Singular behavior of the solution of the heat equation in weighted L^p-Sobolev spaces

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Problem

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\begin{array}{rll}
\partial_t u - \Delta u = h(x,t), & \text{in } \Omega \times]-\pi,\pi[, & \\
u = 0, & \text{on } \partial\Omega \times [-\pi,\pi], & (1)\\
u(\cdot,-\pi) = u(\cdot,\pi), & \text{in } \Omega, &
\end{array}
$$

$\Omega \subset \mathbb{R}^2$ polygonal domain. For $j = 1, \cdots, J$, denote by S_j, the vertices of $\partial\Omega$ enumerated clockwise, ψ_j the interior angle of Ω at the vertex S_j, $\lambda_j = \frac{\pi}{\psi_j}$ and (r_j, θ_j) the polar coordinates centered at S_j.

$h \in L^p(-\pi,\pi; L^p_\mu(\Omega))$ with

$$L^p_{\vec{\mu}}(\Omega) = \{f \in L^p_{loc}(\Omega) \mid wf \in L^p(\Omega)\}$$

where

$w = r_j^{\mu_j}$ on $D_j(1/2)$ and $w = 1$ on $\Omega \backslash \sup_{j=1}^{J} D_j(1)$.

First Strategy (Da Prato-Grisvard 1975)

Let E be a Banach space,
$A : D(A) \subset E \to E$, $B : D(B) \subset E \to E$ be closed linear densely defined operators and

$$L : D(L) := D(A) \cap D(B) \to E : x \mapsto Ax + Bx.$$

$(H_1)\, \exists\ M, R \geq 0$, $\theta_A, \theta_B \in]0, \pi]$ such that

$$\theta_A + \theta_B > \pi,$$
$$S_A := \{\lambda \mid |\lambda| \geq R, |\arg \lambda| \leq \theta_A\} \subset \rho(-A),$$
$$S_B := \{\lambda \mid |\lambda| \geq R, |\arg \lambda| \leq \theta_B\} \subset \rho(-B),$$

and, for all $\lambda \in S_A$ and all $\mu \in S_B$,

$$\|(A + \lambda\, I)^{-1}\| \leq \frac{M}{|\lambda|},\ \|(B + \mu\, I)^{-1}\| \leq \frac{M}{|\mu|};$$

$(H_2)\, \sigma(-A) \cap \sigma(B) = \emptyset$;
$(H_3)\, \forall\ \lambda \in \rho(-A)$ and $\forall\ \mu \in \rho(-B)$,

$$(A+\lambda\, I)^{-1}(B+\mu\, I)^{-1} = (B+\mu\, I)^{-1}(A+\lambda\, I)^{-1}.$$

Then L has an invertible closure $\bar{L}$.

Definition $\bar{L}$ is defined by $x \in D(\bar{L})$ and $\bar{L}x = y$ if $\exists\ (x_n)_n \subset D(L)$ s.t. $x_n \to x$ and $Lx_n \to y$.

A solution of $\bar{L}x = y$ is called a strong solution of $Lx = y$.

The inverse of $\bar{L}$ is given by

$$(\bar{L})^{-1} = \frac{1}{2i\pi} \int_\gamma (A + \lambda I)^{-1} (\lambda I - B)^{-1} d\lambda,$$

where γ is a path which separates $\sigma(-A)$ and $\sigma(B)$ and joins $\infty e^{-i\theta_\gamma}$ to $\infty e^{i\theta_\gamma}$ where θ_γ is chosen so that $\pi - \theta_B < \theta_\gamma < \theta_A$.

$$h(x,t) = g_1(x) + g(x,t) \text{ with } \int_{-\pi}^{\pi} g(x,t)\,dt = 0.$$

u is sol. of (1) $\Leftrightarrow u(x,t) = \bar{u}(x) + v(x,t)$ with

$$\begin{aligned} -\Delta \bar{u} &= g_1(x), && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned}$$

$$\begin{aligned} \partial_t v - \Delta v &= g(x,t), && \text{in } \Omega \times]-\pi, \pi[, \\ v &= 0, && \text{on } \partial\Omega \times [-\pi, \pi], \\ v(\cdot, -\pi) &= v(\cdot, \pi), && \text{in } \Omega, \\ \int_{-\pi}^{\pi} v(x,t)\,dt &= 0, && \text{for all } x \in \Omega. \end{aligned}$$

