Weak solutions and weak-strong uniqueness to a thermodynamically consistent model describing solid-liquid phase transitions

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We consider one of the simplest diffuse-interface models describing non-isothermal phase transition processes in a thermodynamically consistent setting,

(1a) $$\theta_t + \theta\varphi_t - \kappa\Delta\theta = |\varphi_t|^2 \qquad \text{in } \Omega \times (0,T)\,,$$

(1b) $$\varphi_t - \Delta\varphi + F'(\varphi) = \theta \qquad \text{in } \Omega \times (0,T)\,.$$

By F we denote an λ-convex interaction potential entering the free energy functional and by $\kappa > 0$ the heat conductivity, assumed to be constant. The unknowns of the system are the temperature θ and the phase-field variable φ. This system is equipped with homogeneous NEUMANN boundary conditions and initial conditions, *i.e.,*

(1c) $$\mathbf{n}\cdot\nabla\theta = 0 = \mathbf{n}\cdot\nabla\varphi \quad \text{on } \partial\Omega\times(0,T)\,, \qquad \theta(0)=\theta_0\,, \quad \varphi(0)=\varphi_0 \qquad \text{in } \Omega\,.$$

A physical derivation of this system describing liquid solid phase transitions is provided in the paper [1]. From the mathematical viewpoint, there has not been any global-in-time well-posedness result for the initial-boundary value problem (1) in more than one space dimension [2]. A global existence result in two or three space dimensions has only been proved for power-like type growth of the heat flux law ($\kappa(\theta) \sim \theta^\eta$, for η big enough for θ large) [3].

Using a new *a priori* estimate, we prove global existence of weak solutions in any finite dimension [5]. We use a notion of solution inspired by [4], where the pointwise internal energy balance is replaced by the total energy inequality complemented with a weak form of the entropy inequality. Moreover, we prove existence of local-in-time strong solutions [5] and, finally, we prove weak-strong uniqueness of solutions [5], meaning that every weak solution coincides with a local strong solution emanating from the same initial data, as long as the latter exists. The weak-strong uniqueness result relies on the formulation of an appropriate relative energy inequality, which can also be used to derive further information on solutions to this system, like long-time behavior, or identifying singular limits.

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