The Dirichlet problem for non-divergence parabolic equations with discontinuous in time coefficients in a wedge

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joint work with A.Nazarov

$$Lu \equiv \partial_t u - a^{ij}(t) D_i D_j u = f, \quad (1)$$

a^{ij} are measurable real valued functions of t satisfying $a^{ij} = a^{ji}$ and

$$\nu |\xi|^2 \leqslant a^{ij} \xi_i \xi_j \leqslant \nu^{-1} |\xi|^2, \qquad \xi \in \mathbb{R}^n$$

$\nu = const > 0$. We use the space $L_{p,q}(\Omega \times \mathbb{R})$ with the norm

$$\|f\|_{p,q} = \Big(\int_{\mathbb{R}} \Big(\int_{\Omega} |f(x,t)|^p dx \Big)^{q/p} dt \Big)^{1/q}.$$

N.V. Krylov (2001): for $f \in L_{p,q}(\mathbb{R}^n \times \mathbb{R})$, $1 < p, q < \infty$, equation (1) has a unique solution s.t.

$$\|\partial_t u\|_{p,q} + \sum_{ij} \|D_i D_j u\|_{p,q} \leq C \|f\|_{p,q} .$$

He proved also coercive estimates for u in spaces $L^q(\mathbb{R}; C^{2+\alpha})$, $\alpha \in (0,1)$.

The Dirichlet BVP in the half-space $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$.

Now equation (1) is satisfied for $x_n > 0$ and $u = 0$ for $x_n = 0$. The weighted coercive estimate

$$\|x_n^\mu \partial_t u\|_{p,q} + \sum_{ij} \|x_n^\mu D_i D_j u\|_{p,q} \le C \|x_n^\mu f\|_{p,q}\ , \quad (2)$$

was proved by Krylov (2001), with $1 < p, q < \infty$ and $\mu \in (1 - 1/p, 2 - 1/p)$.

In

Vladimir Kozlov and Alexander Nazarov, The Dirichlet problem for non-divergence parabolic equations with discontinuous in time coefficients, Math. Nachr. **282** (2009), No. 9, 1220 1241.

estimate (2) is proved for solutions of the Dirichlet problem to (1) for the same p, q and

$$-1/p < \mu < 2 - 1/p\,. \qquad (3)$$

Remarks In the paper [2007] Krylov and Kim proved, in particular, estimate (2) in the half-space for $\mu = 0$ and $p = q$. In

D. Kim, Parabolic Equations with Partially BMO Coefficients and Boundary Value Problems in Sobolev Spaces with Mixed Norms, Potential Anal., published on line in 2009.

estimate (2) is proved for $\mu = 0$ and arbitrary $1 < p, q < \infty$.

Dirichlet problem in bounded domain Ω

Let $Q = \Omega \times \mathbb{R}$. We introduce the spaces $\mathbb{L}_{p,q,(\mu)}(Q)$ with the norm

$$\|f\|_{p,q,(\mu)} = \|(\widehat{d}(x))^{\mu} f\|_{p,q},$$

where $\widehat{d}(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$.

We consider the boundary value problem

$$\partial_t u - a^{ij}(x,t) D_i D_j u + b^i(x,t) D_i u = f(x,t) \text{ in } Q;$$
$$u|_{\partial' Q} = 0;$$

the matrix $(a^{ij}) \in \mathcal{C}(\overline{\Omega} \to L^{\infty}(0,T))$ is symmetric and uniformly elliptic. Here ∂Q is the boundary of Q.

K.-N.,2009: Let $\partial\Omega \in \mathcal{C}^{1,\delta}$ with $\delta \in [0,1]$, $1 < p,q < \infty$, and let $1 - \delta - \frac{1}{p} < \mu < 2 - \frac{1}{p}$. Then, for b^i in a suitable class and for any $f \in \mathbb{L}_{p,q,(\mu)}(Q)$, the above problem has a unique solution in $L_{p,q,(\mu)}(Q)$. Moreover, this solution satisfies

$$\|\partial_t u\|_{p,q,(\mu)} + \sum_{ij} \|D_i D_j u\|_{p,q,(\mu)} \leqslant C\|f\|_{p,q,(\mu)},$$

Remarks. For $p = q$ and $\delta = 0$ this theorem was proved by Kim and Krylov (2004).

