OVERLAPPING DOMAIN PROBLEMS WITH CRACKS AND RIGID INCLUSIONS

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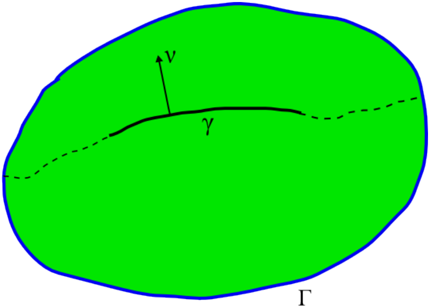
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CRACK PROBLEMS. DIRECTIONS OF INVESTIGATIONS

1. Solvability of boundary value problems, solution smoothness (elastic, viscoelastic, thermoelastic, electro-thermoelastic bodies)
2. Dependence on parameters, shape sensitivity analysis, differentiability of energy functionals
3. Optimal control problems
4. Smooth domain method. Fictitious domain method
5. Contact of elastic bodies of different dimensions
6. Overlapping domain problems
7. Rigid inclusions in elastic bodies

Formulation of crack problem Find functions $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, $\boldsymbol{\sigma} = \{\sigma_{ij}\}, i, j = 1, 2$, such that

$$-\operatorname{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma\,, \tag{1}$$

$$\boldsymbol{\sigma} = \mathbf{A}\varepsilon(\mathbf{u}) \quad \text{in} \quad \Omega_\gamma\,, \tag{2}$$

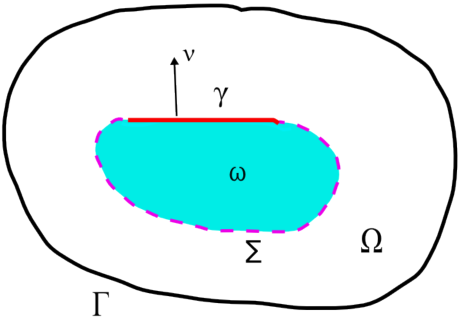
$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma\,, \tag{3}$$

$$[\mathbf{u}]\nu \geq 0, \quad [\sigma_\nu] = 0, \quad [\mathbf{u}]\nu \cdot \sigma_\nu = 0 \quad \text{on} \quad \gamma\,, \tag{4}$$

$$\sigma_\nu \leq 0, \;\; \sigma_\tau = 0 \quad \text{on} \quad \gamma^{\pm}\,, \tag{5}$$

where $[\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-$, $\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu.$

Rigid inclusion with delamination



In domain Ω_γ, find $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$; $\mathbf{u} = \rho_0$ in ω; $\rho_0 \in \mathbf{R}(\omega)$; and in $\Omega \setminus \bar{\omega}$ find $\sigma = \{\sigma_{ij}\}, i, j = 1, 2,$

$$-\operatorname{div}\sigma = \mathbf{f} \quad \text{in} \quad \Omega \setminus \bar{\omega}\ , \tag{6}$$

$$\sigma - \mathbf{A}\varepsilon(\mathbf{u}) = 0 \quad \text{in} \quad \Omega \setminus \bar{\omega}\ , \tag{7}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma\ , \tag{8}$$

$$[\mathbf{u}]\nu \geq 0 \quad \text{on} \quad \gamma, \tag{9}$$

$$\sigma_\tau = 0,\ \sigma_\nu \leq 0\ ,\ \sigma_\nu \cdot [\mathbf{u}]\nu = 0 \quad \text{on} \quad \gamma^+, \tag{10}$$

