Asymptotics Problems for Wave-Particles Interactions; Quantum and Classical Models

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Quantum Model: Unknown=Density Matrix $\rho(t; n, m)$, $n, m \in \mathbb{N}$,

$$i\hbar\, \partial_t \rho + \big[H_{\rm q}, \rho\big] = \frac{1}{\tau}\, Q_{\rm q}(\rho), \qquad \text{Bloch equation}$$

where $\big[H_{\rm q}, \rho\big](n, m) = \sum_{k \in \mathbb{N}} \big(H_{\rm q}(n, k)\rho(k, m) - \rho(n, k) H_{\rm q}(k, m)\big)$,

Classical Model: Unknown=Distribution Function $f(t; x, p) \geq 0$, $x, p \in \mathbb{R}^N$

$$\partial_t f + \big\{H_{\rm c}, f\big\} = \frac{1}{\tau}\, Q_{\rm c}(f),$$

where $\big\{H_{\rm c}, f\big\} = \nabla_p H_{\rm c} \cdot \nabla_x f - \nabla_x H_{\rm c} \cdot \nabla_p f$.

Scaling Issues

$$\text{We split } H = \begin{cases} H_{0\mathrm{c}}(x,p) + \epsilon\, V_{\mathrm{c}}(t, t/\theta; x, p), \\ H_{0\mathrm{q}}(n,m) + \epsilon\, V_{\mathrm{q}}(t, t/\theta; n, m), \end{cases}$$

$H_0 =$ time independent Hamiltonian corresponding to the confining potential of the atomic nucleus,

$\epsilon\, V =$ perturbations by external electro-magnetic waves

Scaling Assumptions:

$$\frac{\mathrm{T} H_{0\mathrm{c}}}{\mathrm{LP}} \simeq \frac{1}{\epsilon^2} \gg 1, \qquad \frac{\mathrm{T} H_{0\mathrm{q}}}{\hbar} \simeq \frac{1}{\epsilon^2} \gg 1, \qquad \frac{\mathrm{T}}{\theta} \simeq \frac{1}{\epsilon^2}, \qquad \frac{\mathrm{T}}{\tau} \simeq \frac{1}{\epsilon^2}.$$

Cutting Edge Questions

$$i\partial_t\rho + \quad \frac{1}{\epsilon^2}\left[H_{0\mathrm{q}} + \epsilon\, V_{\mathrm{q}}(t/\epsilon^2), \rho\right] \qquad = \frac{1}{\epsilon^2}\, Q_{\mathrm{q}}(\rho),$$

$$\partial_t f + \quad \frac{1}{\epsilon^2}\underbrace{\left\{H_{0\mathrm{c}} + \epsilon\, V_{\mathrm{c}}(t/\epsilon^2), f\right\}}_{\substack{\text{Free Hamiltonian } H_0 \\ +\text{ Small Oscillating Perturbation}}} \qquad = \underbrace{\frac{1}{\epsilon^2}\, Q_{\mathrm{c}}(f)}_{\substack{\text{Damping} \\ \text{or Relaxation term}}},$$

- Derivation of a New Classical Model (Relaxation operator)
- Relaxation and Homogenization all together
- Random vs. Deterministic modelling

Asymptotic Behavior

• Quantum Model: $\rho_\epsilon(t;n,m) \to \rho(t;n,n)\ \delta(n,m)$ verifying the Einstein Rate Equation

$$\partial_t \rho(t;n,n) = \sum_k A(t;n,k)\ \big(\rho(t;k,k) - \rho(t;n,n)\big)$$

• Classical Model: $f_\epsilon(t;x,p) \to F(t;H_0(x,p))$ verifying the Diffusion Eq. wrt energy

$$\partial_t \big(h_0(E)\ F(t;E)\big) - \partial_E \big(d(E) h_0(E)\ \partial_E F(t;E)\big) = 0$$

$h_0(E)\ F(t;E)$=# of particles with energy E,

$d(E)h_0(E)\ \partial_E F(t;E)$ = Flux through the energy surface Σ_E.

