Model of zero-range potential with internal structure for Maxwell operator

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Report overview

- A model of the Helmholtz resonator;
- Self-adjoint Maxwell operator;
- Pontryagin space;
- GPI model;
- Problem for coupled domains;
- GPI model for opto-electronic system
- Conclusion;

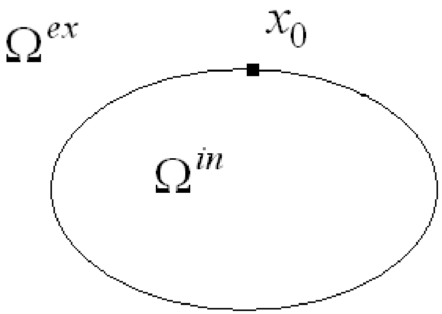
Helmholtz resonator, Neuman

Initial operator: $L_2\left(\Omega^{in}\right)\oplus L_2\left(\Omega^{ex}\right)$ $\Omega^{in},\Omega^{in}\subset R^3(R^2)$

$$-\Delta=-\left(\Delta^{in}\oplus\Delta^{ex}\right)\qquad \left.\frac{\partial U}{\partial x}\right|_{\partial\Omega}=0$$

Restriction $-\Delta_0$:

$$x_0\in\partial\Omega\qquad U\left(x_0\right)=0$$

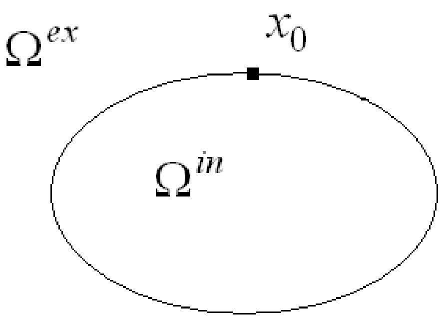


$-\Delta_0$ is symmetric, nonself-adjoint operator

deficiency indices $\left(2,2\right)\Rightarrow$ self-adjoint extensions exist

Helmholtz resonator, Dirichlet

Initial operator: $L_2\left(\Omega^{in}\right) \oplus L_2\left(\Omega^{ex}\right)$



$$-\Delta = -\left(\Delta^{in} \oplus \Delta^{ex}\right) \qquad U\big|_{\partial\Omega} = 0$$

Restriction $-\Delta_0$:

$$x_0 \in \partial\Omega \qquad U\left(x_0\right) = 0$$

$-\Delta_0$ is symmetric, essentially self-adjoint operator

deficiency indices $\left(0,0\right)$

WHY?

Helmholtz resonator, Dirichlet

Neuman case: Green function

$$\begin{pmatrix} G^{in}\left(x, x_0, k\right) \\ G^{ex}\left(x, x_0, k\right) \end{pmatrix}$$

is a deficiency element;

Dirichlet case: derivative of the Green function may be a deficiency element:

$$\begin{pmatrix} \frac{\partial}{\partial n} G^{in}\left(x, x_0, k\right) \\ \frac{\partial}{\partial n} G^{ex}\left(x, x_0, k\right) \end{pmatrix}$$

it does't belong to $L_2\left(\Omega^{in}\right) \oplus L_2\left(\Omega^{ex}\right)$

Helmholtz resonator, Dirichlet

How to construct a model?

Extension theory model for indefinite metric spaces

$$L_2\left(\Omega^{in}\right) \oplus L_2\left(\Omega^{ex}\right) \longrightarrow \text{Pontryagin space } \Pi$$

Shondin Yu.G., Tip A., Dijksma A., Popov I.Yu., et. al

All above for the Laplace operator…

Maxwell operator - ?

Maxwell operator

The starting point – self-adjoint operator

How to introduce self-adjoint Maxwell operator?

Maxwell operator

Birman M. Sh., Solomyak M. Z. Tip A.:

$$M\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = -i\begin{pmatrix}0 & \varepsilon^{-1}\boldsymbol{\varepsilon}\mathbf{p}\mu^{-1}\\ -\boldsymbol{\varepsilon}\mathbf{p} & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix}$$

$$\partial_{\mathbf{x}}(\varepsilon\mathbf{E}) = 0 \qquad \partial_{\mathbf{x}}\mathbf{B} = 0 \qquad \gamma_\tau\mathbf{E} = 0 \qquad \gamma_\nu\mathbf{B} = 0$$

$\varepsilon(\mathbf{x}), \mu(\mathbf{x})$ - smooth strictly positive, bounded functions of $x \in R^3$,

$\mathbf{p} = -i\partial_{\mathbf{x}}$, $\boldsymbol{\varepsilon}$ - Levi-Chivita tensor.

