ the regular part. There exists $g \in L^\infty(\Omega)$ such that

$$\chi(r)\,\phi_R(r,\,\theta) = r\sin(\alpha\theta)\,g(r,\theta), \quad \|g\|_{L^\infty(\Omega)} \leq C\,\|\phi_R\|_{W^{2,p}(\Omega)}.$$

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Consequences:

- The singular term is **dominant** near the corner.
- Let (f_1, ϕ_1) and (f_2, ϕ_2) be two solutions to Problem (5) associated to M_1 and M_2 respectively.

$$\text{If } M_1 \geq M_2, \text{ then } \lambda_1 \geq \lambda_2.$$

- For $\beta \geq -G^{-1}(||\rho_e||_{L^\infty})$, we have $\phi \geq \phi_e$ in Ω and $\lambda \geq \lambda_e$.

Behaviour w.r.t. mass

Now we assume a Maxwellian distribution: $\gamma(s) = \mathrm{e}^{-s}$,
Problem (5) becomes: (Maxwell–Boltzmann problem)

$$-\Delta\phi = \kappa\,\mathrm{e}^{\phi_e - \phi} := \rho, \quad \int_\Omega \rho\, dx = M. \tag{8}$$

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Theorem 3: As $M \to 0$, we have

$$\kappa \sim M\Big(\int_\Omega \mathrm{e}^{\phi_e}\,dx\Big)^{-1} \quad \text{and} \quad \lambda \sim \kappa\int_\Omega \mathrm{e}^{\phi_e}\,P_s\,dx.$$

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Proposition 3: As $M \to \infty$, we have $\kappa \to \infty$ and

$$\phi \to \infty \quad \text{a.e. in } \Omega, \quad \rho/\kappa \to 0 \quad \text{in } L^p(\Omega), \quad \forall p < \infty,$$
$$\phi/\kappa \to 0 \quad \text{in } H^1(\Omega)\cap C(\overline{\Omega}).$$

Remark: a boundary layer appears near Dirichlet boundaries.

Numerical simulation

Pbm (8) $\iff$

minimise $\mathcal{J}[\rho] = \int_\Omega (\rho \ln \rho - \phi_e\, \rho + \frac{1}{2} |\nabla \phi[\rho]|^2)\, dx$

on $L^1_M(\Omega) = \{\rho \in L^1(\Omega) : \int_\Omega \rho = M\}.$

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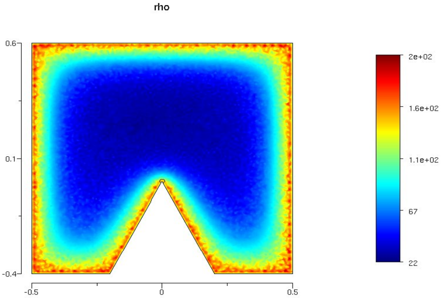
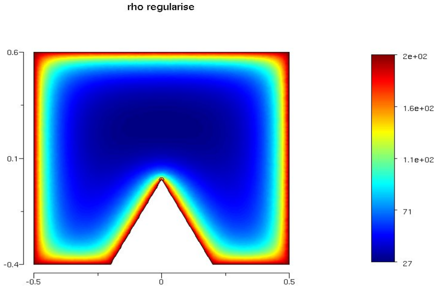
Algorithm:

- Initialization: choose $\rho^0 \in L^1_M(\Omega)$ and $\ell \in \mathbb{N}$.
- Step $n+1$: set $\rho^{n,0} := \rho^n$, then
  - For $j = 1, \ldots, \ell$ compute
    $\rho^{n,j}$ = result of one conjugate gradient iteration
    for $\mathcal{J}$ on $L^1_M(\Omega)$, starting from $\rho^{n,j-1}$.
  - Regularization: solve $-\Delta \phi^{n+1} = \rho^{n,\ell}$, then
    $\rho^{n+1} = M \, \mathrm{e}^{\phi_e - \phi^{n+1}} / (\int_\Omega \mathrm{e}^{\phi_e - \phi^{n+1}})$.
- Stop: $\|\rho^{n+1} - \rho^{n,\ell}\| < \epsilon \, M$.

Remark: solution of Laplacian by singular complement.

$\phi_e \equiv 0$, $\rho \propto \mathrm{e}^{-\phi}$, $\phi = 0$ on Γ and $M = 70$





Without regularization, ρ is very noisy ($L^1(\Omega)$ hardly regular).