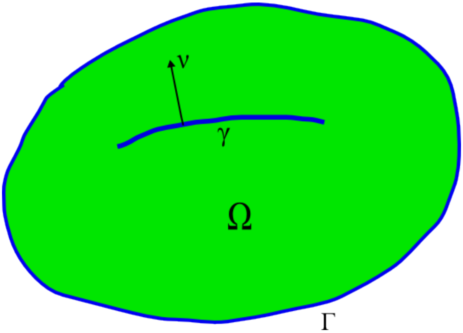
$$-\int_\Sigma \sigma\nu \cdot \rho = \int_\omega \mathbf{f}\rho \quad \forall \rho \in \mathbf{R}(\omega), \tag{11}$$

where

$$\mathbf{R}(\omega) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = \mathbf{B}x + \mathbf{C},\ x \in \omega\},$$

$$\mathbf{B} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},\ \mathbf{C} = (c^1, c^2);\ b, c^1, c^2 = \text{const}.$$

Thin rigid inclusion with delamination



Find functions $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}), \rho_0 \in \mathbf{R}(\gamma)$, $\boldsymbol{\sigma} = \{\sigma_{ij}\}, \mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2}$, such that

$$-\mathrm{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \boldsymbol{\Omega}_\gamma, \tag{12}$$

$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \boldsymbol{\Omega}_\gamma, \tag{13}$$

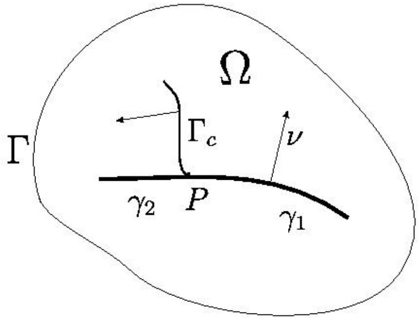
$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \boldsymbol{\Gamma}, \tag{14}$$

$$[\mathbf{u}]\nu \geq \mathbf{0},\ \mathbf{u}^- = \rho_0,\ \boldsymbol{\sigma}_{\boldsymbol{\nu}}^+ \leq \mathbf{0},\ \boldsymbol{\sigma}_{\boldsymbol{\tau}}^+ = \mathbf{0} \quad \text{on} \quad \gamma, \tag{15}$$

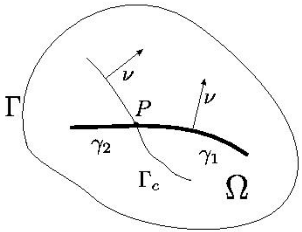
$$\boldsymbol{\sigma}_{\boldsymbol{\nu}}^+ \cdot [\mathbf{u}]\nu = \mathbf{0} \quad \text{on} \quad \gamma, \tag{16}$$

$$\int_\gamma [\sigma\nu]\rho = \mathbf{0} \quad \forall \rho \in \mathbf{R}(\gamma). \tag{17}$$

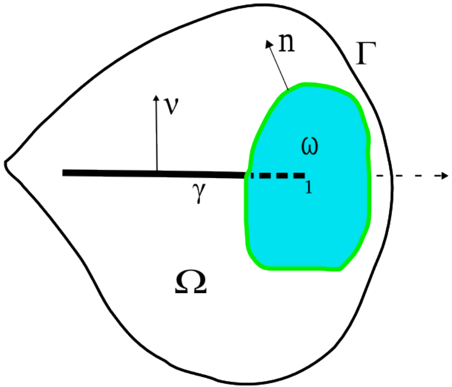
Deviated and bifurcated cracks



Crack crossing a rigid inclusion



"Patch" problem



$$-\,\mathrm{div}\boldsymbol{\sigma}^{\delta} = \mathbf{f} \ \text{ in } \ \boldsymbol{\Omega}_{\gamma} \setminus \partial\omega, \tag{18}$$

$$\boldsymbol{\sigma}^{\delta} = \mathbf{A}\varepsilon(\mathbf{u}^{\delta}) \ \text{ in } \ \boldsymbol{\Omega}_{\gamma}, \tag{19}$$

$$-\,\mathrm{div}\mathbf{p}^{\delta} = \mathbf{0} \ \text{ in } \ \omega, \tag{20}$$

$$\mathbf{p}^{\delta} = \frac{1}{\delta}\mathbf{B}\varepsilon(\mathbf{v}^{\delta}) \ \text{ in } \ \omega, \tag{21}$$

