Scattering and resonances in leaky quantum-wire systems

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Talk overview

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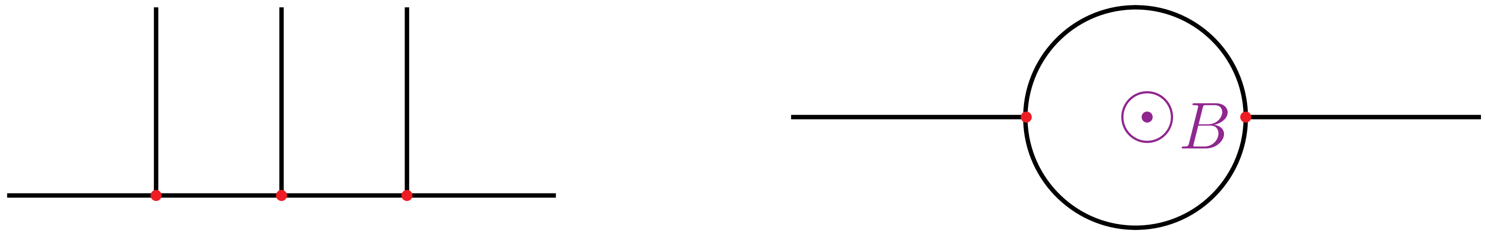
- Approximation by point interaction Hamiltonians and indications for existence of resonances
- Open questions

Scattering on quantum-wire systems

Widely used: scattering on “ideal” graphs, e.g.



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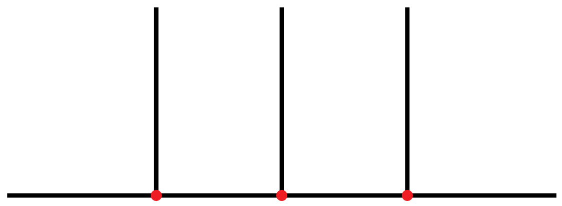

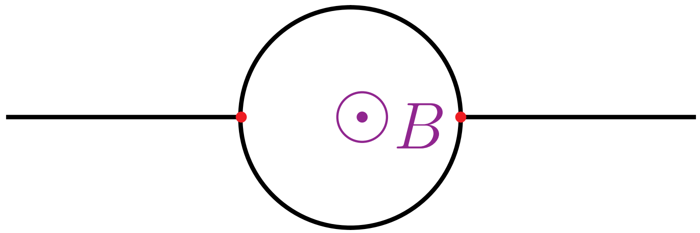

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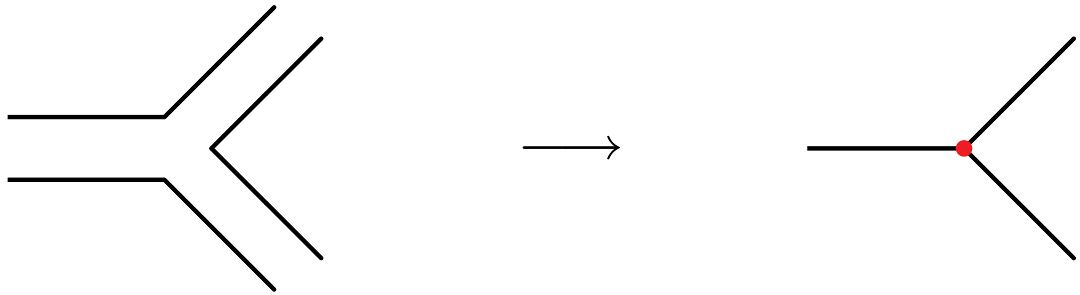
More realistic models of quantum wires treat them as *finite-width channels*, typically with Dirichlet b.c. Various scattering problems studied numerically in many papers.

Rigorous results not so common, for instance, resonances existence in smoothly bent tubes was demonstrated in [Duclos-E.-Šťovíček’95], [Duclos-E.-Meller’98].