• Both limit eq. are Irreversible Equations (decay of L^2 norms)

Role of the Damping Term, Deterministic vs Random Perturbations

Two Difficulties:

- Homogeneization Limit (Fast Oscillations of V)

- Relaxation Limit that pushes towards $\mathrm{Ker}\big([H_0, \cdot] - Q_{\mathrm{q}}\big)$ (resp. $\mathrm{Ker}(\{H_0, \cdot\} - Q_{\mathrm{c}}\big))$ $\sim$ "hydrodynamic limit"

Two Frameworks:

- (Quasi-)Periodic Oscillations and Damping is crucial

- Random Perturbation where Damping term can vanish

Role of the Damping: Cell Problems

Let $\Omega \in \mathbb{R}^D$ with rationally independent components. Consider

$$\lambda R + \Omega \cdot \nabla_\vartheta R = H,$$

with $(0,1)^N = \Theta-$periodic boundary condition and $\lambda \geq 0$.

- If $\lambda = 0$ then $\int_\Theta H \, \mathrm{d}\vartheta = 0$ is a *necessary* condition.

- If $\lambda = 0$ and $H = 0$ then $\Omega \cdot \xi \; \widehat{R}(\xi) = 0$ and R is constant.

- But the Fredholm alternative *does not* apply to $\Omega \cdot \nabla_\vartheta$ due to Small Divisors problems: set $\widehat{R}(\xi) = -i\widehat{H}(\xi)/\Omega \cdot \xi$,

$$\text{If } |\Omega \cdot \xi| \geq C/|\xi|^\gamma \text{ then } \|R\|_{L^2} \leq C\|H\|_{H^\gamma}.$$

- With $\lambda > 0$ we get

$$R(\vartheta) = \int_0^{+\infty} e^{-\lambda\sigma} H(\vartheta - \Omega\sigma) \, \mathrm{d}\sigma \in L^2.$$

How does randomness induce irreversibility ?

Toy Model: $\frac{\mathrm{d}}{\mathrm{d}t}u_\epsilon(t) = i\frac{1}{\epsilon}a(t/\epsilon^2)u_\epsilon(t)$ with a random, $\mathbb{E}a = 0$

Crucial assumption: $a(t)$ and $a(s)$ decorrelate when $|t-s| \geq 1$

Duhamel's Formula: $u_\epsilon(t) = u_\epsilon(t-\epsilon^2) + \frac{1}{\epsilon}\int_{t-\epsilon^2}^{t} ia(s/\epsilon^2)u_\epsilon(s)\,\mathrm{d}s$

$$\frac{a(t/\epsilon^2)}{\epsilon}u_\epsilon(t) = \underbrace{\frac{a(t/\epsilon^2)}{\epsilon}u_\epsilon(t-\epsilon^2)}_{\mathbb{E}(\dots)=0} + \underbrace{\frac{1}{\epsilon^2}\int_{t-\epsilon^2}^{t} ia(t/\epsilon^2)a(s/\epsilon^2)u_\epsilon(s)\,\mathrm{d}s}_{\mathcal{O}(1)}$$

$$\mathbb{E}\frac{i}{\epsilon}a(t/\epsilon^2)u(t) = -\int_0^1 \mathbb{E}\big(a(t/\epsilon^2)a(t/\epsilon^2-\tau)\big)\,\mathrm{d}\tau\ \mathbb{E}u_\epsilon(t) + \text{small terms}$$

so that the limit eq. is

$$\frac{\mathrm{d}}{\mathrm{d}t}u = -\lambda u, \qquad \lambda = \int_0^1 \mathbb{E}\big(a(0)a(-\tau)\big)\,\mathrm{d}\tau > 0.$$

The Quantum Model

$$\begin{cases} \gamma(n,m) \geq 0, \qquad \gamma(n,n) = 0, \qquad \gamma(n,m) = \gamma(m,n), \\ \omega(n,m) = -\omega(m,n) \in \mathbb{R}, \\ V_\epsilon(t;n,k) = V(t, t/\epsilon^2; n,k), \quad V(n,k) = \overline{V(k,n)} + \text{bounds} \end{cases}$$

$Q(\rho)(n,m) = i\gamma(n,m)\ \big(\rho(n,m)\delta(n,m) - \rho(n,m)\big)$

$[H_0, \rho] = -\omega(n,m)\rho(n,m)$

Set $Z(n,m) = \gamma(n,m) + i\omega(n,m)$:

$Z(n,m) = 0$ for $n = m$, and $\overline{Z(n,m)} = Z(m,n)$.

$$\begin{aligned} \partial_t \rho(t;n,m) + \frac{1}{\epsilon^2} Z(n,m)\rho(n,m) &= \frac{1}{\epsilon}\Theta_\epsilon[\rho](t;n,m) \\ &= -\frac{i}{\epsilon}\sum_{k\in\mathbb{N}} \Big[V_\epsilon(n,k)\rho(t;k,m) - V_\epsilon(k,m)\rho(t;n,k)\Big]. \end{aligned}$$

Θ_ϵ is a (uniformly) bounded operator on ℓ^2

The problem is well-posed in $C^0([0,\infty);\ell^2)$ + (uniform) estimates.

