PDEs and Variational Problems with random coefficients II

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Overview

1. Introduction
2. Existence/Nonexistence: Nonnegative solutions for semilinear random PDE
3. Uniqueness: Unique minimizer for random functional with double-well structure.
4. Review of random homogenization
   - Random functionals
   - Second order elliptic

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PDEs with random coefficients

General form:

$$F(D^2u, Du, u, x, \omega) = 0 \quad (= \partial_t u),$$

where the random function

$$F : \mathbb{R}^{nxn}_{\mathrm{sym}} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}^m$$

(here $m = 1$) satisfies **deterministic bounds/structural conditions.** (E.g. continuous, uniformly elliptic etc.)
Probability measure $\mathbb{P}$ on all equations with these bounds
Example:

$$F(M, \xi, u, x, \omega) = tr(M) + f(x, u, \omega) \qquad F(M, \xi, u, x, \omega) = a(x, \omega)tr(M)$$

Usually: Law **translation invariant and ergodic**, so "almost sure" results for large-scale behaviour.
Homogenization: Behaviour of solns. for $F(D^2u, Du, u, x/\epsilon, \omega) = 0$, on bounded domain as $\epsilon \to 0$.

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Random Functionals

Find minimizer in a suitable function space (e.g. $H^{1,2}(D)$) of

$$u(x) \mapsto \int_D F(Du, u, x, \omega) dx$$

Minimizer will be random function.
$D = \mathbb{R}^n$: Minimizer under compact perturbations.

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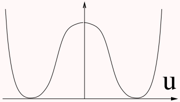
Minimisers of random of energy (with E. Orlandi)

$$\text{Area}(\Sigma \cap \Lambda) + \int_{\Lambda \cap E} f(X)dX \quad \text{where } \Sigma = \partial E.$$

$$F_\epsilon(u) = \int_\Lambda \left(\frac{\epsilon}{2}|\nabla u(x)|^2 + \frac{1}{\epsilon} W(u(x)) + \frac{\alpha_\epsilon}{\epsilon} h\left(\frac{x}{\epsilon}, \omega\right) u(x) \right) dx$$

h bounded random field, short correlation length
W double-well potential, two minimizers ± 1.



- Idea: u^ϵ minimiser $\Rightarrow u^\epsilon \to \pm 1$ on $\mathbb{R}^d \setminus \Sigma$ as $\epsilon \to 0$, F_ϵ converges to (possibly anisotropic) area functional.

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- Replace gradient term by nonlocal term

$$E_\Lambda(m, m_0) = \int_{\Lambda\times\Lambda} \mathrm{d}x\mathrm{d}y \frac{|\mathrm{m(x)} - \mathrm{m(y)}|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}} + \underbrace{2 \int_\Lambda \mathrm{d}x \int_{\mathbb{R}^d\setminus\Lambda} \mathrm{d}y \frac{|\mathrm{m(x)} - \mathrm{m_0(y)}|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}}}_{\text{boundary cond. } \mathrm{m}_0}$$

$d = 2$, $s \in (\frac{1}{2}, 1)$ **or** $d = 1$, $s \in [\frac{1}{4}, 1)$: Unique minimiser (comp. pert.)

The functional

Randomness: $(g(z,\omega))_{z\in\mathbb{Z}^d}$, d space dimension family of uniformly bounded i.i.d. r.v. with mean zero and variance 1 and **Lebesgue-continuous** and symmetric distribution.

$$g(x,\omega) := \sum_{z\in\mathbb{Z}^d} g(z,\omega) 1_{(z+[-\frac{1}{2},\frac{1}{2}]^d)\cap\Lambda}(x),$$

Energy:

$$\mathcal{K}(v,\omega,\Lambda) = \int_{\Lambda\times\Lambda} dxdy \frac{|v(x)-v(y)|^2}{|x-y|^{d+2s}} + \int_\Lambda W(v(x))\mathrm{dx} - \int_{\sim} \mathrm{g(x,\omega)v(x)dx}.$$