Coercive estimates for the heat equation with constant coefficients in a wedge.

Conical points

1. V. A. Kozlov and V. G.Maz'ya, On singularities of a solution to the first boundary-value problem for the heat equation in domains with conical points, Izv. Vyssh. Uchebn. Zaved., Ser. Mat., No. 2, 38-46 (1987) and No.3, 37-44 (1987).

2. V. A. Kozlov, On asymptotic of the Green function and the Poisson kernels of the mixed parabolic problem in a cone, Zeitschr. Anal. Anw., 8 (1989), No. 2, 131-151 and 10 (1991), No. 1, 27-42.

Dihedral angles and wedges

5. Solonnikov, V. A., *L_p-estimates for solutions of the heat equation in a dihedral angle*, Rend. Mat. Appl. (7) **21** (2001), N1-4, 1-15.

6. Nazarov, A. I., *L_p-estimates for a solution to the Dirichlet problem and to the Neumann problem for the heat equation in a wedge with edge of arbitrary codimension*, Probl. Mat. Anal., **22** (2001), 126-159 (Russian); English transl.: J. Math. Sci., **106** (2001), N3, 2989-3014.

We use the notation $(x', x'') \in \mathbb{R}^n$, where $x' \in \mathbb{R}$ and $x'' \in \mathbb{R}^{n-m}$. Let K be a cone in $\mathbb{R}^m$ such that the boundary $\partial K \setminus \mathcal{O}$ is of class C^2. We put $\mathcal{K} = K \times \mathbb{R}^{n-m}$. For $\mu \in \mathbb{R}$ and $1 < p, q < \infty$ we introduce spaces $L_{p,q,\mu} = L_{p,q,\mu}(\mathcal{K} \times \mathbb{R})$ with the norm

$$||u||_{p,q,\mu} = \left(\int_{\mathbb{R}} \left(\int_{\mathcal{K}} |x'|^{\mu p} |u(x,t)|^p dx \right)^{q/p} dt \right)^{1/q}$$

Let also

$$Q_R^{\mathcal{K}}(t_0) = \left(B_R(0) \cap \mathcal{K} \right) \times (t_0 - R^2, t_0)$$

where $B_R(x_0)$ is the ball $|x - x_0| < R$.

By $V(Q_R^K(t_0))$ we denote the set of functions u with finite norm

$$||u||_{V(Q_R^K(t_0))} = \sup_{\tau \in (t_0 - R^2, t_0)} ||u(\tau, \cdot)||_{L^2(B_R(0) \cap \mathcal{K})}$$
$$+ ||D_x u||_{L^2(Q_R^K(t_0)))}$$
$$+ \int_{t_0 - R^2}^{t_0} ||D_t u(t, \cdot)||_{W^{-1}(B_R(0) \cap \mathcal{K})} dt.$$

We define the critical exponent for the operator L and the wedge $\mathcal{K}$ as the supremum of all λ such that

$$|u(x,t)| \leq C_\lambda \left(\frac{|x|}{R} \right)^\lambda \sup_{(y,\tau) \in Q_R^K(t_0)} |u(y, \tau)| \qquad (4)$$

for $(x,t) \in Q_{R/2}^K(t_0)$. This inequality must be satisfied for all t_0, $R > 0$ and $u \in V(Q_R^K(t_0))$ subject to

$$Lu = 0 \quad \text{in } Q_R^K(t_0) \qquad (5)$$

and

$$u = 0 \quad \text{on } Q_R^K(t_0) \cap \partial\mathcal{K} \times \mathbb{R}.$$

We shall denote this critical exponent by λ_c. Since $\lambda = 0$ satisfies (4) we conclude that $\lambda_c \geq 0$. Below we give some estimates for λ_c for various geometries of K.

Estimates for the critical exponent

1. Using weighted energy estimates one can show that

$$\lambda_c \geq \frac{2-m}{2} + \nu\sqrt{\Lambda_D + (m-2)^2/4},$$

where Λ_D is the first positive eigenvalue of the Dirichlet-Laplacian on $K \cap S^{m-1}$.

