Approximate Models for the Scattering by a Thin Periodic Layer

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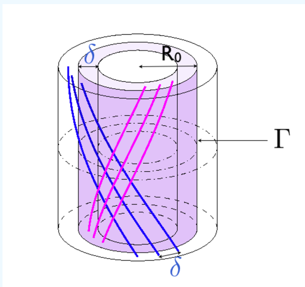
Context



- Thickness of the ring δ, Angular periodicity $\approx \delta$.
- $\lambda = \frac{2\pi}{\omega} \gg \delta$, $\delta \ll R_0$.
- **Difficulty**: **two different scales** δ, λ.

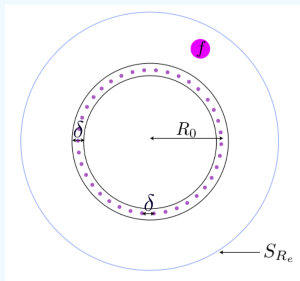
Goal: replacing the periodic ring by an approximate transmission condition across Γ.

Method:
- **Asymptotic expansion** of the solution with respect to the small parameter δ: matched asymptotic expansion / homogenisation (see [1], [2],[3]).
- **Construction of stable** approximate models using this expansion.

Two Dimensional Model Problem

Description

$$\begin{cases} \nabla\cdot(\mu^\delta \nabla u^\delta) + \omega^2 \rho^\delta u^\delta = f, \\ \partial_r u^\delta + i\omega u^\delta = 0 \text{ on } S_{R_e} \end{cases} \qquad \lambda = \frac{2\pi}{\omega} \gg \delta, \quad R_0 \gg \delta$$

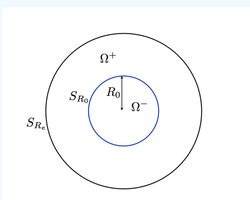


- $(\rho^\delta, \mu^\delta) = (\rho_\infty, \mu_\infty)$ outside the periodic ring, $(\rho^\delta, \mu^\delta)$ periodic in θ of periodicity $\frac{\delta}{R_0}$
- Thickness of the ring δ ($\approx$ periodicity of the ring)

Asymptotic Expansion

Far field:

$$(1) \quad u^\delta = \sum_{n\in\mathbb{N}} \delta^n u_n(r,\theta)$$

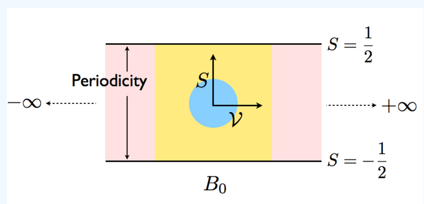


$$\mu_\infty \Delta u_n^\pm + \omega^2 \rho_\infty u_n^\pm = f\delta_0(n) \text{ in } \Omega^\pm$$

Near field:

$$(2) \quad u^\delta = \sum_{n\in\mathbb{N}} \delta^n U_n(\mathcal{V}, S; \theta)$$

$S = \frac{R_0\theta}{\delta}$, $\mathcal{V} = \frac{r-R_0}{\delta}$, U_n 1-periodic in S



$$\frac{\partial}{\partial S}\left(\mu \frac{\partial U_n}{\partial S}\right) + \frac{\partial}{\partial \mathcal{V}}\left(\mu \frac{\partial U_n}{\partial \mathcal{V}}\right) = \sum_{j=1}^{4} \mathcal{A}_j U_{n-j} \text{ in } B_0$$

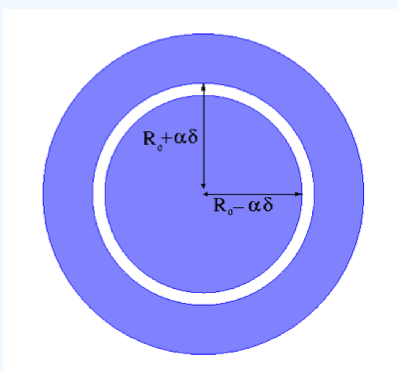
where $\mathcal{A}_j$ are differential operators in $S, \mathcal{V}, \theta$.

+ matching conditions: (1) and (2) coincide in an overlapping area.

