New regularity theorems for nonautonomous anisotropic variational problems

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Non-autonomous integrals (1)

- Problem: regularity results for local minimizers of functionals

\[ J[w] := \int_{\Omega} F(\cdot, \nabla w) \, dx \]  \hspace{1cm} (0.1)

with a function \( F : \Omega \times \mathbb{R}^{nN} \rightarrow [0, \infty) \) and a domain \( \Omega \subset \mathbb{R}^n \).

- Anisotropic growth conditions: for all \( Z, Q \in \mathbb{R}^{nN} \) and all \( x \in \Omega \) we have

\[ C_1 |Z|^p - c_1 \leq F(x, Z) \leq C_2 |Z|^q + c_2 \]

with constants \( C_1, C_2 > 0, c_1, c_2 \geq 0 \).

- If \( p = q \) there is no problem to extend the regularity statements from the autonomous case to the situation with \( x \)-dependence.
Non-autonomous integrals (2)

Before Esposito, Leonetti und Mingione found rather surprising counterexamples (see [ELM]) most authors ignored \(x\)-dependence for a technical simplification of their proofs.

We assume \((p, q)\)-ellipticity:

\[
\lambda(1+|Z|^2)^{p-2} |Q|^2 \leq D_p F(x, Z)(Q, Q) \leq \Lambda(1+|Z|^2)^{q-2} |Q|^2
\]

(A1)

for all \(Z, Q \in \mathbb{R}^{nN}\) and all \(x \in \Omega\) with positive constants \(\lambda, \Lambda\) and exponents \(1 < p \leq q < \infty\).

We suppose for all \(Z \in \mathbb{R}^{nN}\) and all \(x \in \Omega\)

\[
|\partial_\gamma D_p F(x, Z)| \leq \Lambda_2(1 + |Z|^2)^{q-1 \over 2}
\]

(0.2)

with \(\Lambda_2 > 0\) and \(\gamma \in \{1, \ldots, n\}\).
Gap between both cases (1)

- In [ELM] Esposito, Leonetti and Mingione examine the Lavrentiev gap functional, which is defined as

\[ \mathcal{L} := \inf_{u_0 + W^{1,q}_0(B, \mathbb{R}^N)} J - \inf_{u_0 + W^{1,p}_0(B, \mathbb{R}^N)} J \]

on a ball \( B \subset \Omega \) with boundary data \( u_0 \in W^{1,p}(B, \mathbb{R}^N) \).

- The results of the studies from [ELM] provide the sharpness of the bound

\[ q < p \frac{n + \alpha}{n} \]

for higher integrability of solutions (assuming that \( D_P F(x, Z) \) is \( \alpha \)-Hölder continuous with respect to \( x \)).

- Without this condition they have examples for Lavrentiev-phenomenon.
Gap between both cases (2)

- Under the condition
  \[ q < p \frac{n+1}{n} \]  \hspace{1cm} (0.3)

Bildhauer and Fuchs [BF1] prove full $C^{1,\alpha}$-regularity for $N = 1$ or $n = 2$ and partial regularity in the general vector case.

- This statement is in accordance with the results of [ELM].

- Under several structure conditions Bildhauer and Fuchs can improve the last result to full regularity (see [BF1]).

- Without $x$-dependence we know from [BF2] that the better bound
  \[ q < p \frac{n+2}{n} \]  \hspace{1cm} (A2)
  
is sufficient for regularity.
Two problems

► If one have a look at the proof in [BF1], one see two main differences to the case of autonomous.

► The first obstacle is that the standard-regularization $u_\delta$ does not converge against the minimum $u$ without (0.3). Thereby $u_\delta$ is defined as the unique minimizer of

$$\int_B \left[ F(\cdot, \nabla w) + \delta (1 + |\nabla w|^2)^{\frac{\tilde{q}}{2}} \right] \, dx$$

in $(u)_\epsilon + W^{1,\tilde{q}}_0(B, \mathbb{R}^N)$ with $\tilde{q} > q$ and $B \subseteq \Omega$.

► The second obstacle in the proof in [BF1] is estimating the term

$$\int \eta^2 \partial_\gamma D_P F(\cdot, \nabla u) : \partial_\gamma \nabla u \, dx.$$
Solving the first one (1)

- To solve the first problem we work with a regularization from below: we need a function $F_M$ such that

\[
F_M(x, Z) = F(x, Z) \text{ if } |Z| \leq M \\
F_M(x, Z) \leq F(x, Z).
\]

- Such a regularization from below is based on a construction from [CGM].

- We have to extend all growth conditions assumed for $F$ uniformly in $M$ to $F_M$ and show isotropic growth (i.e. $F_M$ is $p$-elliptic).

- A necessary assumption for the construction of $F_M$ is

\[
F(x, P) = g(x, |P|). \tag{A3}
\]
Solving the first one (2)

- We define the regularization $u_M$ as the unique minimizer of
  \[ J_M[w] = \int_B F_M(\cdot, \nabla w) \, dx \]
  in $u + W^{1,p}_0(B, \mathbb{R}^N)$ with a ball $B \subseteq \Omega$.
- This is the minimizer of an isotropic problem and so we have several regularity properties of $u_M$. 

Solving the second one (1)

- To handle to critical integral we suppose for all $P, Z \in \mathbb{R}^{nN}$

$$\left| \partial_{\gamma} D_{P}^{2} F(x, Z)(P, Z) \right| \leq \Lambda_{3} \left| D_{P}^{2} F(x, Z)(P, Z) \right| (1 + |Z|^{2})^{\frac{\epsilon}{2}} + \Lambda_{3} (1 + |Z|^{2})^{\frac{p+q-2}{4}} |P|$$

for $0 \leq \epsilon \ll 1$.