Eq. for the populations:

$$\partial_t \rho(t;n,n) = -\frac{i}{\epsilon} \sum_{k \in \mathbb{N}} \Big[V_\epsilon(n,k)\rho(t;k,n) - V_\epsilon(k,n)\rho(t;n,k) \Big].$$

depends only on the coherences

Eq. for the quantum coherences: $(n \neq m)$

$$\partial_t \rho(t;n,m) = -\frac{1}{\epsilon^2} Z(n,m)\rho(n,m) - \frac{i}{\epsilon} \sum_{k \in \mathbb{N}} \Big[V_\epsilon(n,k)\rho(t;k,m) - V_\epsilon(k,m)\rho(t;n,k) \Big].$$

Quantum Coherences are Damped when $\mathrm{Re} Z(n,m) = \gamma(n,m) \neq 0$,

Hence the solution relaxes to $\rho(t;n,m)\ \delta(n,m)$.

The Two-level Model

"1"=Ground state, "2"=Excited state

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{11}^{\epsilon} = -\frac{2}{\epsilon}\mathrm{Im}\big(V_{12}^{\epsilon}\rho_{21}^{\epsilon}\big), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\rho_{21}^{\epsilon} = -\frac{1}{\epsilon^2}(i\omega+\gamma)\rho_{21}^{\epsilon}+\frac{i}{\epsilon}V_{21}^{\epsilon}(2\rho_{11}^{\epsilon}-1).$$

Perturbation $V_{12}^{\epsilon}(t) = \exp\left(i(\Delta+\omega)t/\epsilon^2\right)$.

Set $\rho_{21}^{\epsilon}(0) = 0$, then as $\epsilon \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{11} = \frac{2\gamma}{\gamma^2+\Delta^2}(\rho_{22}-\rho_{11}), \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\rho_{11}+\rho_{22}) = 0$$

($\Delta = 0$ as well.)

However, when $\gamma = 0$, we have

$$\rho_{11}^{\epsilon}(t) = \frac{4}{4+\Delta^2/\epsilon^2}(\rho_{11}(0)-1/2)\cos\left(\sqrt{4+\Delta^2/\epsilon^2}\; t/\epsilon\right) + \frac{1}{4+\Delta^2/\epsilon^2}\left(\frac{\Delta^2}{\epsilon^2}\rho_{11}(0)+2\right),$$

Rabi's oscillations

Quantum Model: (Quasi-)Periodic oscillations

$V_\epsilon(t;n,m) = \mathcal{V}(t, \Omega t/\epsilon^2; n,m)$ with $\Omega \in \mathbb{R}^d \setminus \{0\}$, rationally independent components and $\gamma(n,m) \geq \gamma_\star > 0$ for $n \neq m$.

Multiscale Ansatz: $\rho_\epsilon(t;n,m) = \sum_j \epsilon^j \; \rho^{(j)}(t, \Omega t/\epsilon^2; n,m)$.

$\partial_t \longrightarrow \partial_t + \frac{1}{\epsilon^2}\Omega \cdot \nabla_\vartheta$ yields

$$\begin{aligned}
1/\epsilon^2 \text{ terms:} \quad & \Omega \cdot \nabla_\vartheta \rho^{(0)} + Z(n,m)\rho^{(0)} = 0, \\
1/\epsilon^1 \text{ terms:} \quad & \Omega \cdot \nabla_\vartheta \rho^{(1)} + Z(n,m)\rho^{(1)} = \Theta(t,\vartheta)[\rho^{(0)}], \\
1/\epsilon^0 \text{ terms:} \quad & \Omega \cdot \nabla_\vartheta \rho^{(2)} + Z(n,m)\rho^{(2)} = -\partial_t \rho^{(0)} + \Theta(t,\vartheta)[\rho^{(1)}], \ldots
\end{aligned}$$