$$\mathbf{u}^{\delta} = \mathbf{0} \ \text{ on } \ \boldsymbol{\Gamma}, \tag{22}$$

$$[\mathbf{u}^{\delta}]\nu \geq \mathbf{0},\ [\sigma_{\nu}^{\delta}] = \mathbf{0},\ \sigma_{\nu}^{\delta} \leq \mathbf{0},\ \boldsymbol{\sigma}_{\boldsymbol{\tau}}^{\delta} = \mathbf{0},\ \sigma_{\nu}^{\delta}\cdot[\mathbf{u}^{\delta}]\nu = \mathbf{0} \ \text{ on } \ \gamma, \tag{23}$$

$$\mathbf{u}^{\delta} = \mathbf{v}^{\delta},\ [\sigma^{\delta}\mathbf{n}] = \mathbf{p}^{\delta}\mathbf{n} \ \text{ on } \ \partial\omega. \tag{24}$$

Limit problem

$$-\,\mathrm{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma \setminus \partial\omega, \tag{25}$$

$$\boldsymbol{\sigma} = \mathbf{A}\varepsilon(\mathbf{u}) \quad \text{in} \quad \Omega_\gamma, \tag{26}$$

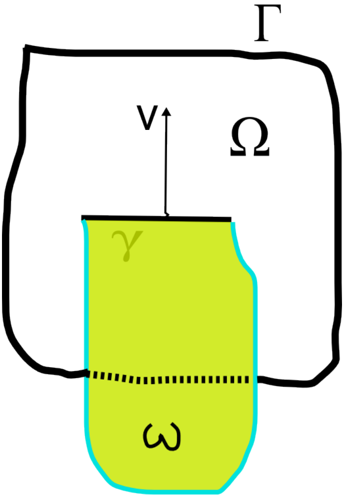
$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \tag{27}$$

$$\mathbf{u} = \rho_0 \quad \text{on} \quad \partial\omega, \tag{28}$$

$$[\mathbf{u}]\nu \geq 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \leq 0,\ \sigma_\tau = 0,\ \sigma_\nu \cdot [\mathbf{u}]\nu = 0 \quad \text{on} \quad \gamma, \tag{29}$$

$$\int_{\partial\omega} [\sigma \mathbf{n}]\rho = 0 \quad \forall \rho \in \mathbf{R}(\omega), \tag{30}$$

Two layer structure



The set of admissible displacements

$$\mathbf{K} = \{(\mathbf{v}, \rho) \in \mathbf{H} \mid [\mathbf{v}]\nu \geq \mathbf{0} \text{ on } \gamma;\ \mathbf{v}|_{\gamma^-} = \rho,\ \rho \in \mathbf{R}(\omega)\}$$

$$\mathbf{H}^1_\Gamma(\Omega_\gamma) = \{\mathbf{v} \in \mathbf{H}^1(\Omega_\gamma) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma\},$$
$$\mathbf{H} = \mathbf{H}^1_\Gamma(\Omega_\gamma)^2 \times \mathbf{H}^1(\omega)^2$$

Problem formulation

Find functions $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2), \rho_0 \in \mathbf{R}(\omega),\ \boldsymbol{\sigma} = \{\sigma_{\mathbf{ij}}\}, \mathbf{i}, \mathbf{j} = 1, 2,$

$$-\mathrm{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma, \tag{31}$$

$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{32}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma, \tag{33}$$

$$\mathbf{u}^- = \rho_0, \quad [\mathbf{u}]\nu \geq 0 \quad \text{on} \quad \gamma, \tag{34}$$

$$\int\limits_\gamma [\sigma\nu \cdot \mathbf{u}] + \int\limits_\omega \mathbf{g}\rho_0 = 0, \tag{35}$$