$\bar{u}$ admits the decomposition

$$\bar{u} = \bar{u}_R + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N}, \lambda'_j = k\lambda_j}} \bar{c}_{\lambda'_j}\, r^{\lambda'_j} \sin(\lambda'_j \theta)$$

with $\bar{u}_R \in V^{2,p}_{\vec{\mu}}(\Omega)$,

$\|\bar{u}_R\|_{V^{2,p}_{\vec{\mu}}(\Omega)} \lesssim \|g_1\|_{L^p_{\vec{\mu}}(\Omega)}$ and $|\bar{c}_{\lambda'_j}| \lesssim \|g_1\|_{L^p_{\vec{\mu}}(\Omega)}$.

Application of the First Strategy to

$$\begin{aligned} E &= L^p_m(I; L^p_{\vec{\mu}}(\Omega)) \\ &= \{h \in L^p(I; L^p_{\vec{\mu}}(\Omega)) \mid \int_{-\pi}^{\pi} h(x,t)\, dt = 0\}. \end{aligned}$$

$$A : D(A) \subset E \to E : u \mapsto -\Delta u, \quad \text{with}$$
$$D(A) = L^p_m(I; D(\Delta_{p,\vec{\mu}})) \text{ where}$$
$$D(\Delta_{p,\vec{\mu}}) = \{u \in H^1_0(\Omega) \,|\, \Delta u \in L^p_{\vec{\mu}}(\Omega)\},$$

$$B_0 : D(B_0) \subset E \to E : u \mapsto \partial_t u, \quad \text{with}$$
$$D(B_0) = \{u \in E \mid \partial_t u \in L^p(I; L^p_{\vec{\mu}}(\Omega)), \\ u(\cdot, -\pi) = u(\cdot, \pi)\}.$$

$\sigma(-A) = \{-\nu_k \mid k \in \mathbb{N}\}$ with $-\nu_k \le -\nu_0 < 0$ and by Serge's talk, for all $\lambda > 0$,

$$\|(A + \lambda I)^{-1}\| \le \frac{1}{\lambda}.$$

$\sigma(B_0) = i\mathbb{Z}^*$. Moreover, $\forall \theta_B < \frac{\pi}{2}$, $\exists M \geq 0$ s.t., $\forall \mu \in S_{B_0} = \{\mu \in \mathbb{C} \mid |\arg(\mu)| \leq \theta_B\}$,

$$\|(B_0 + \mu\, I)^{-1}\| \leq \frac{M}{|\mu|}.$$

We just have to multiply the equation

$$\partial_t u(x,t) + \mu\, u(x,t) = f(x,t), \text{in } \Omega \times]-\pi,\pi[,$$

by $v := w^p |u|^{p-2}\overline{u}$ and integrating.

The condition (H_3) is satisfied as the variables are separate in these two operators.

Hence the operator $A + B_0$ has an inverse closure.

Moreover we have

$$v = \frac{1}{2\pi i}\int_\gamma (A + z\, I)^{-1}(z\, I - B_0)^{-1} g\, dz,$$

with $\gamma : \mathbb{R} \to \mathbb{C}$ defined for example by

$$\begin{aligned} \gamma(z) &= |z|\, e^{-i(\frac{\pi}{2}+\delta)}, && \text{for } z < -1, \\ &= \tfrac{1}{2}\, e^{i(\frac{\pi}{2}+\delta)z}, && \text{for } z \in [-1,1], \\ &= |z|\, e^{i(\frac{\pi}{2}+\delta)}, && \text{for } z > 1. \end{aligned}$$

By Serge's talk and

$$v = \frac{1}{2\pi i}\int_\gamma (A + z\,I)^{-1}(z\,I - B_0)^{-1} g\,dz,$$

we have the decomposition

$$v = v_R + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda_j'<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N}, \lambda_j'=k\lambda_j}} v_{\lambda_j'}$$

with

$$v_R(x,t) = \frac{1}{2\pi i}\int_\gamma R(z)(z\,I - B_0)^{-1} g\,dz$$

$$v_{\lambda_j'}(x,t) = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda_j'}(z), (z\,I - B_0)^{-1} g\right\rangle P_{j,\lambda_j'}(r\sqrt{z})e^{-r\sqrt{z}}\, r^{\lambda_j'} \sin(\lambda_j'\theta)\,dz.$$

Structure of $v_{\lambda'_j}$.