Boundary Cost:

$$\mathcal{W}((v,\Lambda),(u,\Lambda^c)) = 2\int_\Lambda \mathrm{dx} \int_{\mathbb{R}^d\setminus\sim} \mathrm{dy} \frac{|\mathrm{v(x)}-\mathrm{u(y)}|^2}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d}+2\mathrm{s}}}$$

$$G^{v_0}(v,\omega,\Lambda) = \mathcal{K}(v,\omega,\Lambda) + \mathcal{W}((v,\Lambda)(v_0,\Lambda^c))$$

Minimizer under compact perturbation

$u : \mathbb{R}^d \to \mathbb{R}$ **Minimizer under compact perturbations:** For any compact subdomain $U \subset$ we have

$$G^u(u, \omega, U) < \infty, \quad a.s.$$

and

$$G^u(u, \omega, U) \leq G^v(v, \omega, U) \quad a.s.$$

for any v which coincides with u in $\mathbb{R}^d \setminus U$.

$u : \Lambda \to \mathbb{R}$ is v^0-minimizer if it minimizes G^{v_0} among all functions which coincide with v^0 on $\mathbb{R}^d \setminus \Lambda$.
These exist by standard arguments.

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Minimizers are ordered

u min. of $G^u(\cdot,\Lambda)$, v min. of $G^v(\cdot,\Lambda)$, then

- if $u = v$ on $\Lambda^c \Rightarrow u \leq v$ on Λ or $v \leq u$ on Λ
- if $u < v$ on open subset of Λ^c, then $u \leq v$ on Λ.

In general no uniqueness even on compact domains!
Idea:

$$G(u \vee v,\Lambda) + G(u \wedge v,\Lambda) \leq G(u,\Lambda) + G(v,\Lambda).$$

Extremal K-minimizers

On compact domain with b.c. in general no uniqueness, but there exists maximal and minimal minimizer.

Consider now constant b.c. $\pm K$ for $K \gg 1$ and let $u^{\pm,K,\Lambda_n}$ be the extremal min. with b.c. $\pm K$ on $\Lambda_n := (-n,n)^d$.

Define:

$$u^{\pm K}(x,\omega) := \lim_{n\to\infty} u^{\pm,K,\Lambda_n}(x,\omega)$$

Pointwise increasing bounded sequence, converges in better function spaces, consequence:

$u^{\pm K}(x,\omega)$ are min. under compact perturbations!

Moreover: Translation covariant
i.e. $u^{\pm K}(x,\omega)$ and $u^{\pm K}(y,\omega)$ are the same in law.

Extremal ergodic states

WANTED: Extremal min. under compact pert. on $\mathbb{R}^n$. If they are unique, all min. are equal.

Consequence of min. property of $u^{\pm K}$ and translation covariance: uniform bounds on $\|u^{\pm K}\|_\infty$ which do not depend on K.

Consequence:

$$u^{\pm}(x,\omega) := \lim_{K\to\infty} u^{\pm K}(x,\omega)$$

well defined, uniformly bounded and min. under compact pert.
Show: $u^+ = u^-$ a.s.
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Bound on difference of optimal energies

$$\left|G^{v^+}(v^+,\Lambda) - G_1^{v^-}(v^-,\Lambda)\right| \leq C \begin{cases} |\Lambda|^{\frac{d-1}{d}} & \text{if } s \in (\frac{1}{2}, 1) \\ |\Lambda|^{\frac{d-2s}{d}} & \text{if } s \in (0, \frac{1}{2}) \\ |\Lambda|^{\frac{d-1}{d}} \log|\Lambda| & \text{if } s = \frac{1}{2} \end{cases} .$$

Note: $|\Lambda_n| \sim n^d$.

Idea: Interpolate on the boundary between u^+ and u^-, estimate "cost" by estimating singular integrals.

Central Limit Theorem: Set-up

Note: Minimal energy and minimizer depend in complicated way on all random variables $g(z,\omega)$.

σ-algebras:

• $\mathcal{B}_{n,i} = \sigma\left(\{g(z), z \in \Lambda_n,\ z \leq i\}\right)$ where $\leq$ refers to lexicographic ordering in $\mathbb{Z}^d$.