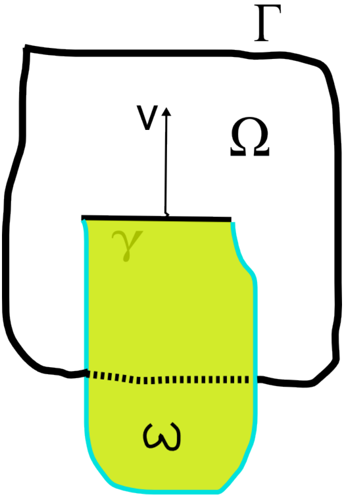
$$\int\limits_\gamma [\sigma\nu \cdot \mathbf{v}] + \int\limits_\omega \mathbf{g}\rho \leq 0 \quad \forall (\mathbf{v}, \rho) \in \mathbf{K}. \tag{36}$$

Variational inequality

$$(\mathbf{u}, \rho_0) \in \mathbf{K}, \tag{37}$$

$$\int\limits_{\Omega_\gamma} \sigma(\mathbf{u})\varepsilon(\mathbf{v}-\mathbf{u}) - \int\limits_{\Omega_\gamma} \mathbf{f}(\mathbf{v}-\mathbf{u}) - \int\limits_{\omega} \mathbf{g}(\rho-\rho_0) \geq 0 \quad \forall(\mathbf{v}, \rho) \in \mathbf{K} \tag{38}$$

Asymptotic analysis



Find functions $\mathbf{u}^\delta = (\mathbf{u}_1^\delta, \mathbf{u}_2^\delta)$, $\mathbf{w}^\delta = (\mathbf{w}_1^\delta, \mathbf{w}_2^\delta)$, $\boldsymbol{\sigma}^\delta = \{\sigma_{ij}^\delta\}$, $\mathbf{p}^\delta = \{\mathbf{p}_{ij}^\delta\}, \mathbf{i}, \mathbf{j} = 1, 2$,

$$-\mathrm{div}\boldsymbol{\sigma}^\delta = \mathbf{f} \quad \text{in} \quad \Omega_\gamma, \tag{39}$$

$$\boldsymbol{\sigma}^\delta - \mathbf{A}\varepsilon(\mathbf{u}^\delta) = \mathbf{0} \quad \text{in} \quad \Omega_\gamma, \tag{40}$$

$$-\mathrm{div}\mathbf{p}^\delta = \mathbf{g} \quad \text{in} \quad \omega, \tag{41}$$

$$\mathbf{p}^\delta - \frac{1}{\delta}\mathbf{B}\varepsilon(\mathbf{w}^\delta) = \mathbf{0} \quad \text{in} \quad \omega, \tag{42}$$

$$\mathbf{u}^\delta = \mathbf{0} \quad \text{on} \quad \Gamma; \;\; \mathbf{p}^\delta\nu = \mathbf{0} \quad \text{on} \quad \partial\omega \setminus \bar{\gamma}, \tag{43}$$

$$\mathbf{u}^{\delta -} = \mathbf{w}^\delta, \quad [\mathbf{u}^\delta]\nu \ge \mathbf{0} \quad \text{on} \quad \gamma, \tag{44}$$

$$\int\limits_\gamma [\boldsymbol{\sigma}^\delta\nu \cdot \mathbf{u}^\delta] - \int\limits_\gamma \mathbf{p}^\delta\nu \cdot \mathbf{w}^\delta = \mathbf{0}, \tag{45}$$

$$\int\limits_\gamma [\boldsymbol{\sigma}^\delta\nu \cdot \bar{\mathbf{u}}] - \int\limits_\gamma \mathbf{p}^\delta\nu \cdot \bar{\mathbf{w}} \le \mathbf{0} \quad \forall(\bar{\mathbf{u}}, \bar{\mathbf{w}}) \in \mathbf{K}^*. \tag{46}$$