Let $\sigma_j := 1 - \frac{1}{p} - \frac{\mu_j + \lambda'_j}{2}$. Under the assumptions of Serge's talk, $\forall s \sim 0$, $g \in W_m^{s,p}(I, L^p_\mu(\Omega))$, $\exists \tilde{q}_{\lambda'_j} \in W_m^{s+\sigma_j,p}(I)$ and $\tilde{E}_{\lambda'_j}$ s.t.

$$v_{\lambda'_j} = (\tilde{E}_{\lambda'_j} *_t \tilde{q}_{\lambda'_j})\, r^{\lambda'_j} \sin(\lambda'_j \theta).$$

$$\tilde{q}_{\lambda'_j}(t) = \frac{1}{2\pi i} \int_{\gamma_3} \left\langle T_{\lambda'_j}(z), (z\, I - B_0)^{-1} g \right\rangle dz,$$

$$\tilde{E}_{\lambda'_j}(x,t) = \sum_{k \in \mathbb{Z}^*} e^{ikt} P_{j,\lambda'_j}(r\sqrt{ik})\, e^{-r\sqrt{ik}},$$

where $\gamma_3 : \mathbb{R} \to \mathbb{C}$ is defined by

$$\begin{aligned} \gamma_3(z) &= |z| e^{-i(\frac{\pi}{2}+\delta)}, \quad \text{for } z \leq 0, \\ &= |z| e^{i(\frac{\pi}{2}+\delta)}, \quad \text{for } z > 0. \end{aligned}$$

Ideas of the proof By partial Fourier serie in t and Cauchy representation formula.

Regularity of $u_{\lambda'_j}$.

Let $\sigma_j = 1 - \frac{1}{p} - \frac{\mu_j + \lambda'_j}{2}$. Under the assumptions of Serge's talk, $\forall h \in L^p(I, L^p_{\vec{\mu}}(\Omega))$, the problem (1) has a unique strong solution u with

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k \in \mathbb{N}, \lambda'_j = k\lambda_j}} u_{\lambda'_j}$$

and

$$u_R(x,t) = \frac{1}{2\pi i} \int_\gamma R(z)(z\,I - B_0)^{-1} \tilde{h}\, dz + \bar{u}_R(x)$$

$$u_{\lambda'_j}(x,t) = (E_{\lambda'_j} *_t q_{\lambda'_j})\, r^{\lambda'_j} \sin(\lambda'_j \theta)$$

$\tilde{h} = h - \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\cdot, s)\, ds$, $q_{\lambda'_j} \in W^{\sigma_j,p}(I)$ and $E_{\lambda'_j}$ verifying

$$\begin{aligned} E_{\lambda'_j}(x,t) &= \tilde{E}_{\lambda'_j}(x,t) + \tfrac{1}{2\pi} \\ &= \sum_{k \in \mathbb{Z}^*} e^{ikt} P_{j,\lambda'_j}(r\sqrt{ik})\, e^{-r\sqrt{ik}} + \frac{1}{2\pi}. \end{aligned}$$

Proof by interpolation.

To consider the regularity of u_R, observe that

$$\partial_t u_R + u_R - \Delta u_R = h + u_R - \sum_{j=1}^{J} \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N}, \lambda'_j = k\lambda_j}} (\partial_t(\eta_j u_{\lambda'_j}) - \Delta(\eta_j u_{\lambda'_j}))$$

Case $P_{j,\lambda'_j} \equiv 1$. Take $f = \eta_j(\frac{\partial}{\partial t} - \Delta)u_{\lambda'_j}$. Let $A'_j = 2\lambda'_j + 1$. The Fourier series in t give

$$\widehat{f}_k = \widehat{q}_{\lambda'_j}(k)\, A'_j \sqrt{ik}\, e^{-r\sqrt{ik}}\, r^{\lambda'_j - 1} \sin(\lambda'_j\theta)\, \eta_j(r),$$

$H(r,t) = \sum_{k\in\mathbb{Z}} \sqrt{ik}\, e^{-r\sqrt{ik}}\, e^{ikt}$ is s.t.