$\gamma_\tau\mathbf{E}$ and $\gamma_\nu\mathbf{B}$ are tangential and normal components of the corresponding fields.

GPI model

Scale of Hilbert spaces

$$\ldots \subseteq H_2 \subseteq H_1 \subseteq H_0 \subseteq H_{-1} \subseteq H_{-2} \subseteq \ldots$$

$$H_0 = L_2\left(\Omega^{in}, d\mathbf{x}, \square^6\right) \oplus L_2\left(\Omega^{ex}, d\mathbf{x}, \square^6\right) \qquad H_k = \mathbf{R}_0\left(z_0\right)^k H_0$$

$$\mathbf{R}_0\left(z_0\right) = \left(M - z_0\right)^{-1} \qquad M = M^{in} \oplus M^{ex}$$

Let $\chi_h \in H_{-3} \setminus H_{-2}$ and $\chi_{hk} \in \mathbf{R}\left(z_0\right)^{k+3} \chi_h \quad k = -2, -1, 0, 1$

$h = 1 \ldots 6$

The elements of pre-Pontryagin space are:

$$\mathbf{F} = \mathbf{F}_2 + \sum_{h=1}^{6} \sum_{k=-2}^{1} F_{hk} \chi_{hk} \qquad \mathbf{F}_2 \in H_2 \qquad F_{hk} \in C$$

Pontryagin space

Pre-Pontryagin space doesn't contain the whole H_0

It should be completed into the Pontryagin space.

Let

$$\Pi_m = \left\{ \left(\varphi_0, \tilde{\mathbf{c}}, \mathbf{c} \right) \middle| \varphi_0 \in H_0; \tilde{\mathbf{c}}, \mathbf{c} \in \mathbb{C}^m \right\}$$

with inner product:

$$\langle \varphi | \varphi' \rangle = \langle \varphi_0 | \varphi_0' \rangle + \tilde{\mathbf{c}}^* \mathbf{c}' + \mathbf{c}^* \tilde{\mathbf{c}}' + \mathbf{c}^* g \mathbf{c}'$$

g is a Hermitian matrix

GPI model

The inner product:

$$\left[\mathbf{F},\mathbf{G}\right]=\left(\mathbf{F}_2,\mathbf{G}_2\right)+\sum_{h=1}^{6}\sum_{k=-2}^{1}\left\{F_{hk}\left(\chi_{hk},\mathbf{G}_2\right)+\bar{G}_{hk}\left(\mathbf{F}_2,\chi_{hk}\right)\right\}+$$

$$+\sum_{j,h=1}^{6}\sum_{k,l=-2}^{1}F_{jk}\bar{G}_{hk}\left[\chi_{jk},\chi_{hl}\right]$$

here

$$\left[\chi_{jk},\chi_{hl}\right]=\begin{cases}\left(\chi_{jk},\chi_{hl}\right), k+l\geq 0\\ g_{kl}^{(jh)}, k+l<0\end{cases}$$

and

$$g_{kl}^{(jh)}=\bar{g}_{lk}^{(jh)} \qquad g_{k+1,l}^{(jh)}-\bar{g}_{k,l+1}^{(jh)}=\left(z_0-\bar{z}\right)g_{kl}^{(jh)}$$

GPI model

The resolvent for self-adjoint extension of symmetric operator in Pointryagin space is given by Krein's resolvent formula:

$$\mathbf{R}(z,\Lambda)=\mathbf{R}_0(z)-\mathbf{R}_0(z)\sum_{j,h=1}^{6}\left|\chi_j\right\rangle\left(\Gamma^{-1}\right)_{jh}\left\langle\chi_h\right|\mathbf{R}_0(z)$$

where

$$\Gamma(z,\Lambda)_{jh}=\left(\Lambda^{-1}\right)_{jh}+\left[\left\{\mathbf{R}_0(z)-\frac{1}{2}\left(\mathbf{R}_0(z)+\mathbf{R}_0(\overline{z}_0)\right)\right\}\boldsymbol{\chi}_j,\boldsymbol{\chi}_h\right]$$

GPI model

For free space we can represent the resolvent kernel in the form:

$$\langle \mathbf{x} | \mathbf{R}_0(z) | \mathbf{y} \rangle =$$

$$\left\{ z^{-1}\left(\partial_{\mathbf{x}}\partial_{\mathbf{x}} + z^2 U\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\boldsymbol{\varepsilon}\partial_{\mathbf{x}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \frac{\exp\left(iz|\mathbf{x}-\mathbf{y}|\right)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

where

$$U = \left(\boldsymbol{\varepsilon}\mathbf{p}\right)^2 - \frac{\mathbf{pp}}{p^2}$$

