The low Mach number limit for complex fluids: recent results on strong and weak solvability

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The average volume $\mathfrak{v}$ of a fluid mixture depends essentially on temperature, pressure and *composition*. Unlike the well studied case of single component fluids, the incompressible limit $\frac{\partial \mathfrak{v}}{\partial p} = 0$ by this more complex molecular structure is not correctly expressed by the total mass density of the fluid being a constant. For a mixture of $N > 1$ chemical substances, the usual incompressibility constraint is to be substituted by the condition

$$\sum_{i=1}^{N} \bar{\mathfrak{v}}_i \, \rho_i = 1 \,. \tag{1}$$

Here $\rho_1, \ldots, \rho_N$ are the mass densities of the constituents in the mixture, while $\bar{\mathfrak{v}}_1, \ldots, \bar{\mathfrak{v}}_N$ are reference partial volumes of the species at mean temperature and pressure.

In the talk, we present a model accounting for mass transfer and momentum balance in multi-component isothermal fluids that are incompressible in the sense of (1): On the one hand N advection–diffusion equations $\partial_t \rho_i + \text{div}\,(\rho_i \, v + J_i) = 0$ for the mass densities of the species, with consistent Fickean closure for the diffusion fluxes $J^i = -M_{i,j} \, \nabla \mu_j$ ($\mu =$ chemical potentials, $M =$ Onsager matrix); On the other hand, the Navier-Stokes equations for the barycentric velocity of the fluid. We will mainly discuss the mathematical structure of the PDEs and diverse issues of their well-posedness analysis.

In order to exhibit the structure, we substitute the mass densities in the advection–diffusion equations by equivalent variables: The total mass density $\varrho := \sum_{i=1}^{N} \rho_i$ and $N-1$ linear combinations of chemical potentials $q_1, \ldots, q_{N-1}$ that drive the dissipative mechanisms. This way to look at the problem possesses the important advantage that the components of q are not subject to pointwise positivity and incompressibility constraints like the mass densities. After the change of variables, the barycentric velocity v of the fluid and the variable ϱ are subject to a Navier-Stokes sytem (of 'compressible' type by the way, to employ the usual mathematical classification) while the components of q are driven by a non diagonal system of quasilinear parabolic–elliptic type.

In comparison to purely compressible models, the condition (1) is the cause of two additional singularities affecting analysis. At first, the total mass density is confined to a bounded interval $[\varrho_{\min}, \ \varrho_{\max}]$, where the thresholds are defined by the inverse largest and smallest reference volume $\bar{\mathfrak{v}}_i$ in (1). At second, the thermodynamic pressure in the Navier-Stokes equations is given by a special algebraic formula $p = \hat{P}(\varrho, \, q_1, \ldots, q_{N-1})$, such that $\hat{P}(\varrho, \cdot) \to \pm\infty$ logarithmically for $\varrho \to \{\varrho_{\min}, \varrho_{\max}\}$.

We are able to prove the local-in-time existence well-posedness for smooth initial data satisfying (1). Global-in-time weak solutions can be constructed too. Their pressure field is affected by a defect measure accounting for the singularities $\varrho \in \{\varrho_{\min}, \varrho_{\max}\}$. This will be briefly discussed in the talk.

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