Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Gemeinschaft

Hans-Christoph Kaiser & Joachim Rehberg

Optimal elliptic regularity for some polygonal, multi-material constellations in three-dimensional real space

Singularities

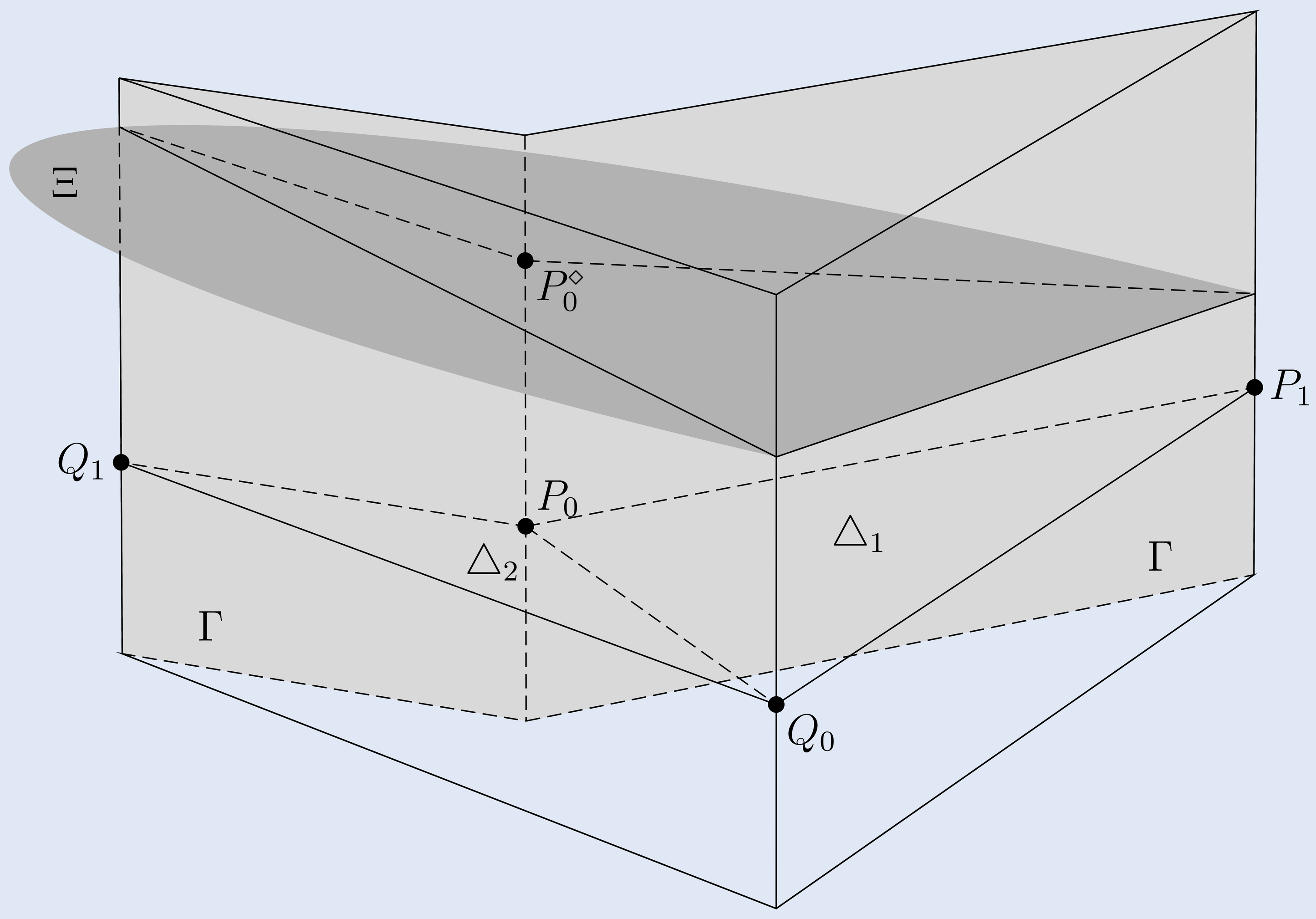
The following phenomena—alone or in combination—can cause singularities in the solution of an elliptic equation:

- Edges and vertices of the spatial domain.
- Crossings of Dirichlet and Neumann boundary conditions.
- Material interfaces.

We present optimal regularity of elliptic div–grad operators

$$-\nabla \cdot \mu \nabla : W^{1,p}_{\Gamma}(\Omega) \to W^{-1,p}_{\Gamma}(\Omega) \qquad (1)$$

for several model constellations. The elliptic coefficient function μ on a right prism Ω takes its values in the set of real, symmetric, positive definite 3×3 matrices. It is constant in each of the halfspaces separated by the plane Ξ. On Γ is a homogeneous Neumann b.c.