- On account of (A3) this means

$$\left| \partial_{\gamma} g''(x, t) \right| \leq \Lambda_{4} \left[ g''(x, t)(1 + t^{2})^{\frac{\epsilon}{2}} + (1 + t^{2})^{\frac{p+q}{4} - 1} \right] \quad (A4)$$

- Example: for $f : \Omega \rightarrow (1, \infty)$ consider

$$\int_{\Omega} (1 + |\nabla w|^{2})^{\frac{f(x)}{2}} \, dx.$$
Solving the second one (2)

To extend our growth conditions to $F_M$ we have to suppose

$$|\partial_\gamma^2 g''(x, t)| \leq \Lambda_5 (1 + t^2)^{\frac{q-2}{2}}$$

as a last assumption.

This is in accordance with

$$g''(x, t) \leq \Lambda_5 (1 + t^2)^{\frac{q-2}{2}}.$$
Theorems (1)

If we assume (A1)-(A5) we have the following result for local minimizers of (0.1):

- Full regularity if $n = 2$,
- Full regularity if $N = 1$,
- Partial regularity in general vector case.
Theorems (2)

To achieve full regularity in the general vector case we need further assumptions:

- Suppose for all $P, Q \in \mathbb{R}^{nN}$, all $x \in \Omega$ with $\alpha \in (0, 1)$

\[
|D^2F(x, P) - D^2F(x, Q)| \leq c(1+|P|^2+|Q|^2)^{\frac{q-2-\alpha}{2}} |P - Q|^\alpha. 
\]

(A6)

This condition is also needed in the isotropic situation.

- One of the following two conditions (only for $n \geq 5$)

(i) \quad $q < p \frac{n - 1}{n - 2}$ \quad (A7)

(ii) \quad $g'(x, t) \leq cg''(x, t)(1+t^2)^{\frac{\omega}{2}}$ \quad (A8)

for \quad $\omega < \left( \frac{pn}{n - 2} - q \right) + 1$. 


Locally bounded minimizers (1)

- If we assume $u \in L^\infty_{loc}(\Omega, \mathbb{R}^N)$ we have dimensionless conditions between $p$ and $q$: In the autonomous situation from [BF2]

$$q < p + 2,$$

(A9)

whereas the non-autonomous situation requires the much more restrictive bound (see [BF1])

$$q < p + 1.$$  

- How to close this gap?
Locally bounded minimizers (2)

- We get full regularity if we suppose (A1), (A3)-(A6), (A9) and
  \[ g'(x, t) \leq cg''(x, t)(1 + t^2)^{\frac{\omega}{2}} \]
  for \( \omega < (p + 2 - q) + 1 \).

- It is not possible to extend (A10) to \( g_M \) (note \( F_M(x, Z) = g_M(x, |Z|) \)) uniformly in \( M \).

- Therefore we use the \( M \)-regularization to show \( \nabla u \in L^{p+2}_{loc}(\Omega, \mathbb{R}^{nN}) \) which is possible without (A10).

- Then we have a \( W^{1,q}_{loc} \)-minimizer and thereby the \( \delta \)-regularization converge.

- Use this to show local boundedness of \( \nabla u \).
New regularity theorems for nonautonomous anisotropic variational problems

Overview

<table>
<thead>
<tr>
<th>known results</th>
<th>new results</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q &lt; p \frac{n+1}{n}$</td>
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Nonlinear Stokes problem (1)

- Minimizing functionals of the form
  \[
  \widetilde{J}[v] := \int_{\Omega} \{ H(\varepsilon(v)) - f \cdot v \} \, dx, \quad \varepsilon(v) = \frac{1}{2}(\nabla v + \nabla v^T)
  \]
  subject to the constraint \( \text{div}(v) = 0 \).

- Applications: mathematical fluid mechanics.

- Minimizers correspond to the following system of partial differential equations
  \[
  \begin{aligned}
  \text{div} \{ \nabla H(\varepsilon(v)) \} &= \nabla \pi - f \quad \text{on } \Omega, \\
  \text{div} v &= 0 \quad \text{on } \Omega,
  \end{aligned}
  \quad (0.2)
  \]

- The solution \( v : \Omega \rightarrow \mathbb{R}^n \) is the velocity field and \( \pi : \Omega \rightarrow \mathbb{R} \) is the pressure.

- Here \( \Omega \) denotes a domain in \( \mathbb{R}^n \) (\( n \in \{2, 3\} \)), \( f : \Omega \rightarrow \mathbb{R}^n \) is a system of volume forces.
Nonlinear Stokes problem (2)

Examples for the density $H$

- Classical Stokes problem: $H(\epsilon) = |\epsilon|^2$
- Power law fluids: $H(\epsilon) = (1 + |\epsilon|^2)^{\frac{p}{2}}, 1 < p < \infty$
- Non-Newtonian fluids: $H$ has anisotropic behaviour in $\epsilon$
- Especially Electrorheological fluids: $H(\epsilon) = (1 + |\epsilon|^2)^{\frac{p(x)}{2}}$

Assume that $H$ satisfies the conditions (A1)-(A5) and consider minimizers of

$$\widetilde{J}[w] := \int_{\Omega} \{H(\cdot, \epsilon(w)) - f \cdot w\} \, dx, \quad \text{div}(w) = 0.$$  

The results about full regularity for $n = 2$ and partial regularity in the general vector case extend to this situation.
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