Variational inequality

$$(\mathbf{u}^\delta, \mathbf{w}^\delta) \in \mathbf{K}^*, \tag{47}$$

$$\int_{\Omega_\gamma} \sigma(\mathbf{u}^\delta)\varepsilon(\bar{\mathbf{u}} - \mathbf{u}^\delta) - \int_{\Omega_\gamma} \mathbf{f}(\bar{\mathbf{u}} - \mathbf{u}^\delta) + \int_{\omega} \mathbf{p}^\delta(\mathbf{w}^\delta)\varepsilon(\bar{\mathbf{w}} - \mathbf{w}^\delta) - \int_{\omega} \mathbf{g}(\bar{\mathbf{w}} - \mathbf{w}^\delta) \geq 0 \quad \forall (\bar{\mathbf{u}}, \bar{\mathbf{w}}) \in \mathbf{K}^* \tag{48}$$

Estimates

$$\|\mathbf{u}^\delta\|^2_{H^1_\Gamma(\Omega_\gamma)^2} + \|\mathbf{w}^\delta\|^2_{H^1(\omega)^2} \leq \mathbf{c},$$

$$\int_{\omega} \mathbf{p}(\mathbf{w}^\delta)\varepsilon(\mathbf{w}^\delta) \leq \mathbf{c}\delta,$$

uniform with respect to δ, $\delta \in (0, \delta_0)$

$$\mathbf{u}^{\delta} \longrightarrow \mathbf{u} \quad \text{weakly in} \quad \mathbf{H}^1_{\Gamma}(\Omega_\gamma)^2 \tag{49}$$

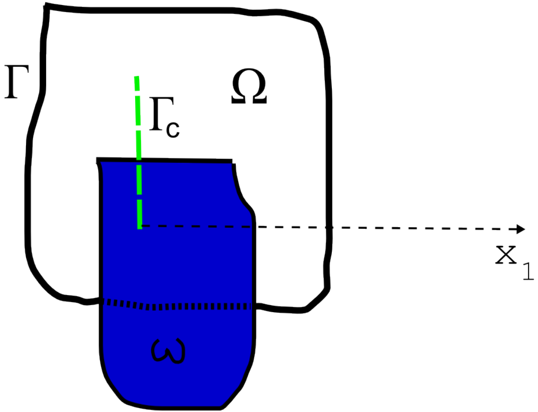
$$\mathbf{w}^{\delta} \longrightarrow \rho_0 \quad \text{weakly in} \quad \mathbf{H}^1(\omega)^2,\ \rho_0 \in \mathbf{R}(\omega) \tag{50}$$

Variational inequality

$$(\mathbf{u}, \rho_0) \in \mathbf{K},$$

$$\int\limits_{\Omega_\gamma} \sigma(\mathbf{u})\varepsilon(\bar{\mathbf{u}} - \mathbf{u}) - \int\limits_{\Omega_\gamma} \mathbf{f}(\bar{\mathbf{u}} - \mathbf{u}) - \int\limits_{\omega} \mathbf{g}(\rho - \rho_0) \geq \mathbf{0} \quad \forall(\bar{\mathbf{u}}, \rho) \in \mathbf{K}$$

Optimal choice of safe loading



Find functions $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}), \rho_0 \in \mathbf{R}(\omega), \sigma = \{\sigma_{ij}\}, \mathbf{i}, \mathbf{j} = 1, 2,$

$$-\mathrm{div}\sigma = \mathbf{f} \quad \text{in} \quad \Omega_\gamma^c, \tag{51}$$

$$\sigma - \mathbf{A}\varepsilon(\mathbf{u}) = 0 \quad \text{in} \quad \Omega^c, \tag{52}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma; \;\; \mathbf{u} = \rho_0 \quad \text{on} \quad \gamma, \tag{53}$$

$$[\mathbf{u}]\mathbf{n} \geq 0 \quad \text{on} \quad \Gamma_c, \tag{54}$$

$$\int\limits_{\Gamma_c} [\sigma \mathbf{n} \cdot \mathbf{u}] + \int\limits_{\gamma} [\sigma\nu]\rho_0 + \int\limits_{\omega} \mathbf{g}\rho_0 = 0, \tag{55}$$

$$\int\limits_{\Gamma_c} [\sigma \mathbf{n} \cdot \bar{\mathbf{u}}] + \int\limits_{\gamma} [\sigma\nu]\bar{\rho} + \int\limits_{\omega} \mathbf{g}\bar{\rho} \leq 0 \quad \forall (\bar{\mathbf{u}}, \bar{\rho}) \in \mathbf{K_0}. \tag{56}$$