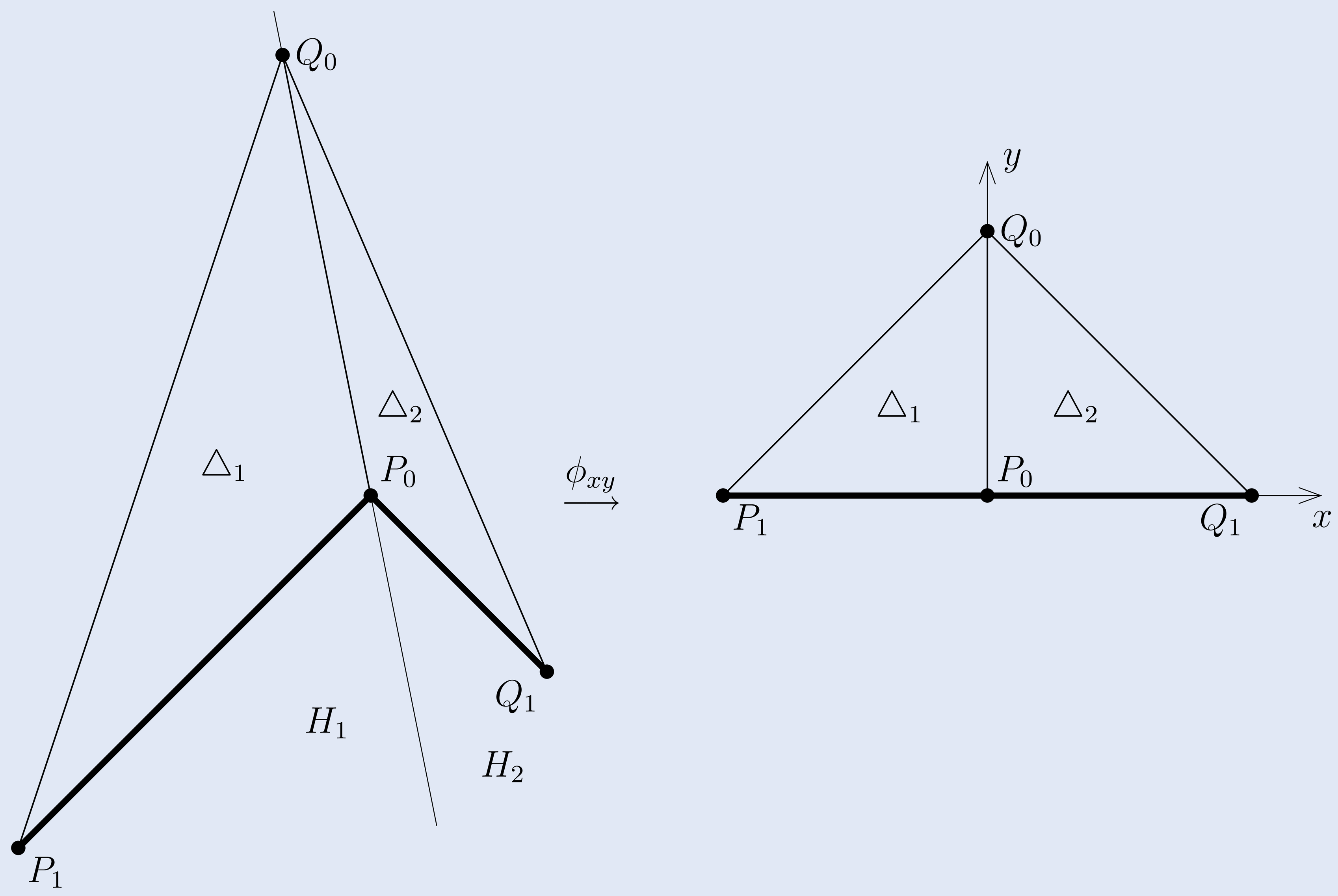
Theorem: For the model constellation on the left there is a $p > 3$ such that (1) is a topological isomorphism.

Proposition 1: (Maz'ya et. al.) For a Lipschitzian, multi–material, polyhedral compound $\Omega \subset \mathbb{R}^3$ the operator (1) with $\Gamma = \emptyset$ is a topological isomorphism for some $p > 3$, if for every edge E the associated operator A_λ of the edge pencil has a trivial kernel for all λ with $0 < \Re\lambda < 1/3 + \varepsilon$.

