Decomposition of vector fields in weighted L^q-spaces

Maria Specovius, and Damian Bartocha

University of Kassel, Institute for Mathematics
Heinrich-Plett-Straße 40, D-34132 Kassel

e-mail: specovi@mathematik.uni-kassel.de

The decompositions of vector fields into a divergence free part and a gradient field play a fundamental role in the theory of the continuum mechanics, in particular for the Navier-Stokes system. Starting from the representation of smooth vector fields in $\mathbb{R}^3$ by means of Helmholtz potentials:

$$u = \nabla \times \Phi - \nabla p \quad (\text{ assuming } |u(x)| = O(\frac{1}{|x|}) \text{ as } |x| \to \infty)$$

a vast amount of literature related to such decompositions has appeared. Here the decompositions of L^q vector fields are of particular interest. A famous result of de Rham - here applied in the formulation of Sohr ([2, Lemma II.2.2.1]) - implies $f = \nabla p, p \in L^q(\Omega)$ for any field of distributions $f \in (W^{-1,q}_{loc}(\Omega))$ vanishing on the space $C^\infty_{0,\sigma}(\Omega)$ (smooth solenoidal vector fields). Using this result it is easy to handle the Hilbert space case $q = 2$ for abitrary domains $\Omega \subseteq \mathbb{R}^n$: we always have

$$L^2(\Omega)^n = \overline{C^\infty_{0,\sigma}(\Omega)}^{L^2} \oplus \{\nabla p; p \in W^{1,2}_{loc}(\Omega)\}.$$

For $q \neq 2$ the situation is quite different in particular in domains with nonsmooth boundaries or unbounded domains, even if the boundary is smooth. In the latter case one idea to overcome the difficulities is to consider intersections of L^q-spaces. On the other hand, for domains with all kind of boundary singularities it is quite natural to use weighted L^q- (and Sobolev-) spaces in order to control the asymptotic behavior of the functions in question near the singularities including the “point” infinity.

Furthermore, these kind of decompositions are closely related to boundary value problems for the Laplace equation, which are also vey well understood in weighted Sobolev spaces (with an equal vast amount of literature, see e.g. [1]). In this lecture some results about the Helmholtz decomposition in weighted L^q-spaces are collected and garnished with some new aspects especially concerning the behavior of the potential p. In particular for so called model problems it is astonishingly easy to combine well known results with duality arguments to close some gaps which still existed in the literature.

References

[1] V. A. Kozlov, V. G. Mazya, and J. Rossmann. *Elliptic Boundary Value Problems in domains with point singularities* AMS, University of California, Berkeley 1997.

[2] H. Sohr. *The Navier-Stokes Equations. An Elementary Functional Analytic Approach* Birkhäuser Verlag, Basel-Boston-Berlin, 2001.