Set of admissible displacements

$$\mathbf{K_0} = \{(\mathbf{v}, \rho) \in \mathbf{H_0} \mid [\mathbf{v}]\mathbf{n} \geq \mathbf{0} \text{ on } \mathbf{\Gamma_c};\ \mathbf{v}|_\gamma = \rho,\ \rho \in \mathbf{R}(\omega)\}$$

$$\mathbf{H_0} = \mathbf{H}^1_\Gamma(\Omega^c)^2 \times \mathbf{H}^1(\omega)^2$$

Variational inequality

$$(\mathbf{u}, \rho_0) \in \mathbf{K_0}, \tag{57}$$

$$\int\limits_{\Omega^c} \sigma(\mathbf{u})\varepsilon(\mathbf{v}-\mathbf{u}) - \int\limits_{\Omega^c} \mathbf{f}(\mathbf{v}-\mathbf{u}) - \int\limits_{\omega} \mathbf{g}(\rho-\rho_0) \geq \mathbf{0}\ \ \forall(\mathbf{v}, \rho) \in \mathbf{K_0}. \tag{58}$$

Denote $\Omega_\gamma^{c\lambda} = \Omega \setminus (\bar{\Gamma}_c^\lambda \cup \bar{\gamma})$, $\Omega^{c\lambda} = \Omega \setminus \bar{\Gamma}_c^\lambda$
Find functions
$u^\lambda = (u_1^\lambda, u_2^\lambda), \rho_0^\lambda \in R(\omega),\ \sigma^\lambda = \{\sigma_{ij}^\lambda\}, i, j = 1, 2,$ such that

$$-\mathrm{div}\sigma^\lambda = f \quad \text{in} \quad \Omega_\gamma^{c\lambda}, \tag{59}$$

$$\sigma^\lambda - A\varepsilon(u^\lambda) = 0 \quad \text{in} \quad \Omega^{c\lambda}, \tag{60}$$

$$u^\lambda = 0 \quad \text{on} \quad \Gamma;\ u^\lambda = \rho_0^\lambda \quad \text{on} \quad \gamma, \tag{61}$$

$$[u^\lambda]n \geq 0 \quad \text{on} \quad \Gamma_c^\lambda, \tag{62}$$

$$\int\limits_{\Gamma_c^\lambda} [\sigma^\lambda n \cdot u] + \int\limits_{\gamma} [\sigma^\lambda \nu] \rho_0^\lambda + \int\limits_{\omega} g \rho_0^\lambda = 0, \tag{63}$$

$$\int\limits_{\Gamma_c^\lambda} [\sigma^\lambda n \cdot \bar{u}] + \int\limits_{\gamma} [\sigma^\lambda \nu] \bar{\rho} + \int\limits_{\omega} g \bar{\rho} \leq 0 \quad \forall (\bar{u}, \bar{\rho}) \in K_0^\lambda. \tag{64}$$

Set of admissible displacements

$$\mathbf{K}_0^\lambda = \{(\mathbf{v}, \rho) \in \mathbf{H}_0^\lambda \mid [\mathbf{v}]\mathbf{n} \geq 0 \text{ on } \Gamma_c^\lambda; \;\; \mathbf{v}|_\gamma = \rho, \; \rho \in \mathbf{R}(\omega)\}$$

$$\mathbf{H}_0^\lambda = \mathbf{H}_\Gamma^1(\Omega^{c\lambda})^2 \times \mathbf{H}^1(\omega)^2$$