2. Using barrier technique one can show that

a) the critical exponent is positive provided the complement of $\overline{K}$ is non-empty;

b) if K is contained in a half-space then $\lambda_c > 1$.

3). If $L = \partial_t - \Delta$ then

$$\lambda_c = \frac{2-m}{2} + \sqrt{\Lambda_D + (m-2)^2/4}$$

Theorem Let λ_c be the critical exponent. Then for

$$\left|\mu + \frac{n}{p} - \frac{m+2}{2}\right| < \lambda_c + \frac{m-2}{2}$$

the following estimate holds:

$$||u_t||_{p,q,\mu} + ||\nabla\nabla u||_{p,q,\mu} \leq C||f||_{p,q,\mu}$$

For $\delta > 0$ we define $K_\delta = \{x' \in K : \text{dist}(x', \partial K) > \delta|x'|\}$ and $\mathcal{K}_\delta = K_\delta \times \mathbb{R}$.

The next statement can be found (up to scaling) in [LSU].

Proposition 1. *(i)Let $u \in W^{2,1}(Q_R(x_0,t_0))$ solve the equation $L\,u = 0$ in $Q_R(x_0,t_0)$. Then*

$$|Du| \leqslant \frac{C}{R} \sup_{Q_R(x_0,t_0)} |u| \qquad \text{in} \quad Q_{R/2}(x_0,t_0).$$

(ii) For sufficiently small $\delta > 0$, $x_0' \in K \setminus K_\delta$ and $|x_0'| = 1$ the following assertion is valid. Let $u \in W_2^{2,1}(Q_R^+(x_0,t_0))$ solve the equation $L\,u = 0$ in $Q_R^+(x_0,t_0)$, where $R \le 1/2$, and let $u(x,t) = 0$ for $x \in \partial\mathcal{K}$. Then

$$|Du| \leqslant \frac{C}{R} \sup_{Q_R^+(x_0,t_0)} |u| \qquad \text{in} \quad Q_{R/2}^+(x_0,t_0).$$

Here C depends only on ν and K and δ.

We used the notations

$$Q_R(x_0,t_0) = B_R(x_0) \times (t_0 - R^2, t_0)$$

and

$$Q_R^+(x_0,t_0) = \Big(B_R(x_0) \cap \mathcal{K}\Big) \times (t_0 - R^2, t_0).$$

Iterating the inequality from Proposition (i) we arrive at

Lemma 1. *Let* $u \in W_2^{2,1}(Q_R(x_0,t_0))$ *solve the equation* $L\,u = 0$ *in* $Q_R(x_0,t_0)$. *Then*

$$|D^\alpha u| \leqslant \frac{C}{R^{|\alpha|}} \sup_{Q_R(x_0,t_0)} |u| \qquad \text{in} \quad Q_{R/2}(x_0,t_0).$$

Next Lemma is actually proved in [KN].

Lemma 2. *For sufficiently small* $\delta > 0$, $x_0' \in K \setminus K_\delta$, $x_0'' \in \mathbb{R}^{n-m}$ *and* $|x_0'| = 1$ *the following assertion is valid. Let* $u \in W_2^{2,1}(Q_R^+(x_0,t_0))$ *solve the equation* $L\,u = 0$ *in* $Q_R^+(x_0,t_0)$, *where* $R < 1/2$, *and let* $u(x,t) = 0$ *for* $x \in \partial\mathcal{K}$. *For* $|\alpha| \geqslant 2$ *and arbitrary small* $\varepsilon > 0$

$$d(x)^{|\alpha|-2+\varepsilon}|D_x^\alpha u| \leqslant \frac{C}{R^{2-\varepsilon}} \sup_{Q_R^+(x_0,t_0)} |u| \qquad , \quad (6)$$

in $Q^+_{R/8^{|\alpha|}}(x_0,t_0)$, *where* C *is a positive constant depending on* ν, $|\alpha|$, K, δ *and* ε.