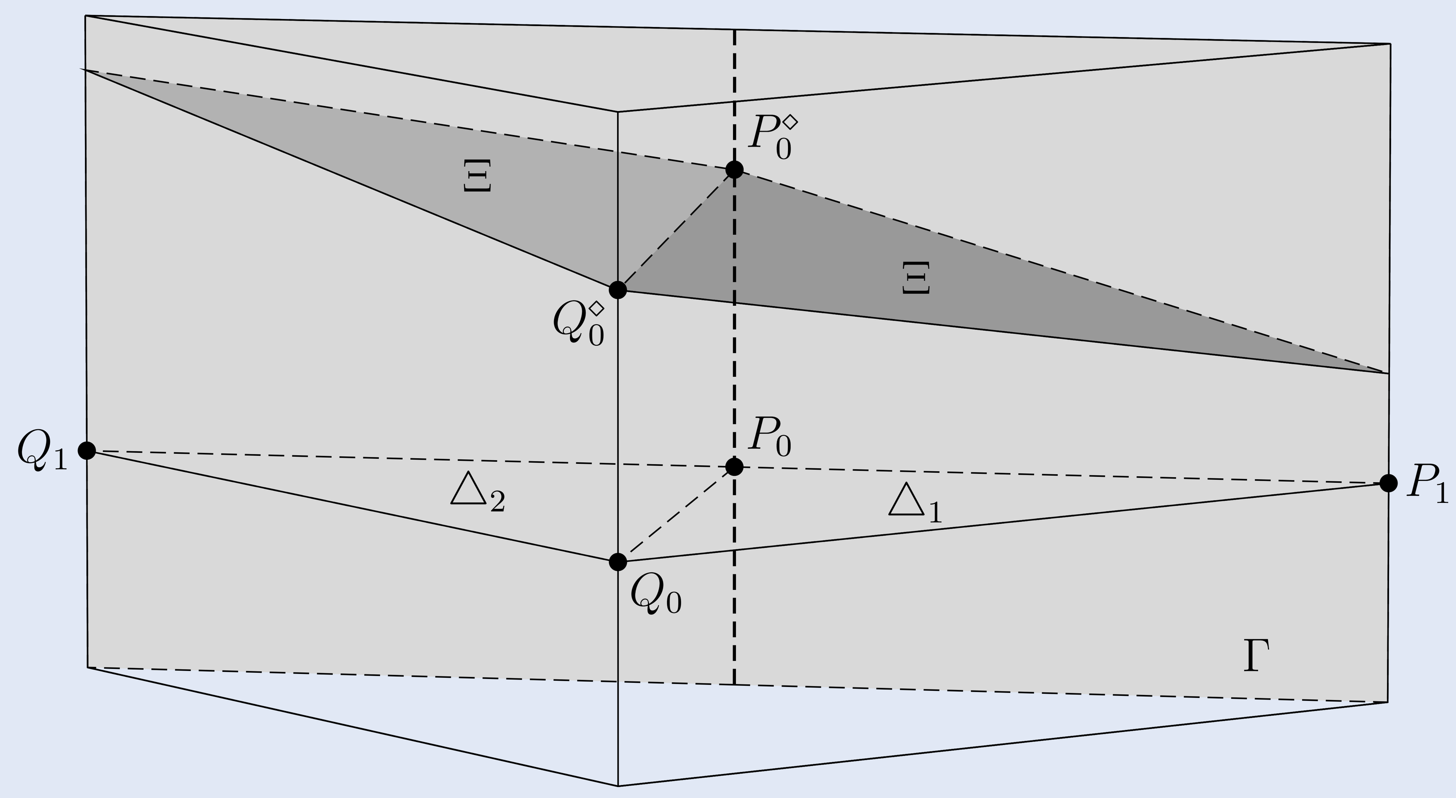
Proposition 2: (Bi–Lipschitz transformation) The regularity of the operator (1) is invariant under bi–Lipschitz transformations of the spatio–material constellation.

Proposition 3: (Material reflection) The regularity of the operator (1) inherits the regularity of the operator corresponding to the problem obtained by even material reflection of the spatio–material constellation at a Neumann face.

Two–dimensional constellation which generates the right prism under consideration, originally (left) and after piecewise linear transformation (right). The trace of the Neumann boundary face Γ is indicated by bold lines.