Approximate Model

- **Main idea**: we want to find an approximate well-posed problem whose solution u_1^δ is close to the two first terms of the far field asymptotic expansion $u_0 + \delta u_1$.
- **Method**: we use the fact that $[u_1^\delta] \approx [u_0] + \delta\, [u_0]$.

$$\mathcal{P} \begin{cases} \Delta u_1^\delta + \frac{\omega^2 \rho_\infty}{\mu_\infty} u_1^\delta = f \text{ in } \Omega_{\alpha\delta}^\pm \\ \left[u_1^\delta\right]_\alpha = \delta A_0^\alpha \langle r \frac{\partial u_1^\delta}{\partial r}\rangle_\alpha \\ \left[r\frac{\partial u_1^\delta}{\partial r}\right]_\alpha = \delta\left(B_0^\alpha \langle u_1^\delta\rangle_\alpha + B_2^\alpha \langle \frac{\partial^2 u_1^\delta}{\partial \theta^2}\rangle_\alpha\right) \\ \partial_r u_1^\delta + i\omega u_1^\delta = 0 \quad \text{on } S_{R_e} \end{cases}$$



$\Omega_{\alpha\delta} = \Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-$

where $g_\alpha^\pm = g(R_0 \pm \alpha\delta)$, $[g]_\alpha = g_\alpha^+ - g_\alpha^-$, $\langle g\rangle_\alpha = \frac{1}{2}(g_\alpha^+ + g_\alpha^-)$ and

$$A_0^\alpha = \frac{2\alpha - 1}{R_0} + \int_{-1/2}^{1/2} \frac{W_0^0(S,\frac{1}{2}) - W_0^0(S,-\frac{1}{2})}{R_0} dS \qquad B_2^\alpha = \frac{1-2\alpha}{R_0} - \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{\mu}{\mu_\infty}\left(\frac{\partial V_1^1}{\partial S} + \frac{1}{R_0}\right)$$

$$B_0^\alpha = \frac{(1-2\alpha)\omega^2 R_0 \rho_\infty}{\mu_\infty} - \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{\omega^2 R_0 \rho}{\mu_\infty}$$

where V_1^1 and W_0^0 are solutions of Laplace canonical problems posed in B_0:

$$\begin{cases} \frac{\partial}{\partial S}(\mu \frac{\partial V_1^1}{\partial S}) + \frac{\partial}{\partial \mathcal{V}}(\mu \frac{\partial V_1^1}{\partial \mathcal{V}}) = -\frac{1}{R_0}\frac{\partial \mu}{\partial S} \\ V_1^1(R, Z) \sim C \text{ when } \mathcal{V} \to \pm\infty \end{cases} \qquad \begin{cases} \frac{\partial}{\partial S}(\mu \frac{\partial W_0^0}{\partial S}) + \frac{\partial}{\partial \mathcal{V}}(\mu \frac{\partial W_0^0}{\partial \mathcal{V}}) = 0 \\ \partial_{\mathcal{V}} W_0^0 \sim 1 \text{ when } \mathcal{V} \to \pm\infty \end{cases}$$

Proposition: *Problem $\mathcal{P}$ is well-posed as soon as $B_1^\alpha < 0$ and $A_0^\alpha > 0$, i.e when α exceeds a critical value $\alpha^\star > 0$. Moreover, for any $\alpha > \alpha^\star$, for any $\gamma > 0$, there exists $C(\alpha) > 0$, such that, for $\delta \le \delta_0$*