Green's function in $\mathcal{K} \times \mathbb{R}$

Let us consider (1) in the whole space. Using the Fourier transform with respect to x we obtain:

$$u(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} \Gamma(x,y;t,s) f(y,s)\ ds, \qquad (7)$$

where Γ is the Green function of the operator $\mathcal{L}_0$ given by

$$\Gamma(x,y;t,s) = \frac{\det\left(\int_s^t A(\tau)d\tau\right)^{-\frac{1}{2}}}{(4\pi)^{\frac{n}{2}}}$$
$$\times \exp\left(- \frac{\left(\left(\int_s^t A(\tau)d\tau\right)^{-1}(x-y),(x-y)\right)}{4}\right)$$

for $t > s$ and 0 otherwise. Here by $A(t)$ is denoted the matrix $\{a_{ij}(t)\}$. The above representation implies:

$$\left|\partial_t^k D_x^\alpha D_y^\beta \Gamma(x,y;t,s)\right| \le \frac{C_{k,\alpha,\beta}}{(t-s)^{(n+2k+|\alpha|+|\beta|)/2}}$$
$$\times \exp\left(-\frac{\sigma|x-y|^2}{t-s}\right), \qquad (8)$$

where $k \le 1$ and α and β are arbitrary indexes. Here σ is a positive constant depending on ν.

We denote by $\Gamma_{\mathcal{K}} = \Gamma_{\mathcal{K}}(x, y; t, s)$ Green's function to the homogeneous Dirichlet problem of (1), in the half-space. Clearly, $\Gamma_{\mathcal{K}}(x, y; t, s) \le \Gamma(x, y; t, s)$ and therefore

$$\Gamma_{\mathcal{K}}(x, y; t, s) \le \frac{C}{(t-s)^{n/2}} \exp\left(-\frac{\sigma|x-y|^2}{t-s}\right) \quad \text{in } \mathcal{K} \times \mathbb{R}. \tag{9}$$

We shall use the notations

$$\mathcal{R}_x = \frac{|x'|}{|x'| + \sqrt{t-s}}, \quad \mathcal{R}_y = \frac{|y'|}{|y'| + \sqrt{t-s}}$$

and

$$r_x = \frac{d(x)(|x'| + \sqrt{t-s})}{|x'|\sqrt{t-s}}, \quad r_y = \frac{d(y)(|y'| + \sqrt{t-s})}{|y'|\sqrt{t-s}},$$

where $d(x)$ is the distance from x to the boundary $\partial\mathcal{K}$.

Proposition 2. *The following inequality*

$$|\Gamma_{\mathcal{K}}(x,y;t,s)| \leqslant C\,\mathcal{R}_x^{\lambda}\mathcal{R}_y^{\lambda}(t-s)^{-n/2}\,\mathsf{exp}\left(-\frac{\sigma_1|x-y|^2}{t-s}\right) \tag{10}$$

holds for $x, y \in \mathcal{K}$ and $s < t$. Here σ_1 is a positive constant depending only on the ellipticity constant ν and C may depend on ν and λ.

The proof of this proposition and the next theorem essentially uses the above local estimates and the definition of the critical exponent.

Theorem 1. *Let* $|\alpha|, |\beta| \leq 2$. *For* $x, y \in \mathcal{K}$, $0 \leqslant s < t$ *the following estimate is valid*

$$|D_x^\alpha D_y^\beta \Gamma_{\mathcal{K}}(x,y;t;s)| \leqslant C\,\mathcal{R}_x^{\lambda-|\alpha|}\mathcal{R}_y^{\lambda-|\beta|} r_x^{-\varepsilon} r_y^{-\varepsilon}$$

$$(t-s)^{-\frac{n+|\alpha|+|\beta|}{2}} \exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right), \qquad (11)$$

where σ_1 *is a positive constant depending on* ν, ε *is an arbitrary small positive number and* C *may depend on* ν, α, β *and* ε. *If* $|\alpha| \leqslant 1$ *(or* $|\beta| \leqslant 1$*) then the factor* $r_x^{-\varepsilon}$ *(*$r_y^{-\varepsilon}$*) must be removed from the right-hand side respectively.*