Drawbacks of these models

First, we have *ad hoc parameters* in the b.c. describing branchings. A way out: use a zero-width limit, schematically



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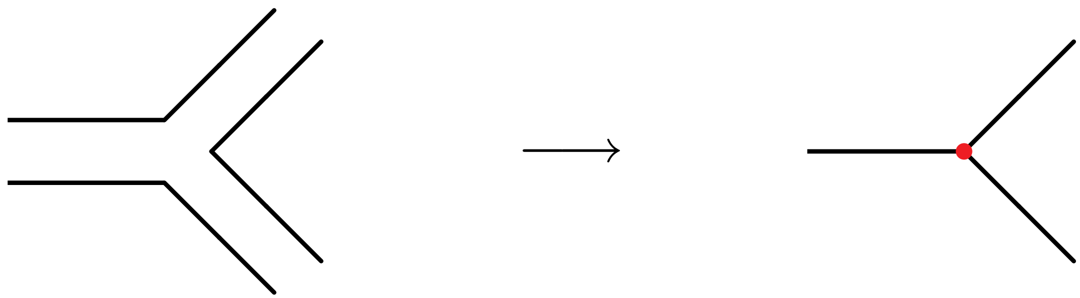

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- after a long effort the *Neumann-like case* is understood, see [Freidlin-Wentzell'93], [Freidlin'96], [Saito'01], [Kuchment-Zeng'01], [Rubinstein-Schatzmann'01], [E.-Post'05, 07], [Post'06], giving basically free b.c. only
- recently a progress achieved in the *Dirichlet case* [Post'05], [Molchanov-Vainberg'07], [Grieser'07], Cacciapuoti-E.'07].

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- At least in principle, the difficulty with the *ad hoc parameters* can be thus solved
- The "ideal" graph models have, however, another flaw, namely that *quantum tunneling is neglected*:

  recall that a true quantum-wire boundary is a finite potential jump, so an electron may pass between different quantum wires by tunneling the classically forbidden region between them

  This motivates us to look for an alternative quantum graph model

Leaky quantum graphs

We consider "leaky" graphs with an *attractive interaction supported by graph edges*. Formally we have

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)\,, \quad \alpha > 0\,,$$

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A proper definition of $H_{\alpha,\Gamma}$: it can be associated naturally with the quadratic form,

$$\psi \mapsto \|\nabla\psi\|^2_{L^2(\mathbb{R}^n)} - \alpha \int_\Gamma |\psi(x)|^2 \mathrm{d}x\,,$$

which is closed and below bounded in $W^{1,2}(\mathbb{R}^n)$; the second term makes sense in view of Sobolev embedding. This definition also works for various "wilder" sets Γ

Leaky quantum-graph Hamiltonians

For Γ with locally finite number of smooth edges and *no cusps* we can use an *alternative definition* by boundary conditions: $H_{\alpha,\Gamma}$ acts as $-\Delta$ on functions from $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial\psi}{\partial n}(x)\Big|_+ - \frac{\partial\psi}{\partial n}(x)\Big|_- = -\alpha\psi(x)$$

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Remarks:

- for graphs in $\mathbb{R}^3$ we use generalized b.c. which define a two-dimensional point interaction in normal plane
- one can combine “edges” of different dimensions as long as $\operatorname{codim}\Gamma$ does not exceed three

Geometrically induced spectrum

(a) *Bending means binding*, i.e. it may create isolated eigenvalues of $H_{\alpha,\Gamma}$. Consider a *piecewise* C^1*-smooth* $\Gamma:\ \mathbb{R}\to\mathbb{R}^2$ parameterized by its arc length, and assume:

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- $|\Gamma(s)-\Gamma(s')|\geq c|s-s'|$ holds for some $c\in(0,1)$
- Γ is asymptotically straight: there are $d>0,\;\mu>\frac{1}{2}$ and $\omega\in(0,1)$ such that

$$1-\frac{|\Gamma(s)-\Gamma(s')|}{|s-s'|}\leq d\left[1+|s+s'|^{2\mu}\right]^{-1/2}$$

  in the sector $S_\omega:=\left\{(s,s'):\;\omega<\frac{s}{s'}<\omega^{-1}\right\}$

