On the Keldysh formalism applied to mesoscopic quantum transport

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- Small system: $h_S = \sum t_{mn}|m\rangle\langle n|$
- Coupling: $h_T(t) = \sum_{\gamma=\pm} V_\gamma(t)\{|0_\gamma\rangle\langle m_\gamma| + |m_\gamma\rangle\langle 0_\gamma|\}$

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- Expectations: $\langle d\Gamma(o)\rangle_{\mathrm{ref}} = \mathrm{Tr}_{\mathcal{F}}\{\rho_0 d\Gamma(o)\} = \sum_{\gamma=\pm} \mathrm{Tr}_{\mathcal{H}}\{f_\gamma(h_L^{(\gamma)})P_\gamma o\} + \sum_\lambda n_\lambda \langle \phi_\lambda, o\phi_\lambda\rangle$, with $h_S\phi_\lambda = \lambda\phi_\lambda$

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  $U(t,t_0) = \mathrm{Id}$
- The current at time t: $I_\alpha(t) = \mathrm{Tr}_{\mathcal{F}}\{U(t,t_0)\rho_0U^*(t,t_0)J_\alpha(t)\} =$
  $\langle U^*(t,t_0)J_\alpha(t)U(t,t_0)\rangle_{\mathrm{ref}}$

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- The Keldysh ”lesser” function:
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- The current formula: $I_\alpha(t) = 2\text{Re}\{V_\alpha(t) G^<_{m_\alpha, 0_\alpha}(t, t)\}.$

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- or: $G^{<}_{x,x'}(t,t') = i\sum_\gamma \langle x, u(t,t_0) f_\gamma(h_L) P_\gamma u^*(t',t_0)x'\rangle +$
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- (self-energy of the leads): $\Sigma^<_\gamma(s,s') := V_\gamma(s)V_\gamma(s')\langle 0_\gamma, f_\gamma(h_L^{(\gamma)})e^{i(s-s')h_L^{(\gamma)}}0_\gamma\rangle$

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- Keldysh: $G^{<}_{m,m'}(t,t') =$
  $\sum_\gamma \int_{t_0}^{t} ds G^{R}_{m,m_\gamma}(t,s) \int_{t_0}^{t'} ds' \Sigma^{<}_{\gamma}(s,s') G^{A}_{m_\gamma,m'}(s',t') +$
  $i\sum_\lambda n_\lambda \langle \phi_\lambda, u^*(t',t_0)m' \rangle \, \langle u^*(t,t_0)m, \phi_\lambda \rangle$

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\begin{aligned}
I_\alpha(t) &= 2\mathrm{Re}\{V_\alpha(t) G^<_{m_\alpha,0_\alpha}(t,t)\} \\
&= \int_{t_0}^{t} dt' \int_{-2}^{2} dE \sqrt{4-E^2} V_\alpha(t') V_\alpha(t) e^{i(t-t')E} \\
&\times \{G^<_{m_\alpha,m_\alpha}(t,t') + f_\alpha(E) G^R_{m_\alpha,m_\alpha}(t,t')\}
\end{aligned}
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- $u(t,t_0) = e^{-i(t-t_1)h}u(t_1,t_0)$ and
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- $u(t,t_0) = e^{-i(t-t_1)h} u(t_1,t_0)$ and $u(t-s,t_0) = e^{-i(t-s-t_1)h} u(t_1,t_0)$
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- $u(t_1,t_0) - e^{-i(t_1-t_0)h_0}$ and $u^*(t_1,t_0) - e^{i(t_1-t_0)h_0}$ are compact operators!

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- $G^{<}_{x,x'}(t, t-s) \approx i \sum_\gamma \langle x, \omega_- f_\gamma(h_L) e^{-ish_L} P_\gamma \omega_-^* x' \rangle$

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- The Jauho-Meir current formula is higly dependent of the composed form of h
- The computations tend to be much more complicated than in the scattering approach

Thank you!