$$\| u^\delta - u_1^\delta \|_{H^1(\Omega_\gamma)} \le C(\alpha)\, \delta^2$$

Numerical Results

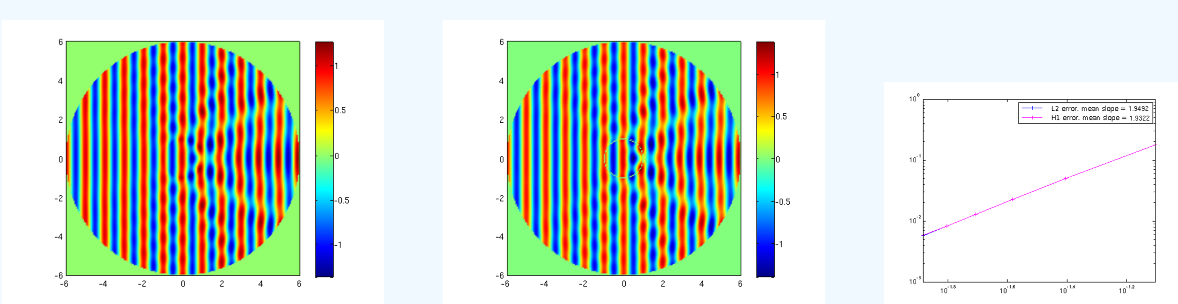
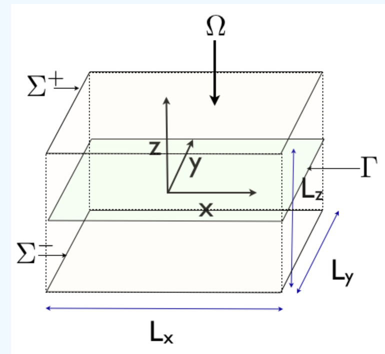


Figure 1: Scattering of a plane wave : 'exact' solution, approximate solution, convergence rate

3D Maxwell Problem

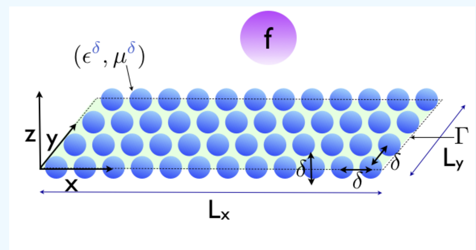
Description

$$\begin{cases} \text{curl}\left(\frac{1}{\mu^\delta}\text{curl}\mathbf{E}^\delta\right) - \omega^2 \epsilon^\delta \mathbf{E}^\delta = F \text{ in } \Omega \\ \mathbf{E}^\delta \ L_x\text{-periodic in } x \\ \mathbf{E}^\delta \ L_y\text{-periodic in } y \\ \frac{1}{i\omega\mu_\infty}\text{curl}\mathbf{E}^\delta \times n - \mathbf{E}_T^\delta = 0 \text{ on } \Sigma^\pm \end{cases}$$



$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3, |x| \le \tfrac{L_x}{2},\ |y| \le \tfrac{L_y}{2},\ |z| \le \tfrac{L_z}{2}\right\}$$

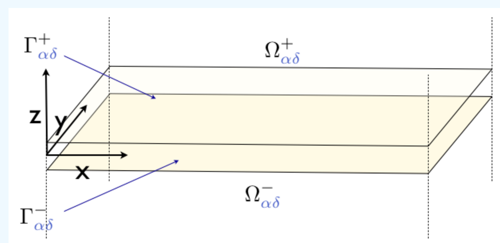
- $(\epsilon^\delta, \mu^\delta)$ δ-periodic in x and y
- $(\mu^\delta, \epsilon^\delta) = (\mu_\infty, \rho_\infty)$ if $|z| \ge 1/2$



Approximate Model

$$\mathcal{P}_1 \begin{cases} \text{curlcurl}\mathbf{E}_1^\delta - \omega^2 \mathbf{E}_1^\delta = F \text{ in } \Omega_{\alpha\delta}^\pm \\ \left[n \times \mathbf{E}_1^\delta\right]_\alpha + \delta\left(\frac{a^\alpha}{\omega^2}\, \vec{\text{curl}}_\Gamma \text{curl}_\Gamma \lambda - D_1^\alpha \lambda\right) = 0 \\ \left[n \times \text{curl}\mathbf{E}_1^\delta\right]_\alpha + \delta\left(b^\alpha\, \vec{\text{curl}}_\Gamma\text{curl}_\Gamma \langle (\mathbf{E}_1^\delta)_T\rangle_\alpha - \omega^2 D_2^\alpha \langle (\mathbf{E}_1^\delta)_T\rangle_\alpha\right) = 0 \\ n \times \text{curl}\mathbf{E}_1^\delta = -i\omega\mu_\infty \mathbf{E}_1^\delta{}_T \quad \text{on } \Sigma^\pm \\ \lambda = \langle \text{curl}(\mathbf{E}_1^\delta)_T\rangle \end{cases}$$

For α large enough,
- D_1^α and D_2^α are positive diagonal matrices.
- a^α, b^α are positive constants.