• $\mathcal{B}_{\Lambda_n} = \sigma\left(\{g(z), z \in \Lambda_n\}\right)$

• $\mathcal{B}(0) = \sigma\left(g(0)\right)$

Consider

$$\begin{aligned} F_n(\omega) &:= \mathbb{E}\left[\left\{G(v^+(\omega),\omega,\Lambda_n) - G(v^-(\omega),\omega,\Lambda_n)\right\}|\mathcal{B}_{\Lambda_n}\right] \\ &= \sum_{i\in\mathbb{Z}^d\cap\Lambda_n}\left(\mathbb{E}[F_n|\mathcal{B}_{n,i}] - \mathbb{E}[F_n|\mathcal{B}_{n,i-1}]\right) := \sum_{i\in\mathbb{Z}^d\cap\Lambda_n} Y_{n,i}. \end{aligned}$$

Martingale Difference: CLT $\Rightarrow F_n \sim \sqrt{|\Lambda|}N(0,D^2)$ where

$$D^2 = \mathbb{E}\left[\left(\mathbb{E}\left[F_n|\mathcal{B}(0)\right]\right)^2\right]$$

Central Limit Theorem: Result

Deterministic bound:

$$|F_n| \leq C \begin{cases} n^{d-1} & \text{if } s \in (\frac{1}{2}, 1) \\ n^{d-2s} & \text{if } s \in (0, \frac{1}{2}) \\ n^{d-1} \log n & \text{if } s = \frac{1}{2} \end{cases} .$$

Fluctuations: $n^{d/2}$ unless $D^2 = 0$.

Contradiction if $d = 2$, $s \in (\frac{1}{2}, 1)$ **or** $d = 1,\ s \in [\frac{1}{4}, 1)$ unless $D^2 = 0$.

"derivative" w.r.t. randomness

$$\omega(0) \mapsto \int_{Q(0)} v^+(\omega(0), \omega^{(0)})\mathrm{d}x$$

is nondecreasing.

$$\frac{\partial G(v^\pm(\omega), \omega, \Lambda)}{\partial \omega(0)} = -\int_{(-1/2,1/2)^d} v^\pm(x, \omega)\mathrm{d}x.$$

Absolutely cont. random variables!

Heuristic: Suppose $u(\omega)$ minimises $F(u, \omega)$.

$$\frac{\partial F(u(\omega), \omega)}{\partial \omega}|_{(u(\omega),\omega)} = \frac{\partial F(u, \omega)}{\partial u}|_{(u(\omega),\omega)} + \frac{\partial F(u, \omega)}{\partial \omega}|_{(u(\omega),\omega)}$$

$$= \frac{\partial F(u, \omega)}{\partial \omega}|_{(u(\omega),\omega)}$$

$$G(u, \omega) = \ldots - \int_\Lambda g(x, \omega)u(x)\mathrm{d}x$$

Central Limit Theorem: Conclusion

$$0 = D^2 = \mathbb{E}\left[\left(\mathbb{E}\left[F_n | \mathcal{B}(0)\right]\right)^2\right] = \mathbb{E}\left[f^2(\omega(0))\right]$$

so $0 = f(s)$ a.s.

$$\begin{aligned} f'(s) &= \frac{\partial G(v^+(\omega), \omega, \Lambda)}{\partial \omega(0)}\Big|_{\omega(0)=s} - \frac{\partial G(v^-(\omega), \omega, \Lambda)}{\partial \omega(0)}\Big|_{\omega(0)=s} \\ &= \int_{(-1/2,1/2)^d} \left(v^+(x,\omega) - v^-(x,\omega)\right) \mathrm{d}x. \end{aligned}$$

$f(s) = 0 \Rightarrow$(mon.) $f'(s) = 0$ a.s. $\Rightarrow$ (ordered) $v^+ = v^-$ a.s.

$$u \mapsto F_\epsilon(u) = \int_D \left(a\left(\frac{x}{\epsilon}, \omega\right) |\nabla u(x)|^q + f(x)u(x) \right) dx,$$

$0 < c < a(x, \omega) < C$
D compact, $\Rightarrow$ unique minimizer u_ϵ in $H_0^{1,q}(D)$.
$u_\epsilon \to u_0$ (weakly) as $\epsilon to 0$.