$$\begin{aligned}
& \rho^{(0)}(t,\vartheta;n,n) = \rho^{(0)}(t;n,n), \qquad \rho^{(0)}(t,\vartheta;n,m) = 0 \quad \text{if } n \neq m, \\
& \rho^{(1)}(t,\vartheta;n,m) = i\chi(t,\vartheta;n,m)\,\big(\rho^{(0)}(t;m,m) - \rho^{(0)}(t;n,n)\big), \\
& \chi(t,\vartheta;n,m) = -\int_0^{+\infty} e^{-Z(n,m)\sigma}\; \mathcal{V}(t, \vartheta - \Omega\sigma; n,m)\,\mathrm{d}\sigma, \qquad n \neq m
\end{aligned}$$

Statement for the Quantum Model: (Quasi-)Periodic oscillations

Theorem. ρ_ϵ converges to $\rho(t;n,n)\delta(n,m)$ weakly in $L^2(\mathbb{R}^+;\boldsymbol{\ell}^2)$; the diagonal part $\rho_\epsilon(t;n,n)$ converges to $\rho(t;n,n)$ in $C^0([0,T];\ell^2 - weak)$ and the limit satisfies the Einstein rate equation

$$\partial_t \rho(t;n,n) = \sum_{k\in\mathbb{N}} A(t;n,k)\big(\rho(t;k,k) - \rho(t;n,n)\big),$$

$$\rho(0;n,n) = \lim_{\epsilon\longrightarrow 0} \rho_\epsilon^0(n,n) \qquad \text{weakly in } \ell^2,$$

$$A(t;n,k) = 2\mathrm{Re}\big(Z(n,k)\big) \int_\Theta |\chi(t,\vartheta;n,k)|^2 \,\mathrm{d}\vartheta > 0, \text{for } n \neq k.$$

Method of proof

- Uniform estimates
- Use Double-scale convergence [Nguetseng 89, Allaire 92] (to be adapted to the quasi-periodic framework):

Let u_ϵ be a bounded sequence in $L^2(\mathbb{R})$. Then, there exists a subsequence and $U \in L^2_\#(\mathbb{R} \times \Theta)$ such that for any trial function

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} u_\epsilon(t)\ \psi(t, \Omega t/\epsilon^2)\,\mathrm{d}t = \int_{\mathbb{R}}\int_{\Theta} U(t,\vartheta)\ \psi(t,\vartheta)\ \mathrm{d}\vartheta\,\mathrm{d}t.$$

- Multiply the equation by "oscillating test functions" [Evans 89-92, Tartar 86-89]

Quantum Model: Random Framework

$V_\epsilon(t;n,m) = \mathcal{V}(t/\epsilon^2;n,m)$ with $\mathcal{V}$ bounded random variable such that

$i)$ $\mathbb{E}(\mathcal{V}(\tau;n,m)) = 0$,

$ii)$ $E(\mathcal{V}(\tau;k,l)\ \mathcal{V}(\sigma;m,n)) = \mathcal{R}(\tau-\sigma;k,l,m,n)$

$iii)$ If $|\tau-\sigma| \geq \mathcal{T}$ then $\mathcal{V}(\tau)$ and $\mathcal{V}(\sigma)$ are independent.

Theorem. Let ρ_ϵ^0 be deterministic. Suppose $Z(n,m) = \gamma(n,m) + i\omega(n,m)$ **vanishes iff** $n=m$. Then, $\mathbb{E}\rho_\epsilon(t;n,m)$ converges weakly to $\rho(t;n,n)\delta(n,m)$ and $\mathbb{E}\rho_\epsilon(t;n,n)$ converges in $C^0([0,T];\ell^2-weak)$ to $\rho \in L^\infty(\mathbb{R}^+;\ell^2)$ sol. of the Einstein eq. with

$$A(n,k) = 2\mathrm{Re}\int_0^{\mathcal{T}} \mathcal{R}(\tau;n,k,k,n)\ e^{-Z(k,n)\tau}\,\mathrm{d}\tau.$$

Comments

• If all $\gamma(n,m) = 0$ it means that $\omega(n,m) = H_0(m,m) - H_0(n,n) \neq 0$ where $H_0(n,n)$=eigenvalues of a differential operator: non–degeneracy assumption.