- straight line is excluded, i.e. $|\Gamma(s)-\Gamma(s')|<|s-s'|$ holds for some $s,s'\in\mathbb{R}$

Bending means binding

Theorem [E.-Ichinose, 2001]: Under these assumptions, $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [-\frac{1}{4}\alpha^2, \infty)$ and $H_{\alpha,\Gamma}$ has at least one eigenvalue below the threshold $-\frac{1}{4}\alpha^2$

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- The same for *curves in* $\mathbb{R}^3$, under stronger regularity, with $-\frac{1}{4}\alpha^2$ is replaced by the corresponding 2D p.i. ev
- For *curved surfaces* $\Gamma \subset \mathbb{R}^3$ such a result is proved in the strong coupling asymptotic regime only
- *Implications for graphs:* let $\tilde{\Gamma} \supset \Gamma$ in the set sense, then $H_{\alpha,\tilde{\Gamma}} \le H_{\alpha,\Gamma}$. If the essential spectrum threshold is the same for both graphs and Γ fits the above assumptions, we have $\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma}) \neq \emptyset$ by minimax principle

Geometrically induced spectrum, contd

(b) *Strong coupling asymptotics:* let $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be as above, now supposed to be C^4-*smooth*

Theorem [E.-Yoshitomi, 2001]: The j-th ev of $H_{\alpha,\Gamma}$ is

$$\lambda_j(\alpha) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1}\ln\alpha) \quad \text{as} \quad \alpha \to \infty\,,$$

where μ_j is the j-th ev of $K_\Gamma := -\frac{\mathrm{d}}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$ on $L^2(\mathbb{R})$ and γ is the curvature of Γ.

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where μ_j is the j-th ev of $K_\Gamma := -\frac{\mathrm{d}}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$ on $L^2(\mathbb{R})$ and γ is the curvature of Γ. The same holds if Γ is a loop; then we also have

$$\#\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma}) = \frac{|\Gamma|\alpha}{2\pi} + \mathcal{O}(\ln\alpha) \quad \text{as} \quad \alpha \to \infty$$

Further extensions

- $H_{\alpha,\Gamma}$ with a *periodic* Γ has a band-type spectrum, but analogous asymptotics is valid for its *Floquet components* $H_{\alpha,\Gamma}(\theta)$, with the comparison operator $K_{\Gamma}(\theta)$ satisfying the appropriate b.c. over the period cell. It is important that the error term is uniform w.r.t. θ

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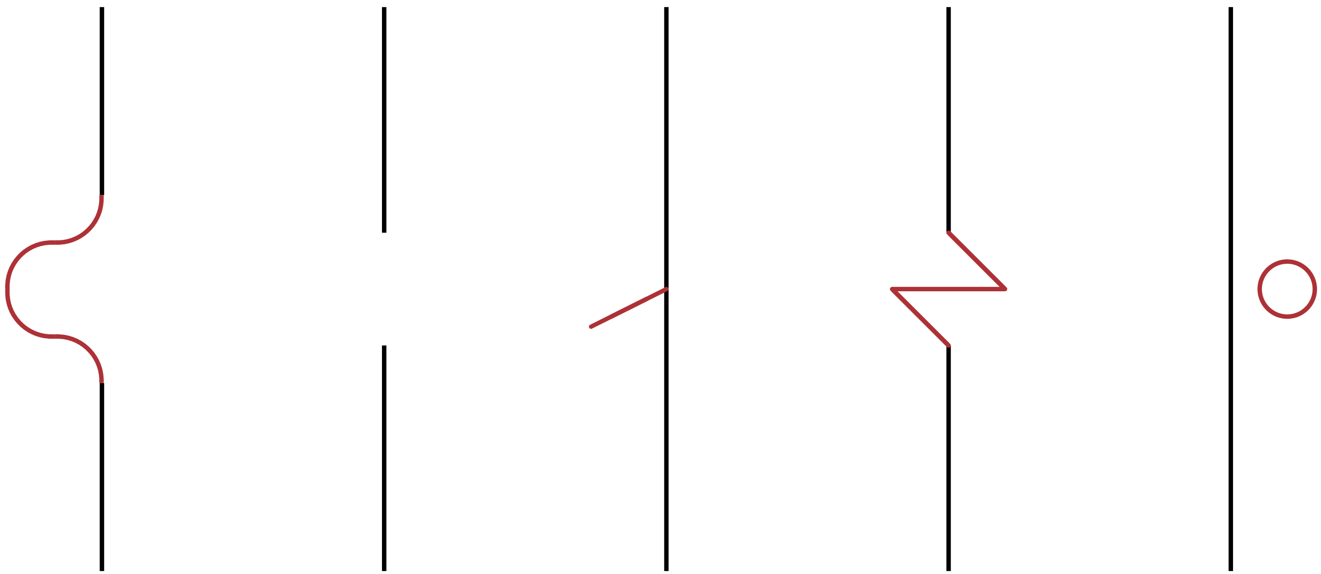
- Similar result holds for planar loops *threaded by mg field*, homogeneous, AB flux line, etc.
- *Higher dimensions:* the results extend to loops, infinite and periodic curves in $\mathbb{R}^3$
- and to *curved surfaces* in $\mathbb{R}^3$; then the comparison operator is $-\Delta_{\mathrm{LB}} + K - M^2$, where K, M, respectively, are the corresponding Gauss and mean curvatures