Is there a homogenized deterministic functional

$$u \mapsto \int_D (\bar{a}|\nabla u(x)|^p + f(x)u(x))\, dx$$

such that u_0 is its unique minimizer?

More general integrand: $f(P, x, \omega)$ with bounds
$c|P|^q < f(P, x, \omega) < C|P|^q$

Important Condition for hom.: Fast decay of correlations in space!
E.g. Dal Maso-Modica: Ex. $M > 0$ s.t. independent for $|x - y| > M$.

Γ-convergence

Convergence $F_\epsilon \to F$ such that minimizers of F_ϵ converge to a minimizer of F.
Suppose F_ϵ, F act on metric space (X, d).
$F_\epsilon \to F$ (d-Γ) if and only if

1. For any sequence $u_\epsilon \to u$ (w.r.t d): $F(u) \leq \liminf F_\epsilon(u_\epsilon)$
2. For any $u \in X$ there exists sequence $v_\epsilon \to u$ (w.r.t. d) s.t. $\lim F_\epsilon(v_\epsilon) = F(u)$.

Makes space of functionals a compact metric space

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Homogenized Functional

In the framework of Dal Maso-Modica: If integrand translation invariant and under independence condition (ergodicity not assumed)

$$F_\epsilon(u, D) = \int_D f(Du, x/\epsilon, \omega)dx \to \ (H^{1,q} - \Gamma)\ F_0(u, D) = \int_D f_0(Du)dx$$

with

$$f_0(P) = \lim_{n\to\infty} (2n)^{-d}\mathbb{E}\left[\min_{u\in H_0^{1,q}((-n,n)^d)} \int_{(-n,n)^d} f(Du + P, x, \omega)dx\right]$$

Necessary condition: u linear function
Additional assumption: Ergodicity w.r.t. spatial translations $\Rightarrow$ no expectation necessary.
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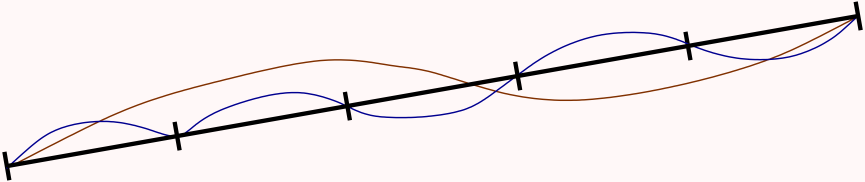
Subadditive ergodic theorem

$$(2kn)^{-d} \min_{u \in H_0^{1,q}((-n,n)^d)} \int_{(-kn,kn)^d} f(Du + P, x, \omega) dx$$

$$\leq n^{-d} \sum_{z \in (-n,n)^d \cap (2\mathbb{Z})^d} (2k)^{-d} \min_{u \in H_0^{1,q}((-k,k)^d)} \int_{(-k,k)^d} f(Du + P, x + z, \omega) dx$$



$\Rightarrow$ Convergence a.s.

Kingman's subadditive ergodic theorem

(Dal Maso-Modica)
Let $m(A, \omega)$ be a random fucntion on bounded subsets of $\mathbb{R}^d$ which is
• subadditive, i.e.

$$A = \bigcup_k A_k \;\Rightarrow\; m(A, \omega) \leq \sum_k m(A_k) \text{ a.s.}$$

• translation invariant: $m(z + A, \omega) = m(A, \omega)$

Then there ex. $\varphi(\omega)$ s.t. for almost all ω

$$\lim_{n \to \infty} \frac{1}{|nQ|} m(nQ, \omega) = \varphi(\omega)$$

Ergodic: φ is constant

Problem

$$F(D^2u, x/\epsilon, \omega) = 0 \quad \text{on } D$$
$$u = g \quad \text{on } \partial D$$

Heuristic Ansatz:

$$u_\epsilon(x,\omega) = u_0(x) + \epsilon^2 u_1(x, x/\epsilon) + \dots$$

u_1 corrector, treat x/ϵ as independent variable y

$$F(D_x^2 u_0(x) + D_y^2 u_1(x,y), y, \omega) = 0$$

Corrector equation For any $Q \in R_{sym}^{d\times d}$, find $(v, \bar{F})$ such that

$$F(Q + D^2 v(y), y, \omega) = \underbrace{\bar{F}(Q)}_{\text{nonlin.ev.}} \qquad \text{on } \mathbb{R}^d, \; \frac{v(y)}{|y|^2} \to 0 \quad \text{as } |y| \to \infty$$