$$|H(r,t)| \lesssim 1 + \frac{1}{(r^2 + |t|)^{3/2}}.$$

Hence

$$f = (H *_t q_{\lambda'_j})\, A'_j\, r^{\lambda'_j - 1} \sin(\lambda'_j\theta)\, \eta_j(r) \in L^p(I; L^p_{\vec{\mu}}(\Omega))$$

Regularity of u_R

Recall that, by Serge's talk, we have

$$R(z) : L^p_{\vec{\mu}}(\Omega) \to V^{2,p}_{\vec{\mu}}(\Omega) : g \mapsto u_R$$

satisfies $\exists K > 0$ s.t., $\forall z \in \pi^+ \cup S_A$,

$$\|R(z)\|_{L^p_{\vec{\mu}} \to V^{2,p}_{\vec{\mu}}} + (1 + |z|) \|R(z)\|_{L^p_{\vec{\mu}} \to L^p_{\vec{\mu}}} \leq K.$$

Hence by interpolation, for all $\theta \in]0, 1[$,

$$\begin{aligned} u_R(x,t) &= \frac{1}{2\pi i} \int_\gamma R(z)(z\, I - B_0)^{-1} \tilde{h}\, dz + \bar{u}_R(x) \\ &\in L^p(I; (L^p_{\vec{\mu}}, V^{2,p}_{\vec{\mu}})_\theta). \end{aligned}$$

Let us show that

$$u_R \in L^p(I; V^{2,p}_{\vec{\mu}}(\Omega)) \cap W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)).$$

u_R is a strong solution of

$$\partial_t u_R + u_R - \Delta u_R = h_R \in L^p(I; L^p_{\vec{\mu}}(\Omega)).$$

Second Strategy (Dore-Venni 1987)

Let E be a Banach space,
$A : D(A) \subset E \to E$, $B : D(B) \subset E \to E$ be closed linear densely defined operators and

$$L : D(L) := D(A) \cap D(B) \to E : x \mapsto Ax + Bx.$$

(H_4) E is a U.M.D. space;

(H_5) $]-\infty, 0] \subset \rho(A) \cap \rho(B)$ and $\exists M \geq 0$, $\forall t \geq 0$,

$$\|(A + t\,I)^{-1}\| \leq \frac{M}{t+1},\ \|(B + t\,I)^{-1}\| \leq \frac{M}{t+1};$$

(H_6) $\forall s \in \mathbb{R}$, A^{is} and $B^{is} \in \mathcal{L}(E)$ and $\exists K > 0$, $\tau_A > 0$, $\tau_B > 0$ s.t.

$$\tau_A + \tau_B < \pi,$$

and, $\forall s \in \mathbb{R}$,

$$\|A^{is}\| \leq K e^{|s|\tau_A},\ \|B^{is}\| \leq K e^{|s|\tau_B}.$$

Under assumptions (H_3), (H_4), (H_5) and (H_6), the operator L is invertible.

We apply the second strategy with

$$E = L^p(I; L^p_{\vec{\mu}}(\Omega)),$$
$$A : D(A) \subset E \to E : u \mapsto -\Delta u, \text{ with}$$
$$D(A) = L^p(I; D(\Delta_{p,\vec{\mu}})),$$
$$B : D(B) \subset E \to E : u \mapsto \partial_t u + u, \text{ with}$$
$$D(B) = W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)).$$

Verification of (H_6). By Coifman - Weiss (1976)
If $-A$ is the infinitesimal generator of a strongly continuous contraction semi-group in E which preserves the positivity, then $\exists K > 0$, $\forall s \in \mathbb{R}$,

$$\|A^{is}\| \leq K(1 + |s|)\, e^{\frac{\pi}{2}|s|}.$$

By max. principle, A satisfies this condition.

$-A$ is symmetric on $L^2(I, L^2(\Omega))$. Hence

$$\|A^{is}\|_{L^2(I,L^2(\Omega))} \leq 1.$$

By interpolation

$$\|A^{is}\|_{L^p(I,L^p_{\vec{\mu}}(\Omega))} = 0(e^{\tau_A|s|}), \text{ with } \tau_A < \pi/2.$$

For B, $\sigma(-B) = \{-(ki+1) \mid k \in \mathbb{Z}\}$, $\forall \lambda \in \mathbb{R}^+$,

$$\|(\lambda I + B)^{-1}\| \leq \frac{1}{\lambda + 1},$$

and $-B$ is the generator of a C_0 semigroup $S(t)$ of contraction. Moreover $S(t)$ preserves the positivity and hence $\exists \tau_B \in]\pi/2, \pi - \tau_A[$ s.t.