Need relaxation for energy levels corresponding to multidimensional eigenspaces.

• Alternative: Same statement with possibly vanishing damping coefficients

$$\gamma_\epsilon(n,m) \geq \underline{\gamma}_\epsilon > 0, \qquad \gamma_\epsilon(n,m) \xrightarrow[\epsilon \to 0]{} \gamma(n,m) \geq 0, \qquad \frac{\epsilon^2}{\underline{\gamma}_\epsilon} \xrightarrow[\epsilon \to 0]{} 0.$$

Trick: use the "Entropy Estimate"

$$\rho_\epsilon(t;n,m) = \rho_\epsilon(t;n,n)\delta(n,m) + \frac{\epsilon}{\sqrt{\underline{\gamma}_\epsilon}} r_\epsilon(t;n,m)$$

with $r_\epsilon(t;n,m)$ bounded.

A New Classical Model

Goal: Mimic the quantum relaxation operator

Populations $\simeq$ # of particles on a energy shell $\Sigma_E = \{H_0(x,p) = E\}$.

Hypothesis (think of $H_0(x,p) = x^2 + p^2$)

• $H_0 \in C^\infty(\mathbb{R}^{2D})$, $\quad \lim_{|(x,p)|\to\infty} H_0(x,p) = +\infty$ (Confining).

• For a.a. $E \in \mathbb{R}$, Σ_E is a smooth orientable $(2D-1)$ submanifold of $\mathbb{R}^{2D}$. We set $\delta(H_0(x,p) - E) := \dfrac{\mathrm{d}\Sigma_E(x,p)}{\left|\nabla_{x,p} H_0(x,p)\right|}$ (Liouville's measure) and suppose $h_0(E) := \int_{\Sigma_E} \delta(H_0(x,p) - E) < +\infty$.

Define $Pf(x,p) = \dfrac{1}{h_0(E)} \displaystyle\int_{\Sigma_E} f(y,q)\ \delta(H_0(y,q) - E)\ \Big|_{E=H_0(x,p)}$

Classical Model

Fundamental properties follow from the coarea formula

$$\int_{\mathbb{R}^{2D}} f(x,p)\,\mathrm{d}p\,\mathrm{d}x = \int_{\mathbb{R}} \left(\int_{\Sigma_E} f(x,p)\,\delta(H_0(x,p)-E) \right) \mathrm{d}E$$

which yields $\begin{cases} P \text{ is a bounded operator on } L^r, & P(Pf) = Pf, \\ \displaystyle\int Pf \ \mathrm{d}p\ dx = \int f \ \mathrm{d}p\ dx, & P\{H_0, f\} = 0 \end{cases}$

$$\underbrace{\partial_t f_\epsilon + \frac{1}{\epsilon^2}\{H_0, f_\epsilon\}}_{\text{Transport along } X_\epsilon(t),\, P_\epsilon(t)} \quad \underbrace{+\frac{1}{\epsilon}\{V_\epsilon, f_\epsilon\}}_{\text{Fast Varying Perturbation}} \quad \underbrace{= \frac{\gamma}{\epsilon^2}(Pf_\epsilon - f_\epsilon)}_{\text{Resonant Interaction}}$$

where $\dfrac{\mathrm{d}}{\mathrm{d}t}(X_\epsilon, P_\epsilon) = \dfrac{1}{\epsilon^2}(\nabla_p H_0, -\nabla_x H_0)(X_\epsilon, P_\epsilon)$

“H-Theorem”: $\|f_\epsilon(t)\|^2_{L^2(\mathbb{R}^{2N})} + \dfrac{\gamma}{\epsilon^2}\|f_\epsilon - Pf_\epsilon\|^2_{L^2((0,\infty)\times\mathbb{R}^{2N})} \le \|f_0\|^2_{L^2(\mathbb{R}^{2N})}$

Write $f_\epsilon = Pf_\epsilon + \epsilon g_\epsilon$ where

$$\partial_t Pf_\epsilon = -P\{V_\epsilon, g_\epsilon\}$$

$$\partial_t g_\epsilon = -\frac{\gamma}{\epsilon^2} g_\epsilon - \frac{1}{\epsilon^2}\{H_0, g_\epsilon\} - \frac{1}{\epsilon^2}\{V_\epsilon, Pf_\epsilon\} - \frac{1}{\epsilon}(I-P)\{V_\epsilon, g_\epsilon\}$$

The remainder is damped for $\gamma > 0$

and the solution relaxes to $Pf(t, H_0(x,p))$.