Existence, Uniqueness, and Stability

Operator G: $G : \begin{cases} H^{-1/2}(\text{div}_\Gamma, \Gamma) \to H_t(\text{curl}_\Gamma, \Gamma) \\ f \mapsto \lambda \text{ such that } a^\alpha\, \vec{\text{curl}}_\Gamma \text{curl}_\Gamma \lambda - \omega^2 D_1^\alpha \lambda = f \end{cases}$

➜ Assumption : ω is such that G is well defined.

Variational formulation: $X := \left\{\mathbf{E} \in H_{\text{per}}(\text{curl}, \Omega_{\alpha\delta}^\pm), \langle \mathbf{E}_T\rangle_\alpha \in H(\text{curl}_\Gamma, \Gamma),\ \mathbf{E}_t \in L_t^2(\Sigma^\pm)\right\}$

$$(\mathcal{P}_1) \quad \Leftrightarrow \quad a^+(\mathbf{E}_1^\delta, \varphi) + b(\mathbf{E}_1^\delta, \varphi) = \int_{\Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-} F \cdot \bar{\varphi}\, dx \quad \forall \varphi \in X$$

$$a^+(E,\varphi) := \int_{\Omega_{\alpha\delta}^+\cup\Omega_{\alpha\delta}^-} \text{curl}E \cdot \overline{\text{curl}\varphi}\, dx + \delta\, b^\alpha \int_\Gamma \text{curl}_\Gamma \langle E_T\rangle_\alpha \cdot \overline{\text{curl}_\Gamma \langle\varphi\rangle_\alpha}\, ds - i\omega \int_{\Sigma^+\cup\Sigma^-} E_T \cdot \overline{\varphi_T}\, ds$$

$$b(E,\varphi) := -\omega^2 \int_{\Omega_{\alpha\delta}^+\cup\Omega_{\alpha\delta}^-} E\cdot\bar{\varphi}\, dx - \delta\,\omega^2 \int_\Gamma D_2^\alpha \langle E_T\rangle_\alpha \cdot \overline{\langle\varphi\rangle_\alpha}\, ds + \frac{\omega^2}{\delta}\langle G([n\times E]), [n\times\varphi]\rangle$$

Proposition: *$\mathcal{P}_1$ is well posed. Moreover, there exist $C > 0$ and $\delta_0 > 0$ such that, for any $\delta \le \delta_0$,*

$$\left\|\mathbf{E}_1^\delta\right\|_{H(\text{curl},\Omega_{\alpha\delta}^+)} + \left\|\mathbf{E}_1^\delta\right\|_{H(\text{curl},\Omega_{\alpha\delta}^-)} \le C\left(\|F\|_{L^2(\Omega_{\alpha\delta}^+)} + \|F\|_{L^2(\Omega_{\alpha\delta}^-)}\right)$$

Main ideas of the proof (based on the ideas of [4])

- Helmholtz decomposition: $X = X_0 \oplus \nabla S$

$$S := \left\{p \in H_{per}^1(\Omega_{\alpha\delta}^+) \cap H_{per}^1(\Omega_{\alpha\delta}^-), \langle p\rangle_\alpha \in H_{per}^1(\Gamma), p_{|\Sigma^\pm} = 0\right\}$$

$$X_0 := \{E \in X, \text{div}E = 0 \in \Omega_{\alpha\delta}^\pm,\ \omega^2\langle E\cdot n\rangle_\alpha = -\frac{\omega^2}{\delta}\text{div}_\Gamma(n\times G([n\times E]_\alpha)), \quad -[n\cdot E]_\alpha = \delta\, \text{div}_\Gamma D_2^\alpha\langle E_T\rangle_\alpha \}$$

- Uniform estimate: proof by contradiction $\Rightarrow$ uniqueness.
- Existence: compactness of b on X_0, Fredholm Alternative.

References

[1] M. Van Dyke. *perturbation methods in fluid mechanics*. Academic press, 1964.

[2] G. Allaire. *Shape Optimization by the Homogenization Method*. Springer.

[3] Yves Achdou. Etude de la réflexion d'une onde électromagnétique par un métal recouvert d'un revêtement métallisé. Technical report, INRIA, 1989.

[4] Peter Monk. *Finite Element Methods for Maxwell's Equations*. Oxford science publications, 2003.