$$\|B^{is}\| = 0(e^{\tau_B |s|}).$$

By the second strategy, $\exists w_R \in W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)) \cap L^p(I; D(\Delta_{p,\vec{\mu}}))$ solution of

$$\begin{array}{cl} \partial_t w + w - \Delta w = h_R, & \text{in } \Omega \times]-\pi, \pi[, \\ w = 0, & \text{on } \partial\Omega \times [-\pi, \pi], \\ w(\cdot, -\pi) = w(\cdot, \pi), & \text{in } \Omega. \end{array}$$

Hence

$$u_R \in W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}) \cap L^p(I; D(\Delta_{p,\vec{\mu}})) \cap L^p(I; (L^p_{\vec{\mu}}, V^{2,p}_{\vec{\mu}})_\theta).$$

$$D(\Delta_{p,\mu}) \cap (L^p_{\vec{\mu}}, V^{2,p}_{\vec{\mu}})_\theta \subset V^{2,p}_{\vec{\mu}}(\Omega)$$

If $u \in D(\Delta_{p,\mu})$ then u admits the decomposition

$$u_R = u_1 + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N},\lambda'_j=k\lambda_j}} c_{\lambda'_j}\, r^{\lambda'_j} \sin(\lambda'_j\theta).$$

Moreover

$$V^{2,p}_{\vec{\mu}}(\Omega) \to W^{2,p}(\Omega) : u \mapsto w\, u$$

as well as

$$L^p_{\vec{\mu}}(\Omega) \to L^p(\Omega) : u \mapsto w\, u$$

are continuous. Hence,

$$u_R \in (L^p_{\vec{\mu}}, V^{2,p}_{\vec{\mu}})_\theta \Rightarrow w\, u_R \in (L^p, W^{2,p})_\theta = W^{2\theta,p}.$$

As $\mu_j + \lambda'_j < 2 - \frac{2}{p}$, for $\theta \sim 1$, $r^{\mu_j+\lambda'_j} \sin(\lambda'_j\theta) \notin W^{2\theta,p}(D_j)$ and $u_R = u_1 \in V^{2,p}_{\vec{\mu}}(\Omega)$.

Theorem

Let $\vec{\mu}$ satisfies, for all $j = 1, \ldots, J$,

$$-\lambda_j < \mu_j < \tfrac{2p-2}{p},$$
$$4(p-1)\lambda_j^2 + 2\mu_j p - \mu_j^2 p^2 > 0$$

and, for all $k \in \mathbb{Z}^*$ and all $j \in \{1, 2, \cdots, J\}$, $2 - \frac{2}{p} - \mu_j \neq k\lambda_j$ and $\mu_j + k\lambda_j \neq 1$.

Then, $\forall h \in L^p(I; L^p_{\vec{\mu}}(\Omega))$, $\exists! u \in L^p(I; L^p_{\vec{\mu}}(\Omega))$ solution of

$$\begin{array}{cl} \partial_t u - \Delta u = h(x,t), & \text{in } \Omega \times]-\pi, \pi[, \\ u = 0, & \text{on } \partial\Omega \times [-\pi, \pi], \\ u(\cdot, -\pi) = u(\cdot, \pi), & \text{in } \Omega. \end{array}$$

Moreover u admits the decomposition

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N}, \lambda'_j = k\lambda_j}} u_{\lambda'_j},$$

with

$$u_R \in L^p(I; V^{2,p}_{\vec{\mu}}(\Omega)) \cap W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega))$$

and

$$u_{\lambda'_j} = (E_{\lambda'_j} *_t q_{\lambda'_j}) r^{\lambda'_j} \sin(\lambda'_j \theta)$$

where

$$q_{\lambda'_j} \in W^{\sigma_j,p}(I) \text{ with } \sigma_j = -\frac{\mu_j + \lambda'_j}{2} + 1 - \frac{1}{p}$$

and

$$E_{\lambda'_j}(x,t) = \sum_{k\in\mathbb{Z}^*} e^{ikt} P_{j,\lambda'_j}(r\sqrt{ik})\, e^{-r\sqrt{ik}} + \frac{1}{2\pi}.$$