New regularity thereoms for nonautonomous anisotropic variational problems

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Non-autonomous integrals (1)

- Problem: regularity results for local minimizers of functionals

$$J[w] := \int_\Omega F(\cdot, \nabla w)\, dx \tag{0.1}$$

  with a function $F : \Omega \times \mathbb{R}^{nN} \to [0, \infty)$ and a domain $\Omega \subset \mathbb{R}^n$.

- Anisotropic growth conditions: for all $Z, Q \in \mathbb{R}^{nN}$ and all $x \in \Omega$ we have

$$C_1 |Z|^p - c_1 \leq F(x, Z) \leq C_2 |Z|^q + c_2$$

  with constantes $C_1, C_2 > 0$, $c_1, c_2 \geq 0$.

- If $p = q$ there is no problem to extend the regularity statements from the autonomous case to the situation with x-dependence.

Non-autonomous integrals (2)

- Before Esposito, Leonetti und Mingione found rather surprising counterexamples (see [ELM]) most authors ignored x-dependence for a technical simplification of their proofs.
- We assume (p, q)-ellipticity:

$$\lambda(1+|Z|^2)^{\frac{p-2}{2}} |Q|^2 \leq D_P^2 F(x, Z)(Q, Q) \leq \Lambda(1+|Z|^2)^{\frac{q-2}{2}} |Q|^2 \tag{A1}$$

for all $Z, Q \in \mathbb{R}^{nN}$ and all $x \in \Omega$ with positive constants λ, Λ and exponents $1 < p \leq q < \infty$.
- We suppose for all $Z \in \mathbb{R}^{nN}$ and all $x \in \Omega$

$$|\partial_\gamma D_P F(x, Z)| \leq \Lambda_2(1 + |Z|^2)^{\frac{q-1}{2}} \tag{0.2}$$

with $\Lambda_2 > 0$ and $\gamma \in \{1, ..., n\}$.

Gap between both cases (1)

- In [ELM] Esposito, Leonetti and Mingione examine the Lavrentiev gap functional, which is defined as

$$\mathcal{L} := \inf_{u_0+W_0^{1,q}(B,\mathbb{R}^N)} J - \inf_{u_0+W_0^{1,p}(B,\mathbb{R}^N)} J$$

  on a ball $B \Subset \Omega$ with boundary data $u_0 \in W^{1,p}(B,\mathbb{R}^N)$.
- The results of the studies from [ELM] provide the sharpness of the bound

$$q < p\,\frac{n+\alpha}{n}$$

  for higher integrability of solutions (assuming that $D_P F(x,Z)$ is α-Hölder continuous with respect to x)
- Without this condition they have examples for Lavrentiev-phenomenon.

Gap between both cases (2)

- Under the condition

$$q < p\,\frac{n+1}{n} \tag{0.3}$$

Bildhauer and Fuchs [BF1] prove full $C^{1,\alpha}$-regularity for $N = 1$ or $n = 2$ and partial regularity in the general vector case.

- This statement is in accordance with the results of [ELM].
- Under several structure conditions Bildhauer and Fuchs can improve the last result to full regularity (see [BF1]).
- Without x-dependence we know from [BF2] that the better bound

$$q < p\,\frac{n+2}{n} \tag{A2}$$

is sufficient for regularity.

Two problems

- If one have a look at the proof in [BF1], one see two main differences to the case of autonomous.
- The first obstacle is that the standard-regularization u_δ does not converge against the minimum u without (0.3). Thereby u_δ is defined as the unique minimizer of

$$\int_B \left[F(\cdot, \nabla w) + \delta \left(1 + |\nabla w|^2\right)^{\frac{\widetilde{q}}{2}} \right] dx$$

in $(u)_\epsilon + W_0^{1,\widetilde{q}}(B, \mathbb{R}^N)$ with $\widetilde{q} > q$ and $B \Subset \Omega$.
- The second obstacle in the proof in [BF1] is estimating the term

$$\int \eta^2 \partial_\gamma D_P F(\cdot, \nabla u) : \partial_\gamma \nabla u \, dx.$$

Solving the first one (1)

- To solve the first problem we work with a regularization from below: we need a function F_M such that

$$F_M(x,Z) = F(x,Z) \text{ if } |Z| \leq M$$
$$F_M(x,Z) \leq F(x,Z).$$

- Such a regularization from below is based on a construction from [CGM].
- We have to extend all growth conditions assumed for F uniformly in M to F_M and show isotropic growth (i.e. F_M is p-elliptic).
- A necessary assumption for the construction of F_M is

$$F(x,P) = g(x,|P|). \tag{A3}$$

Solving the first one (2)

- We define the regularization u_M as the unique minimizer of

$$J_M[w] = \int_B F_M(\cdot, \nabla w)\, dx$$

in $u + W_0^{1,p}(B, \mathbb{R}^N)$ with a ball $B \Subset \Omega$.

- This is the minimizer of an isotropic problem and so we have several regularity properties of u_M.

Solving the second one (1)

- To handle to critical integral we suppose for all $P, Z \in \mathbb{R}^{nN}$

$$\left|\partial_\gamma D_P^2 F(x,Z)(P,Z)\right| \leq \Lambda_3 \left|D_P^2 F(x,Z)(P,Z)\right| (1+|Z|^2)^{\frac{\epsilon}{2}} + \Lambda_3 (1+|Z|^2)^{\frac{p+q-2}{4}} |P|$$

  for $0 \leq \epsilon \ll 1$.