- No proof of existence
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Do not need

$$F(Q + D^2 v(y), y, \omega) = \underbrace{\bar{F}(Q)}_{\text{nonlin.ev.}} \qquad \text{on } \mathbb{R}^d, \ \frac{v(y)}{|y|^2} \to 0 \quad \text{as } |y| \to \infty$$

Need only for any Q unique $\bar{F}(Q)$ such that if some $v_\epsilon(x, \omega)$ solves

$$\begin{aligned} F(D^2 v_\epsilon, y/\epsilon, \omega) &= \bar{F}(Q) \text{ in } B_1 \\ v_\epsilon &= (x, Qx) \quad \text{on } \partial B_1 \end{aligned}$$

then $\|v_\epsilon(x) - (x, Qx)\|_{L^\infty(B_1)} \to 0$.

Obstacle Problem

Rescale

$$F(D^2 w_\epsilon, y, \omega) = \bar{F}(Q) \text{ in} B_{1/\epsilon}$$
$$w_\epsilon = (x, Qx) \quad \text{on } \partial B_{1/\epsilon}$$

and compare with

$$F(D^2 u_\epsilon, y/\epsilon, \omega) = h \text{ in} B_{1/\epsilon}$$
$$u_\epsilon = (x, Qx) \quad \text{on } \partial B_{1/\epsilon}$$
$$u_\epsilon \geq (x, Qx) \text{in } B_{1/\epsilon}$$

Contact set $|\{x: \ u_\epsilon(x, \omega) = (x, Qx)\}|$ Satisfies conditions for subadditive ergodic theorem, so measure of contact set $m(h)$ det.
$m(h) = 0$: Soln. of free and obstacle problem close
$m(h) > 0$: Soln. of obstacle problem and (x, Qx) close (strict ell.!)
Desired $\bar{F}(Q)$: Choose $\sup\{h: \ m(h) = 0\}$.

Obstacle Problem

Rescale

$$F(D^2 w_\epsilon, y, \omega) = \bar{F}(Q) \text{ in} B_{1/\epsilon}$$
$$w_\epsilon = (x, Qx) \quad \text{on } \partial B_{1/\epsilon}$$

and compare with

$$F(D^2 u_\epsilon, y/\epsilon, \omega) = h \text{ in} B_{1/\epsilon}$$
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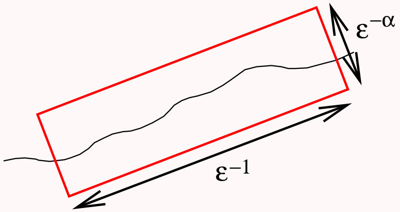
Back to interfaces

- Curve **oscillates sublinearly** in moving frame (kinetic scaling $t = \epsilon^{-1} T, \ r = \epsilon^{-1} x$.)



- positive average speed of subsolutions

Idea: Fastest plane below and slowest plane above graph (in ϵ^{-1}-box) have same average speed, which is deterministic (Obstacles i.i.d.)

$$\begin{aligned} \partial_\tau v(y,\tau,\omega) &= \epsilon \Delta v(y,\tau,\omega) + f(\epsilon^{-1} y, \epsilon^{-1} v(y,\tau,\omega), \omega) + F \\ v(x,0) &= 0 \end{aligned}$$

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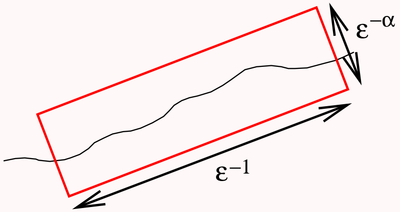

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Open Problems

- Homogenisation for Random Obstacel Model/ randomly forced MCF
- Γ-limit for random functionals with double well potential in $d \geq 3$
- Homogenization for degenerate elliptic second-order PDEs
- Homogenization for Hamilton-Jacobi equations $H(Du, x/\epsilon, \omega) + u = 0$ if H is not convex in P.

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