Weighted analytic regularity in corner domains: 3D polyhedra

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Outline

1. **Hierarchy of singular points and analytic estimates**
   - Hierarchy of points
   - Techniques for analytic estimates
2. **Edges**
   - Isotropic spaces and estimates
   - Anisotropic spaces and estimates
3. **3D polyhedra**
   - Neighborhoods
   - Weighted spaces
   - Analytic regularity

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Hierarchy of points in polyhedra

1. Inner point
2. Smooth boundary point
3. Regular conical point (2D corner; absent on 3D polyhedra; however...)
4. Smooth edge point
5. Polyhedral corner point

History of analytic regularity:

1-2: Morrey-Nirenberg 1957

3-4: Babuška-Guo 1988-1997 (4 only partially)

4-5: CDN 2010

Note: In CDN 2010, 1-3:12 pages, 4-5: 30 pages

Why is this so difficult?

Techniques for analytic estimates

1. Nested Open Sets:
   Consists of
   - Basic (H^2 or H^1) a priori estimate between $\mathcal{V}$ and $\mathcal{V}' \supset\supset \mathcal{V}$
   - Derivatives
   - Nested open sets (ρ-estimates)

   Used for
   - Translation-invariant situation,
     neighborhood of interior and smooth boundary points
2. Dyadic partition:
   Consists of
   - Analytic estimate for smooth case
   - Scaling with powers of 2
   - Covering by dyadic partition

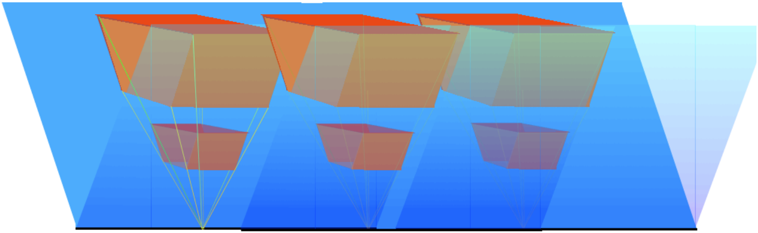
   Used for
   - Dilation-invariant situation,
     neighborhood of regular conical points
3. That's all, nothing else to see !

Outline

Edge points are conical points

Model edge $\boldsymbol{e}$: $\boldsymbol{e} = \mathcal{K} \times \mathbb{R}$,
$\mathcal{K}$ 2D sector, $\boldsymbol{x} = (\boldsymbol{x}_\perp, x_3)$, $\boldsymbol{x}_\perp \in \mathcal{K}$, $x_3 \in \mathbb{R}$, $r := |\boldsymbol{x}_\perp|$.
Dyadic partition technique, starting from analytic estimate between
$\mathcal{V} = \{\frac{1}{4} < r < 1, |x_3| < \frac{1}{2}\}$ and $\mathcal{V}' = \{\frac{1}{4} - \varepsilon < r < 1 + \varepsilon, |x_3| < \frac{1}{2} + \varepsilon\}$.

$\mathcal{V}$ and $\mathcal{V}'$ have smooth boundary components,
therefore the analytic estimates follow from the smooth case



$$\mathcal{V}_{\mu,\nu} = 2^{-\mu}\big(\mathcal{V} + (0,0,\nu/2)\big) \Rightarrow \mathcal{W} = \bigcup_{\mu\in\mathbb{N}, |\nu| < 2^{\mu+1}} \mathcal{V}_{\mu,\nu} = \{r < 1, |z_3| < 1\}$$

Edge points are conical points

Definition (Isotropic weighted Sobolev spaces)

$$K^m_\beta(\mathcal{W}) = \{u \;:\; r^{\beta+|\alpha|}\partial^\alpha_{\boldsymbol{x}} u \in L^2(\mathcal{W}), \quad \forall\alpha,\ |\alpha| \le m\} \text{ (homogeneous)}$$

$$J^m_\beta(\mathcal{W}) = \{u \;:\; r^{\beta+m}\partial^\alpha_{\boldsymbol{x}} u \in L^2(\mathcal{W}), \quad \forall\alpha,\ |\alpha| \le m\} \text{ (non-homogeneous)}$$

Remark: Equivalent "step-weighted" norms in J^m_β

If $\beta + m + 1 > 0$, choose $\gamma \in (-\beta - 1, m]$. Then

$$\begin{aligned}\|u\|^2_{J^m_\beta(\mathcal{W})} &= \sum_{|\alpha|\le m} \|r^{\beta+m}\partial^\alpha_{\boldsymbol{x}} u\|^2_{\mathcal{W}} \\ &\sim \sum_{|\alpha|\le\gamma} \|r^{\beta+\gamma}\partial^\alpha_{\boldsymbol{x}} u\|^2_{\mathcal{W}} + \sum_{\gamma<|\alpha|\le m} \|r^{\beta+|\alpha|}\partial^\alpha_{\boldsymbol{x}} u\|^2_{\mathcal{W}} \\ &\sim \sum_{|\alpha|\le m} \|r^{(\beta+|\alpha|)_+}\partial^\alpha_{\boldsymbol{x}} u\|^2_{\mathcal{W}} \quad \text{if } \beta + m > 0\end{aligned}$$