- On account of (A3) this means

$$|\partial_\gamma g''(x,t)| \leq \Lambda_4 \left[g''(x,t)(1+t^2)^{\frac{\epsilon}{2}} + (1+t^2)^{\frac{p+q}{4}-1} \right] \quad (A4)$$

- Example : for $f : \Omega \to (1,\infty)$ consider

$$\int_\Omega \left(1+|\nabla w|^2\right)^{\frac{f(x)}{2}} dx.$$

Solving the second one (2)

- To extend our growth conditions to F_M we have to suppose

$$|\partial_\gamma^2 g''(x,t)| \leq \Lambda_5(1+t^2)^{\frac{q-2}{2}} \tag{A5}$$

  as a last assumption.

- This is in accordance with

$$g''(x,t) \leq \Lambda_5(1+t^2)^{\frac{q-2}{2}}.$$

Theorems (1)

If we assume (A1)-(A5) we have the following result for local minimizers of (0.1):

- Full regularity if $n = 2$,
- full regularity if $N = 1$,
- partial regularity in general vector case.

Theorems (2)

To achieve full regularity in the general vector case we need further assumptions:

- Suppose for all $P, Q \in \mathbb{R}^{nN}$, all $x \in \overline{\Omega}$ with $\alpha \in (0,1)$

$$\left|D^2F(x,P) - D^2F(x,Q)\right| \le c(1+|P|^2+|Q|^2)^{\frac{q-2-\alpha}{2}}\,|P-Q|^{\alpha}\,. \tag{A6}$$

  This condition is also needed in the isotropic situation.

- One of the following two conditions (only for $n \ge 5$)

$$(i) \qquad q < p\,\frac{n-1}{n-2} \tag{$A7$}$$

$$(ii) \qquad g'(x,t) \le c g''(x,t)(1+t^2)^{\frac{\omega}{2}} \tag{$A8$}$$

$$\text{for} \quad \omega < \left(\frac{pn}{n-2} - q\right) + 1.$$

Locally bounded minimizers (1)

- If we assume $u \in L^{\infty}_{loc}(\Omega, \mathbb{R}^N)$ we have dimensionless conditions between p and q: In the autonomous situation from [BF2]

$$q < p + 2, \tag{A9}$$

whereas the non-autonomous situation requires the much more restrictive bound (see [BF1])

$$q < p + 1.$$

- How to close this gap?

Locally bounded minimizers (2)

- We get full regularity if we suppose (A1), (A3)-(A6), (A9) and

$$g'(x,t) \leq c g''(x,t)(1+t^2)^{\frac{\omega}{2}} \quad \text{for } \omega < (p+2-q)+1. \tag{A10}$$

- It is not possible to extend (A10) to g_M (note $F_M(x,Z) = g_M(x,|Z|)$) uniformly in M.
- Therefore we use the M-regularization to show $\nabla u \in L^{p+2}_{loc}(\Omega, \mathbb{R}^{nN})$ which is possible without (A10).
- Then we have a $W^{1,q}_{loc}$-minimizer and thereby the δ-regularization converge.
- Use this to show local boundedness of ∇u.

Overview

known results	new results
$q < p\,\frac{n+1}{n}$ • FR for $n = 2,\ N = 1$ or GV with SC • PR in GV [BF1], 2005	$q < p\,\frac{n+2}{n}$ $F(x, Z) = g(x, \|Z\|),\ D_x g'' \leq \ldots$ • FR for $n = 2,\ N = 1$ or GV with SC • PR in GV
$q < p + 1,\ u \in L^{\infty}_{loc}(\Omega, \mathbb{R}^N)$ • FR for $N = 1$ or GV with SC [BF1], 2005	$q < p + 2,\ u \in L^{\infty}_{loc}(\Omega, \mathbb{R}^N)$ $F(x, Z) = g(x, \|Z\|)\ ,\ D_x g'' \leq \ldots$ $g'(x, t) \leq c(1 + t^2)^{\frac{\omega}{2}} g''(x, t)$ • FR for $N = 1$ or GV with SC

Nonlinear Stokes problem (1)

- Minimizing functionals of the form

$$\widetilde{J}[v] := \int_\Omega \{H(\epsilon(v)) - f \cdot v\}\, dx, \quad \epsilon(v) = \frac{1}{2}(\nabla v + \nabla v^T)$$

  subject to the constraint $\mathrm{div}(v) = 0$.
- Applications: mathematical fluid mechanics.
- Minimizers correspond to the following system of partial differential equations

$$\begin{cases} \mathrm{div}\,\{\nabla H(\epsilon(v))\} = \nabla\pi - f & \text{on } \Omega, \\ \mathrm{div}\, v = 0 & \text{on } \Omega, \end{cases} \tag{0.2}$$

- The solution $v : \Omega \to \mathbb{R}^n$ is the velocity field and $\pi : \Omega \to \mathbb{R}$ is the pressure.
- Here Ω denotes a domain in $\mathbb{R}^n$ ($n \in \{2, 3\}$), $f : \Omega \to \mathbb{R}^n$ is a system of volume forces.

Nonlinear Stokes problem (2)

Examples for the density H

- Classical Stokes problem: $H(\epsilon) = |\epsilon|^2$
- Power law fluids: $H(\epsilon) = (1 + |\epsilon|^2)^{\frac{p}{2}}$, $1 < p < \infty$
- Non-Newtanion fluids: H has anisotropic behaviour in ϵ
- Especialy Electrorheological fluids: $H(\epsilon) = (1 + |\epsilon|^2)^{\frac{p(x)}{2}}$

Assume that H satisfies the conditions (A1)-(A5) and consider minimizers of

$$\widetilde{J}[w] := \int_\Omega \{H(\cdot, \epsilon(w)) - f \cdot w\} \, dx, \quad \mathrm{div}(w) = 0.$$

The results about full regularity for $n = 2$ and partial regularity in the general vector case extend to this situation.

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