Quasi-periodic Framework

Theorem. Suppose $V_\epsilon(t;x,p) = V(t, \Omega t/\epsilon^2; x, p)$ with Ω having rationaly indep. components Then, $f_\epsilon = Pf_\epsilon + \epsilon g_\epsilon$ where g_ϵ is bounded in $L^2((0,T)\times\mathbb{R}^{2D})$ and, up to a subsequence, $Pf_\epsilon(t;x,p)$ converges to $F(t; H_0(x,p))$ in $C^0([0,T]; L^2(\mathbb{R}^{2D}) - weak)$, with

$$\partial_t(h_0 F) = \partial_E(h_0 d \partial_E F)$$
$$d(t;E) = \Pi\Big(\int_\Theta \{\mathcal{V}, H_0\}\ \chi\,\mathrm{d}\vartheta\Big)(E) \geq 0,$$
$$\chi(t,\vartheta;x,p) = -\int_0^\infty e^{-\gamma s}\{\mathcal{V}, H_0\}(t, \vartheta - \Omega s; \mathcal{X}(-s;x,p), \mathcal{P}(-s;x,p))\,\mathrm{d}s$$

Crucial Assumptions

- Damping $\gamma > 0$

- Stability Property (or increase γ)

$$\sup_{|(x,p)|\leq R} \Big|\nabla_{x,p}(\mathcal{X}(t;x,p), \mathcal{P}(t;x,p))\Big| \leq C_R\ (1+|t|)^{q_R}$$

A Simple Example

$$H_0(x,p) = \frac{x^2+p^2}{2}, \qquad V(t/\epsilon^2, x) = x\cos(\omega t/\epsilon^2)$$

$$\Pi f(E) = \frac{1}{2\pi}\int_0^{2\pi} f\left(\sqrt{2E}\cos(\sigma), \sqrt{2E}\sin(\sigma)\right) d\sigma$$

$$d(E) = \frac{\pi E}{2}\left(\frac{\gamma}{(\omega+1)^2+\gamma^2} + \frac{\gamma}{(\omega-1)^2+\gamma^2}\right)$$

If $\omega = \pm 1$, the coefficient blows up as $\gamma \to 0$: resonance phenomena

Random Framework

Theorem. Let $\gamma = 0$. Suppose that $\{H_0, f\} = 0$ iff $f(y) = F(H_0(y))$. Then, $\mathbb{E}Pf_\epsilon \to F(t; H_0(y))$ in $C^0([0,T]; L^2(\mathbb{R}^{2D}) - weak)$, sol. of a diffusion eq. with

$$d(E) = \Pi \left(\int_0^{\mathcal{T}} \mathcal{R}(\tau; \mathcal{Y}(\tau; y), y) : J\nabla H_0(\mathcal{Y}(\tau; y)) \otimes J\nabla H_0(y)\, \mathrm{d}\tau \right)(E).$$

- Is it interesting ? Not so much!

$H_0(x,p) = (x^2 + p^2)/2$, works in 1D but fails for $D \geq 2$ (since $\{H_0, x \wedge p\} = 0$). Related to ergodicity of H_0 [Knauf 87, Donnay-Liverani 91]

- But we can deal with vanishing damping coefficients

$$\gamma_\epsilon \to \gamma_0 \geq 0, \qquad \gamma_\epsilon > 0, \qquad \epsilon^2/\gamma_\epsilon \xrightarrow[\epsilon \to 0]{} 0$$

- Relaxed Assumptions on H_0: bounded derivatives of order 2, 3.

Comments

• More general oscillating potentials can be dealt with KBM= Krylov-Bogolioubov-Mitropolski type (long time average assumption)

Difficulty: the action of P on Sobolev spaces is unclear.

• Solutions of $\{H_0, f\} = 0$: $H_0, I_1, \ldots, I_K$, then defines P to be a projection onto given involution quantities. This would lead a $K + 1$-dimensional diffusion equation.

• Establish relation between the quantum and the classical models through Semi-Classical Limit.