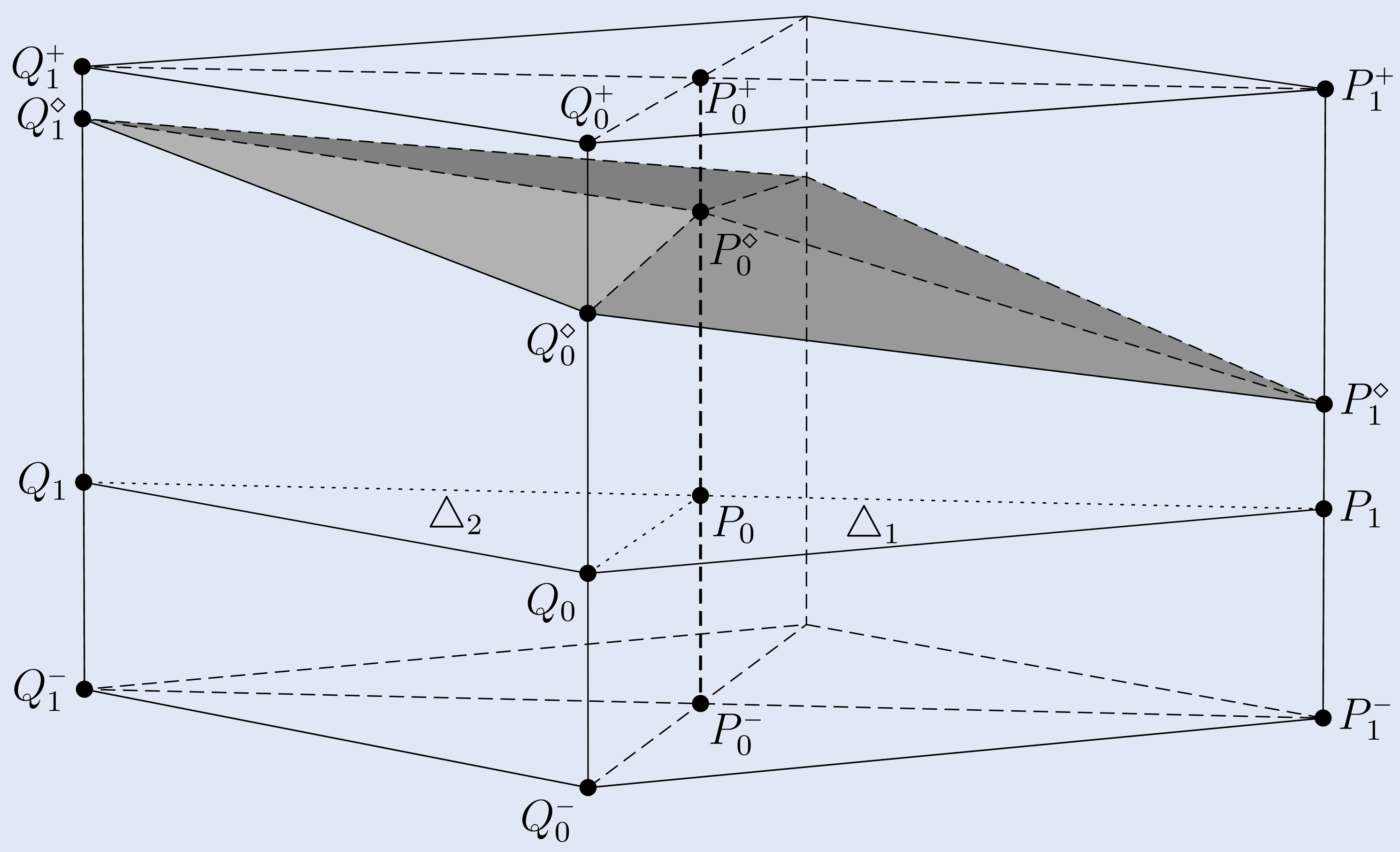


Bi–Lipschitz transformation ϕ (Proposition 2)



Spatio–material constellation after piecewise linear transformation. The Neumann boundary part Γ (fair–grey) is planar after transformation while the material interface Ξ (dark–grey) now is broken along the line through $P_0^\diamond$ and $Q_0^\diamond$. Thus, $\overline{P_0^\diamond Q_0^\diamond}$ is a four–material edge.

Material reflection at Γ (Proposition 3)