Edge points are conical points

Theorem

Let u be a solution of the boundary value problem in $\mathcal{W}'$
(Linear, second order, constant coefficient, right hand side f, zero boundary data).
(i) For all $\beta \in \mathbb{R}$, $n \in \mathbb{N}$:
If $u \in \mathrm{K}^1_\beta(\mathcal{W}_\varepsilon)$ and $f \in \mathrm{K}^n_{\beta+2}(\mathcal{W}')$ then $u \in \mathrm{K}^{n+2}_\beta(\mathcal{W})$.
$\forall\, 0 \le k \le n+2$:

$$\frac{1}{k!} \sum_{|\alpha|=k} \|r^{\beta+|\alpha|}\partial^\alpha_{\boldsymbol{x}} u\|_{\mathcal{W}} \le C^{k+1} \Big\{ \sum_{\ell=0}^{k-2} \frac{1}{\ell!} \sum_{|\alpha|=\ell} \|r^{\beta+2+|\alpha|}\partial^\alpha_{\boldsymbol{x}} f\|_{\mathcal{W}'} + \sum_{|\alpha|\le 1} \|r^{\beta+|\alpha|}\partial^\alpha_{\boldsymbol{x}} u\|_{\mathcal{W}'} \Big\}$$

(ii) Let $m \ge 1$ and $\beta + m > -1$. Let $n \ge m-1$.
If $u \in \mathrm{J}^m_\beta(\mathcal{W}')$ and $f \in \mathrm{J}^n_{\beta+2}(\mathcal{W}')$, then $u \in \mathrm{J}^{n+2}_\beta(\mathcal{W})$ and there are the corresponding Cauchy-type analytic estimates.

What's wrong with this?

Edge singularities

Typical member of $\mathsf{K}^m_\beta(\mathcal{W})$: Principal singularity

$$u(\boldsymbol{x}) = a(x_3)\, r^\lambda\, \psi(\theta), \quad (a \in \mathsf{H}^m,\ \psi \text{ smooth, } \operatorname{Re}\lambda > -\beta - 1)$$

In $\mathsf{K}^m_\beta(\mathcal{W})$, the derivatives in all directions are allowed to be more singular, according to their order.
But here, the derivatives $\partial^\ell_{x_3} u$ have the same singularity at $r = 0$ as u.
This is true in general: One has additional regularity along the edge, because the edge is translation invariant in x_3-direction.

Two consequences to capture this structure:

1. Define anisotropic weighted Sobolev spaces. These will then be suitable for the definition of the spaces of weighted analytic functions.
2. Use Nested Open Sets (ρ-estimates) with derivatives in x_3. For this, we need to start with a basic H^2 a priori estimate, which is non-trivial in this case.

Edges are translation invariant

Write $\partial_{\boldsymbol{x}}^{\alpha} = \partial_{\boldsymbol{x}_\perp}^{\alpha_\perp} \partial_{x_3}^{\alpha_3}$

Definition (Anisotropic weighted Sobolev spaces)

$$\mathrm{M}_\beta^m(\mathcal{W}) = \big\{u \;:\; r^{\beta+|\alpha_\perp|}\partial_{\boldsymbol{x}}^{\alpha} u \in \mathrm{L}^2(\mathcal{W}), \quad \forall\alpha,\ |\alpha| \le m\big\} \text{ (homogeneous)}$$

$$\mathrm{N}_\beta^m(\mathcal{W}) = \big\{u \;:\; r^{(\beta+|\alpha_\perp|)_+}\partial_{\boldsymbol{x}}^{\alpha} u \in \mathrm{L}^2(\mathcal{W}), \quad \forall\alpha,\ |\alpha| \le m\big\} \text{ (non-homogeneous)}$$

For N_β^m, one assumes $\beta + m > 0$, so that the step-weighted definition makes sense.