Energy functional

$$\mathbf{E}(\Omega^{c\lambda}; \mathbf{g}) = \frac{1}{2}\int_{\Omega^{c\lambda}} \sigma(\mathbf{u}^\lambda)\varepsilon(\mathbf{u}^\lambda) - \int_{\Omega^{c\lambda}} \mathbf{f}\mathbf{u}^\lambda - \int_{\omega} \mathbf{g}\rho_0^\lambda \qquad (65)$$

Formula for the derivative

$$\frac{d}{d\lambda}E(\Omega^{c\lambda};g)|_{\lambda=0} = \int_{\Omega^c}\{\frac{1}{2}\varepsilon_{kl}(u)\varepsilon_{ij}(u)(a_{ijkl}\theta)_{,2} - \sigma_{ij}(u)u_{i,2}\theta_{,j}\} - \int_{\Omega^c}(\theta f_i)_{,2}u_i \tag{66}$$

Cost functional

$$J(g) = \frac{d}{d\lambda}E(\Omega^{c\lambda};g)|_{\lambda=0},$$

where $\mathbf{g} \in \mathbf{G}$, and $\mathbf{G} \subset \mathbf{L}^2(\omega)^2$ bounded and weakly closed set

Optimal control problem

$$\sup_{g\in G} J(g) \tag{67}$$

Theorem

There exists a solution of the optimal control problem (67)

$$(\mathbf{u}^n, \rho_0^n) \in \mathbf{K}_0, \tag{68}$$

$$\int_{\Omega^c} \sigma(\mathbf{u}^n)\varepsilon(\mathbf{v}-\mathbf{u}^n) - \int_{\Omega^c} \mathbf{f}(\mathbf{v}-\mathbf{u}^n) - \int_{\omega} \mathbf{g}^n(\rho-\rho_0^n) \geq 0 \quad \forall(\mathbf{v}, \rho) \in \mathbf{K}_0 \tag{69}$$

$$\|\mathbf{u}^n\|_{H^1_\Gamma(\Omega^c)^2} \leq c \tag{70}$$

Limit problem

$$(\mathbf{u}, \rho_0) \in \mathbf{K_0}, \tag{71}$$

$$\int_{\Omega^c} \sigma(\mathbf{u})\varepsilon(\mathbf{v}-\mathbf{u}) - \int_{\Omega^c} \mathbf{f}(\mathbf{v}-\mathbf{u}) - \int_{\omega} \mathbf{g_0}(\rho - \rho_0) \geq \mathbf{0} \quad \forall (\mathbf{v}, \rho) \in \mathbf{K_0} \tag{72}$$

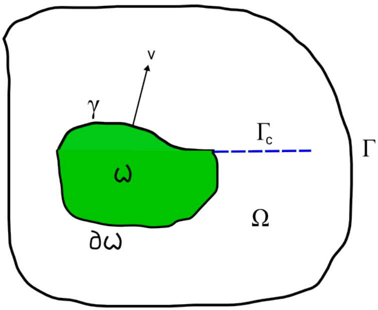
Thus $\mathbf{u} = \mathbf{u}(\mathbf{g_0}), \rho_0 = \rho_0(\mathbf{g_0})$.

$$\mathbf{u}^n \longrightarrow \mathbf{u} \ \text{ strongly in } \mathbf{H}^1_\Gamma(\Omega^c)^2 \tag{73}$$

$$J(g^n) = \int_{\Omega^c} \{\frac{1}{2}\varepsilon_{kl}(u^n)\varepsilon_{ij}(u^n)(a_{ijkl}\theta)_{,2} - \sigma_{ij}(u^n)u^n_{i,2}\theta_{,j}\} - \int_{\Omega^c} (\theta f_i)_{,2}u^n_i$$

$$J(g^n) \longrightarrow J(g_0)$$

Optimal control of crack propagation



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