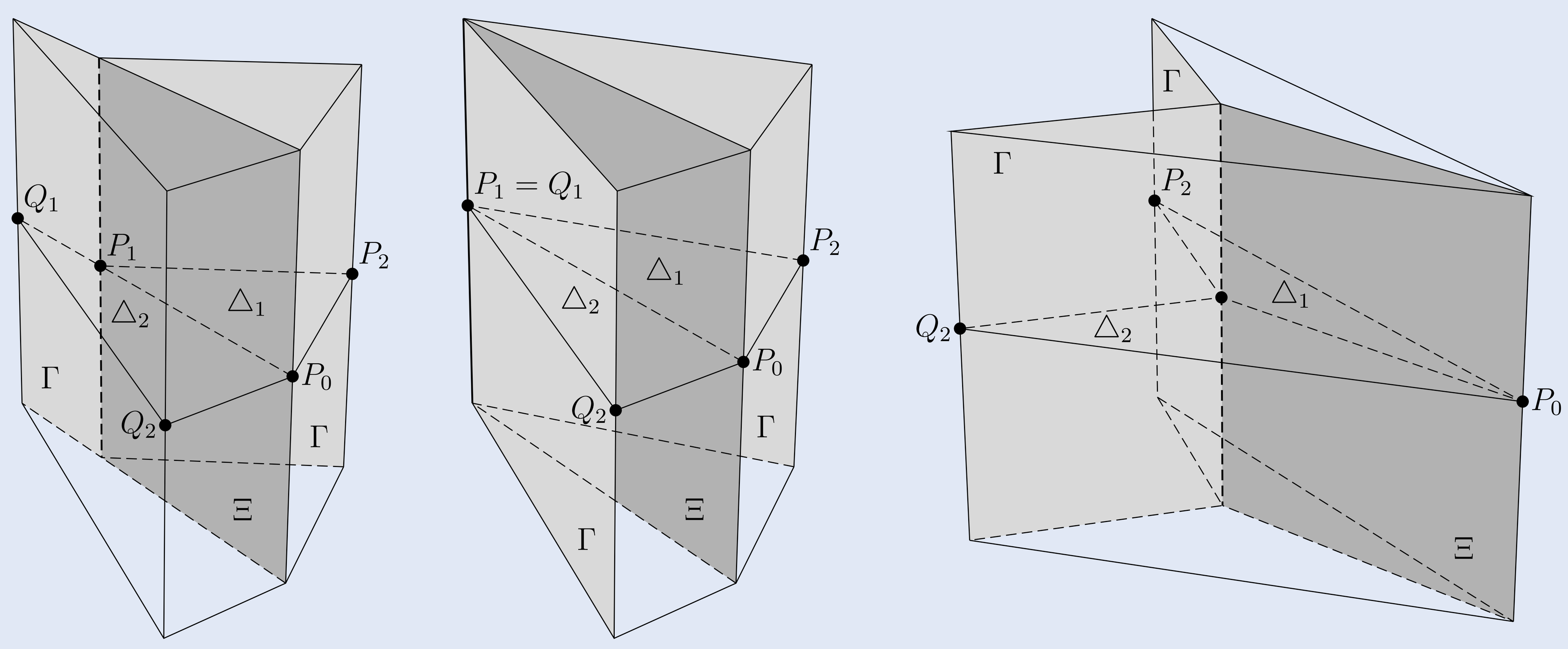
Spatio–material constellation after Bi–Lipschitz transformation and even reflection. The triangles $\triangle_1$ and $\triangle_2$ are the image of the generating two–dimensional constellation. All other lines in the figure represent edges to be discussed with respect to the regularity of the associated edge pencil. The shaded areas are the image of the original material interface.

Now one can show that for every edge E of the transformed constellation the associated edge pencil operator A_λ has a trivial kernel for all λ with $0 < \Re\lambda \le 1/2$ such that Proposition 1 applies.

Discussion of the edge singularities

- Mono–material outer edges like for instance $\overline{Q_1^- Q_0^-}$.
- Bi–material outer edges like for instance $\overline{Q_0^\diamond Q_0^+}$, $\overline{Q_0^+ P_0^+}$, $\overline{P_0^+ P_1^+}$, $\overline{P_1^+ P_1^\diamond}$, and $\overline{P_1^\diamond Q_0^\diamond}$.
- The four–material inner edges $\overline{P_0^\diamond P_0^+}$ and $\overline{P_0^- P_0^\diamond}$
- The four–material inner edges $\overline{P_0^\diamond P_1^\diamond}$ and $\overline{P_0^\diamond Q_1^\diamond}$, the image of the intersection of the original Neumann face and the original material interface.
- The four–material inner edge $\overline{P_0^\diamond Q_0^\diamond}$ and its reflection at the ϕ image of the original Neumann face.

For the following constellations the Theorem also holds true:



For details and references see *Optimal elliptic regularity at the crossing of a material interface and a Neumann boundary edge* by H.-Chr. Kaiser & J. Rehberg, WIAS–Preprint no. 1511, Berlin 2010.

contact: Hans-Christoph Kaiser · WIAS · Mohrenstr. 39, 10117 Berlin · +49 30 20372 455 · Hans-Christoph.Kaiser@wias-berlin.de · www.wias-berlin.de