Definition (Weighted analytic classes for edge neighborhoods)

Homogeneous: $u \in \mathrm{A}_\beta(\mathcal{W})$ if $u \in \mathrm{M}_\beta^m(\mathcal{W})$ for all $m \ge 0$ and

$$\|u\|_{\mathrm{M}_\beta^m(\mathcal{W})} \le C^{m+1}\, m! \quad \forall\, m \ge 0\,.$$

Non-homogeneous: $u \in \mathrm{B}_\beta(\mathcal{W})$ if $u \in \mathrm{N}_\beta^m(\mathcal{W})$ for all $m > -\beta$ and

$$\|u\|_{\mathrm{N}_\beta^m(\mathcal{W})} \le C^{m+1}\, m! \quad \forall\, m > -\beta\,.$$

Edge: Basic a priori estimate

To start the Nested Open Set technique, we need an estimate of the following form

Assumption

Let $u \in \mathsf{K}^2_\beta(\mathcal{W})$ be a solution of the boundary value problem in $\mathcal{W}'$. Then

$$\|u\|_{\mathsf{K}^2_\beta(\mathcal{W})} \le C\left(\|f\|_{\mathsf{K}^0_{\beta+2}(\mathcal{W}')} + \|u\|_{\mathsf{K}^1_{\beta+1}(\mathcal{W}')}\right)$$

with C independent of u.
Similarly for the non-homogeneous case with J instead of K.

Contrary to the case of an interior point or a smooth boundary point, this estimate is not a consequence of ellipticity. It

1. depends on β,
2. is in general not satisfied for all β
   ($-\beta - 1$ must not be a singular exponent for $\mathcal{K}$)
3. holds for some β in the standard problems in variational form.

Edge: Analytic estimates (anisotropic, homogeneous)

Theorem (CDN2010)

Under the Assumption, let $u \in \mathsf{K}^1_\beta(\mathcal{W}')$ *be a solution of the boundary value problem. If* $f \in \mathsf{M}^n_{\beta+2}(\mathcal{W}')$*, then* $u \in \mathsf{M}^n_\beta(\mathcal{W})$*, and there exists a positive constant* C *independent of* u *and* n *such that for all* $0 \le k \le n$ *we have*

$$\frac{1}{k!} \sum_{|\alpha|=k} \| r^{\beta+|\alpha_\perp|} \partial_{\mathbf{x}}^\alpha u \|_{\mathcal{W}} \le C^{k+1} \Big\{ \sum_{\ell=0}^{k} \frac{1}{\ell!} \sum_{|\alpha|=\ell} \| r^{\beta+2+|\alpha_\perp|} \partial_{\mathbf{x}}^\alpha f \|_{\mathcal{W}'} + \|u\|_{\mathsf{K}^1_{\beta+1}(\mathcal{W}')} \Big\}.$$

As a consequence, if $f \in \mathsf{A}_{\beta+2}(\mathcal{W}')$*, then* $u \in \mathsf{A}_\beta(\mathcal{W})$*.*
The analogous result is true for the non-homogeneous case.

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Neighborhoods

Ω: polyhedron
$\mathscr{E}$: set of edges
$\mathscr{C}$: set of corners
For $\boldsymbol{e} \in \mathscr{E}$: $\mathscr{C}_{\boldsymbol{e}}$: extremities of $\boldsymbol{e}$
For $\boldsymbol{c} \in \mathscr{C}$: $\mathscr{E}_{\boldsymbol{c}}$: edges meeting at $\boldsymbol{c}$

$$r_{\boldsymbol{c}}(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \boldsymbol{c}), \quad r_{\boldsymbol{e}}(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \boldsymbol{e}), \quad \rho_{\boldsymbol{ce}}(\boldsymbol{x}) = \frac{r_{\boldsymbol{e}}(\boldsymbol{x})}{r_{\boldsymbol{c}}(\boldsymbol{x})}\,.$$

Neighborhoods, with $0 < \eta < \epsilon$ small enough:

$$\begin{aligned}
&\text{(pure edge)} && \Omega_{\boldsymbol{e}} = \{\boldsymbol{x} \in \Omega \,:\, r_{\boldsymbol{e}}(\boldsymbol{x}) < \epsilon \text{ and } r_{\boldsymbol{c}}(\boldsymbol{x}) > \eta \quad \forall \boldsymbol{c} \in \mathscr{C}_{\boldsymbol{e}}\},\\
&\text{(pure corner)} && \Omega_{\boldsymbol{c}} = \{\boldsymbol{x} \in \Omega \,:\, r_{\boldsymbol{c}}(\boldsymbol{x}) < \epsilon \text{ and } \rho_{\boldsymbol{ce}}(\boldsymbol{x}) > \eta \quad \forall \boldsymbol{e} \in \mathscr{E}_{\boldsymbol{c}}\},\\
&\text{(corner-edge)} && \Omega_{\boldsymbol{ce}} = \{\boldsymbol{x} \in \Omega \,:\, r_{\boldsymbol{c}}(\boldsymbol{x}) < \epsilon \text{ and } \rho_{\boldsymbol{ce}}(\boldsymbol{x}) < \epsilon\},
\end{aligned}$$