Scattering on a locally deformed line

Scattering requires to specify a *free dynamics*. In this talk we suppose that the latter is described by $H_{\alpha,\Sigma}$, where Σ is a *straight line*, $\Sigma = \{(x_1, 0) : x_1 \in \}$, and that the graph Γ in question differs from Σ by a *local deformation* only



Assumptions

We will consider the following class of local deformations:

- there exists a *compact* $M \subset \mathbb{R}^2$ such that $\Gamma \setminus M = \Sigma \setminus M$,
- the set $\Gamma \setminus \Sigma$ admits a finite decomposition,

$$\Gamma \setminus \Sigma = \bigcup_{i=1}^{N} \Gamma_i \,, \quad N < \infty \,,$$

where the Γ_i's are finite C^1 curves such that *no pair* of components of Γ *crosses* at their interior points, neither a component has a *self-intersection*; we allow the components to touch at their endpoints but assume they do not form a *cusp* there

As we have said, $H_{\alpha,\Gamma}$ is then well defined

Krein's formula

Our main tool will be a formula comparing the resolvents of $H_{\alpha,\Gamma}$ and $H_{\alpha,\Sigma}$. We will use the decomposition

$$\Lambda = \Lambda_0 \cup \Lambda_1 \quad \text{with} \quad \Lambda_0 := \Sigma \setminus \Gamma\,, \;\; \Lambda_1 := \Gamma \setminus \Sigma = \bigcup_{i=1}^{N} \Gamma_i\,;$$

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the coupling constant of the perturbation will be naturally equal to α on the "subtracted" set Λ_0 and $-\alpha$ on Λ_1
To construct resolvent of $H_{\alpha,\Sigma}$ we use R^k, the one of $-\Delta$, which is for $k^2 \in \rho(-\Delta)$ an integral operator with the kernel

$$G^k(x-y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{ip(x-y)}}{p^2-k^2}\, \mathrm{d}p = \frac{1}{2\pi} K_0(ik|x-y|)\,,$$

where $K_0(\cdot)$ stands for the Macdonald function

Krein's formula, continued

A straightforward computation shows that the resolvent R^k_Σ of $H_{\alpha,\Sigma}$ has the kernel $G^k_\Sigma(x-y)$ given by

$$G^k(x-y) + \frac{\alpha}{4\pi^3}\int_{3} \frac{\mathrm{e}^{ipx-ip'y}}{(p^2-k^2)(p'^2-k^2)}\,\frac{\tau_k(p_1)}{2\tau_k(p_1)-\alpha}\,\mathrm{d}p\,\mathrm{d}p'_2\,,$$

where $\tau_k(p_1) := (p_1^2 - k^2)^{1/2}$ and $p = (p_1, p_2)$, $p' = (p_1, p'_2)$