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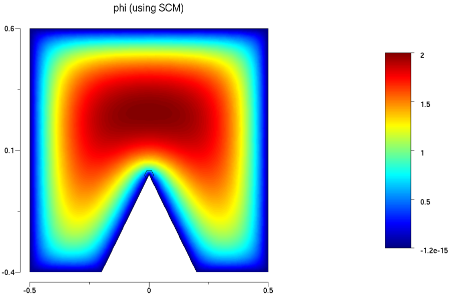
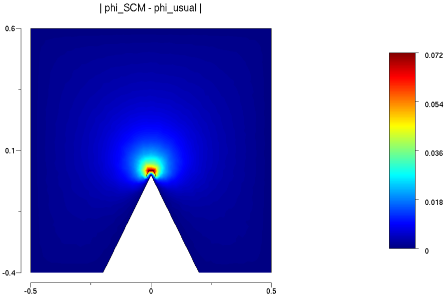
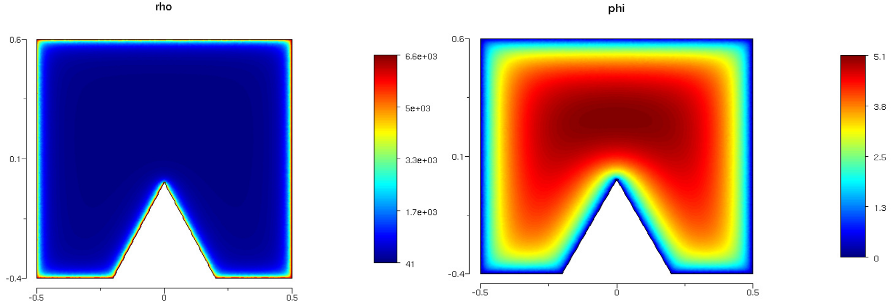




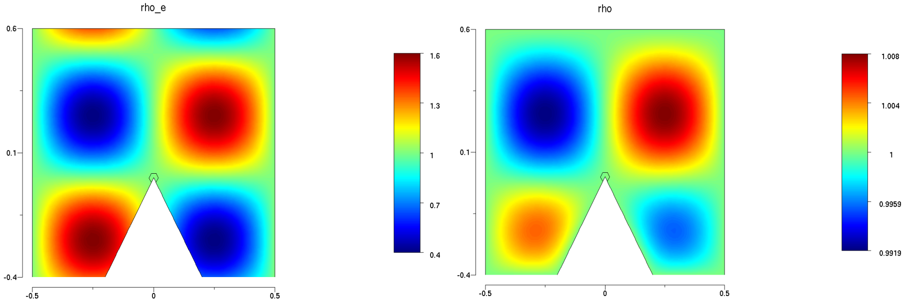
SCM does not dramatically improve computation of ϕ:
non-linear effects stronger than singular behaviour?

$\phi_e \equiv 0$, $\phi = 0$ on Γ and $M = 500$



The corner singularity is “hidden” in the boundary layer.

An example with a neutralising background



$\rho_e = 1 + \epsilon \sin(2\pi x) \sin(2\pi y)$; ρ has the same mass as ρ_e.

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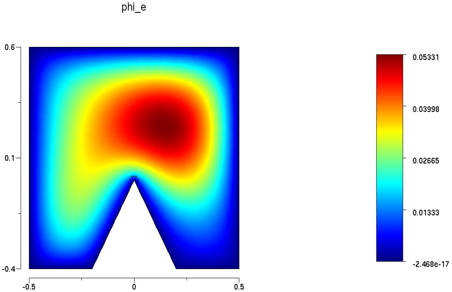

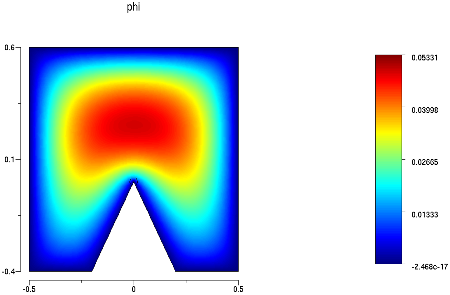

$$-\Delta\phi_e = \rho_e; \quad -\Delta\phi = \rho \propto \mathrm{e}^{\phi_e - \phi}; \quad \phi_e = \phi = 0 \text{ on } \Gamma.$$

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  - Full Vlasov–Poisson... Vlasov–Maxwell...
    Mere **existence of solutions** unknown.
- Realistic modelling of lightning???