Similarly, $\Omega'_{\boldsymbol{e}}$ etc, with $\eta' < \eta$, $\epsilon' > \epsilon$, and finally

$$\Omega_{\mathscr{C}} = \bigcup \Omega_{\boldsymbol{c}}, \quad \Omega_{\mathscr{E}} = \bigcup \Omega_{\boldsymbol{e}}, \quad \Omega_{\mathscr{C}\mathscr{E}} = \bigcup \Omega_{\boldsymbol{ce}}, \quad \Omega_0 = \Omega \setminus \overline{\Omega_{\mathscr{C}} \cup \Omega_{\mathscr{E}} \cup \Omega_{\mathscr{C}\mathscr{E}}}$$

Anisotropic weighted spaces on polyhedra

Definition

On $\mathcal{V} \subset \Omega$, for $m \in \mathbb{N}$ and $\underline{\beta} = \{\beta_{\boldsymbol{c}}\}_{\boldsymbol{c}\in\mathscr{C}} \cup \{\beta_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathscr{E}}$:

$$
\begin{aligned}
\mathrm{M}^m_{\underline{\beta}}(\mathcal{V}) = \Big\{ u :\ \forall\alpha,\ |\alpha| \le m, \quad & \partial^\alpha_{\boldsymbol{x}} u \in \mathrm{L}^2(\mathcal{V}\cap\Omega_0) \quad \text{and} \\
\forall \boldsymbol{c} \in \mathscr{C} : \quad & r_{\boldsymbol{c}}(\boldsymbol{x})^{\beta_{\boldsymbol{c}}+|\alpha|}\, \partial^\alpha_{\boldsymbol{x}} u \in \mathrm{L}^2(\mathcal{V}\cap\Omega_{\boldsymbol{c}}) \\
\forall \boldsymbol{e} \in \mathscr{E} : \quad & r_{\boldsymbol{e}}(\boldsymbol{x})^{\beta_{\boldsymbol{e}}+|\alpha_\perp|}\, \partial^\alpha_{\boldsymbol{x}} u \in \mathrm{L}^2(\mathcal{V}\cap\Omega_{\boldsymbol{e}}) \\
\forall \boldsymbol{c} \in \mathscr{C},\ \forall \boldsymbol{e} \in \mathscr{E}_{\boldsymbol{c}} : \quad & r_{\boldsymbol{c}}(\boldsymbol{x})^{\beta_{\boldsymbol{c}}+|\alpha|}\, \rho_{\boldsymbol{ce}}(\boldsymbol{x})^{\beta_{\boldsymbol{e}}+|\alpha_\perp|}\, \partial^\alpha_{\boldsymbol{x}} u \in \mathrm{L}^2(\mathcal{V}\cap\Omega_{\boldsymbol{ce}}) \quad \Big\},
\end{aligned}
$$

Similarly, one defines the non-homogeneous space $\mathrm{N}^m_{\underline{\beta}}(\mathcal{V})$ and the analytic classes $\mathrm{A}_{\underline{\beta}}(\mathcal{V})$ and $\mathrm{B}_{\underline{\beta}}(\mathcal{V})$.
One can also choose homogeneous norms at some corners and edges and non-homogeneous norms for the other corners and edges.

Analytic estimates on a 3D polyhedron

Techniques that cover the whole polyhedron Ω:

1. On Ω_0: This is the smooth case. Known.
2. On Ω_{e}: This is the edge case, see above.
3. On Ω_{c}: Dyadic partitions, starting from the smooth case (!)
4. On Ω_{ce}: Dyadic partitions, starting from the edge case.

Final result for example:

Theorem (CDN 2010)

Consider a mixed Dirichlet-Neumann problem, defined by a coercive variational form on a subspace V *of* $\mathrm{H}^1(\Omega)$*. There exist* $b_{\mathscr{C}}(\mathrm{V}), b_{\mathscr{E}}(\mathrm{V}) > 0$ *such that for any solution* $u \in \mathrm{V}$ *of the variational problem there holds:*
If for all $\boldsymbol{c}$, $\boldsymbol{e}$: $0 \leq b_{\boldsymbol{c}} < b_{\mathscr{C}}(\mathrm{V})$, $0 \leq b_{\boldsymbol{e}} < b_{\mathscr{E}}(\mathrm{V})$ *and* $\underline{\beta} = -\underline{b} - 1$*, then*

$$f \in \mathrm{B}_{\underline{\beta}+2}(\Omega;\mathrm{V}) \Longrightarrow u \in \mathrm{B}_{\underline{\beta}}(\Omega;\mathrm{V})$$

where the space $\mathrm{B}_{\underline{\beta}}(\Omega;\mathrm{V})$ *is defined using homogeneous norms at the edges lying on faces where Dirichlet conditions are imposed and non-homogeneous norms at all other edges and all corners.*

Thank you for your attention