EM field-electron interaction

Initial operator: $H_0 = L_2\left(R^3, dx, C^6\right) \oplus L_2\left(R^3, dx\right)$

$$H = H_1 \oplus H_2$$

Here

H_1 - Maxwell operator in free space;

$H_2 = -\Delta + V(x)$ - Schrodinger operator of electron ;

Krein resolvent formula

Let φ_s be deficiency element of $H_s : H_s{}^* \varphi_s = i\varphi_s$

Krein resolvent formula for the extension H_Γ of H :

$$\left(H_\Gamma - zI\right)^{-1} - \left(H - zI\right)^{-1} = \frac{H + iI}{H - iI} P \left(Q - P \frac{I + zH}{H - zI} P^{-1} \right)^{-1} P \frac{H - iI}{H - zI}$$

Where $Q = \sum_{s,p=1,2} \varphi_s \Gamma_{sp} \left\langle \cdot, \varphi_p \right\rangle,$ P is the orthogonal projection from H

to the deficiency subspace N_i

Maxwell component

Maxwell operator: $L_2\left(R^3, dx, C^6\right)$

$$H_1\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = -i\begin{pmatrix}0 & \boldsymbol{\varepsilon}\mathbf{p}\\ -\boldsymbol{\varepsilon}\mathbf{p} & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix}$$

Resolvent:

$$\mathbf{R}^1(z) = \left\{-z^{-1}\mathbf{e}_\mathbf{p}\mathbf{e}_\mathbf{p} + z\left[p^2 - z^2\right]\boldsymbol{\Delta}_\mathbf{p}\right\}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} +$$

$$+p\left(p^2 - z^2\right)^{-1}\boldsymbol{\varepsilon}\cdot\mathbf{e}_\mathbf{p}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$\mathbf{e}_\mathbf{p} = \mathbf{p}/p \quad -\boldsymbol{\Delta}_\mathbf{p} = \left(\boldsymbol{\varepsilon}\mathbf{p}\right)^2$$

Resolvent $\left(H_{\Gamma} - zI\right)^{-1}$ construction

Hamiltonian of electron:

$$H_2 = -\Delta + V(x)$$

$$D_1(z) = \left\langle \frac{I + zH_1}{H_1 - zI} \varphi_1, \varphi_1 \right\rangle$$

$$D_2(z) = \lim_{x \to 0} \left(R_z^1(x,0) - \frac{1}{4\pi |x|} \right), \quad R_z^i,\ i = 1, 2,$$

is the resolvent of H_i

Resolvent

$$\mathbf{R}_z^{\Gamma}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} R_z^1 f_1 \\ R_z^2 f_2 \end{pmatrix} + \begin{pmatrix} \dfrac{H_1 + iI}{H_1 - zI} & 0 \\ 0 & R_z^2(x,0) \end{pmatrix} \cdot$$

$$\frac{1}{(\Gamma_{11} - D_1(z))(\Gamma_{22} - D_2(z)) - \left|\Gamma_{12}\right|^2} \cdot$$

$$\begin{pmatrix} \Gamma_{22} - D_2(z) & -\Gamma_{12} \\ -\Gamma_{21} & \Gamma_{11} - D_1(z) \end{pmatrix} \begin{pmatrix} \left\langle (H_1 - iI) R_z^1 f_1, \varphi_1 \right\rangle \\ (R_z^2 f_2)(0) \end{pmatrix}$$

Solution of scattering problem

Taking $f_1 = 0,\ f_2 = \delta(x-y),$

one obtains the "Schrodinger" component of the resolvent kernel:

$$\frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|} + \frac{e^{i\sqrt{z}|x|}}{4\pi|x|} \frac{\Gamma_{11} - D_1(z)}{(\Gamma_{11} - D_1(z))(\Gamma_{22} - D_2(z)) - |\Gamma_{12}|^2}$$

Taking $y \to \nu\infty$

one gets the "Schrodinger" component of the solution of scattering problem

$$\psi_\Gamma(x,\nu) = e^{i\sqrt{z}\langle x,\nu\rangle} + \frac{e^{i\sqrt{z}|x|}}{4\pi|x|} \frac{\Gamma_{11} - D_1(z)}{(\Gamma_{11} - D_1(z))(\Gamma_{22} - D_2(z)) - |\Gamma_{12}|^2}$$

Dispersion equation

Dispersion equation

$$(\Gamma_{11} - D_1(z))(\Gamma_{22} - D_2(z)) - \left|\Gamma_{12}\right|^2 = 0$$

Open problems

- Fitting problem (comparison with realistic problems);
- Spectral problem for complex optical systems;
- Applications to photonic crystals, metamaterials, etc.

Thanks for your time!