Some notes about quasi-incompressible fluids

Tomáš Roubíček

Mathematical Institute, Charles University, Sokolovská 83, CZ-186 75 Praha 8
and Institute of Thermomechanics of the Czech Acad. Sci.
Dolejškova 5, CZ–182 08 Praha 8, Czech Republic

e-mail: tomas.roubicek@mff.cuni.cz

Models for liquids (rather than for gases) where mass density variation can be neglected but propagation of pressure waves (in contrast to ideally incompressible Navier-Stokes equations) with a dispersion are presented. Advantages may be that the pressure can be well determined even pointwise everywhere (including the boundary). One variant is governed by the system

$$(1)\quad \varrho\frac{\partial v}{\partial t}+\varrho(v\cdot\nabla)v-\mathrm{div}\,\nu\nabla v+\frac{\varrho}{2}(\mathrm{div}\,v)\,v+\nabla\pi=g \quad\text{and}\quad \frac{1}{K}\frac{\partial\pi}{\partial t}+\mathrm{div}\,v=\frac{1}{H}\Delta\frac{\partial\pi}{\partial t}\,,$$

where v is the velocity, π pressure, K is the elastic bulk modulus (in the physical units Pa=J/m^3) and H is a "hyper" bulk modulus (in the physical units Pa/m^2=J/m^5). The bulk force $\frac{\varrho}{2}(\mathrm{div}\,v)\,v$ was invented by R. Temam to fit the expected energetics, cf. [7]. This model is usually considered rather as an artificial regularization (or numerical stabilization) of the incompressible model only, possibly with $H=\infty$ or ignoring the force $\frac{\varrho}{2}(\mathrm{div}\,v)\,v$, cf. e.g. [1, 2] or [4]. Yet, it has a good physical relevancy by itself as the modulus K with the mass density ϱ determines the speed of P- (=pressure) waves for low-frequency range, while $H<\infty$ leads to a micro-inertia like [3] and facilitates a normal dispersion for higher frequencies leading to a lower speed of P-waves.
Existence of a weak solution and certain regularity as well as asymptotics towards the incompressible Navier-Stokes model for $K,H\to\infty$ is at disposal in this model.
Various modifications of the basic model (1) will also be presented. For example, $K=K(\pi)$ leading to a pressure-dependent speed of P-waves, or enhancement towards anomalous dispersion, or a coupling with a phase field χ governed by the Cahn-Hilliard equation (like e.g. [5] for incompressible case) giving rise to a Korteweg-type stress and then $K=K(\chi)$ leading to a phase-dependent P-wave speed in a two-phase flow or, e.g., to a salinity-dependent P-wave speed if χ is concentration of salt in seawater.
Other (likely physical relevant) modifications to be discussed are replacement of the Temam's force by a Bernoulli-type pressure, or making also the second equation in (1) parabolic $\frac{1}{K}\frac{\partial\pi}{\partial t}+\mathrm{div}\,v=\frac{1}{H}\Delta\pi$ as in [6], or possibly also the fully convective variant, i.e.

$$\varrho\frac{\partial v}{\partial t}+\varrho(v\cdot\nabla)v-\mathrm{div}\Big(\nu\nabla v+\big(\frac{\varrho}{2}|v|^2+\frac{\pi^2}{2K}-\pi\big)\mathbb{I}\Big)=g \quad\text{and}\quad \frac{1}{K}\Big(\frac{\partial\pi}{\partial t}+v\cdot\nabla\pi\Big)+\mathrm{div}\,v=\frac{\Delta\pi}{H}.$$

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