Krein's formula, continued

A straightforward computation shows that the resolvent R_Σ^k of $H_{\alpha,\Sigma}$ has the kernel $G_\Sigma^k(x-y)$ given by

$$G^k(x-y) + \frac{\alpha}{4\pi^3}\int_3 \frac{e^{ipx-ip'y}}{(p^2-k^2)(p'^2-k^2)}\,\frac{\tau_k(p_1)}{2\tau_k(p_1)-\alpha}\,\mathrm{d}p\,\mathrm{d}p_2',$$

where $\tau_k(p_1) := (p_1^2-k^2)^{1/2}$ and $p=(p_1,p_2)$, $p'=(p_1,p_2')$

We need embeddings of R_Σ^k to $L^2(\nu)$, where $\nu \equiv \nu_\Lambda$ is the Dirac measure on Λ. It can be written as $\nu_\Lambda = \nu_0 + \sum_{i=1}^N \nu_i$, where ν_0 is the Dirac measure on Λ_0. It convenient also to introduce the space $\mathrm{h} \equiv L^2(\nu)$ which decomposes into

$$\mathrm{h} = \mathrm{h}_0 \oplus \mathrm{h}_1 \quad \text{with} \quad \mathrm{h}_0 \equiv L^2(\nu_0) \quad \text{and} \quad \mathrm{h}_1 \equiv \bigoplus_{i=1}^N L^2(\nu_i)$$

Embeddings

Now we are able to introduce the operator

$$\mathrm{R}^k_{\Sigma,\nu} : \mathrm{h} \to L^2\,, \quad \mathrm{R}^k_{\Sigma,\nu} f = G^k_\Sigma * f\nu \quad \text{for} \quad f \in \mathrm{h}$$

defined for suitable values of k. Similarly, $(\mathrm{R}^k_{\Sigma,\nu})^* : L^2 \to \mathrm{h}$ is its adjoint and $\mathrm{R}^k_{\Sigma,\nu\nu}$ denotes the operator-valued matrix in h with the "block elements" $G^k_{\Sigma,ij} \equiv G^k_{\Sigma,\nu_i\nu_j} : L^2(\nu_j) \to L^2(\nu_i)$

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They have the following properties:

- For any $\kappa \in (\alpha/2, \infty)$ the operator $\mathrm{R}^{i\kappa}_{\Sigma,\nu}$ is bounded. In fact, $\mathrm{R}^{i\kappa}_{\Sigma,\nu}$ is a continuous embedding into $W^{1,2}$
- For any $\sigma > 0$ there exists κ_σ such that for $\kappa > \kappa_\sigma$ the operator $\mathrm{R}^{i\kappa}_{\Sigma,\nu\nu}$ is bounded with the norm less than σ

Krein's formula, continued

Introduce an operator-valued matrix in $\mathrm{h} = \mathrm{h}_0 \oplus \mathrm{h}_1$ as

$$\Theta^k = -(\alpha^{-1}\check{\mathbb{I}} + \mathrm{R}^k_{\Sigma,\nu\nu}) \quad \text{with} \quad \check{\mathbb{I}} = \begin{pmatrix} \mathbb{I}_0 & 0 \\ 0 & -\mathbb{I}_1 \end{pmatrix},$$

where $\mathbb{I}_i$ are the unit operators in h_i. Using the properties of the embeddings we prove the following claim:

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Theorem [E.-Kondej, 2005]: Let Θ^k have inverse in $\mathcal{B}(\mathrm{h})$ for $k \in \mathbb{C}^+$ and let the operator

$$R^k_\Gamma = R^k_\Sigma + \mathrm{R}^k_{\Sigma,\nu}(\Theta^k)^{-1}(\mathrm{R}^k_{\Sigma,\nu})^*$$

be defined everywhere on L^2. Then k^2 belongs to $\rho(H_{\alpha,\Gamma})$ and the resolvent $(H_{\alpha,\Gamma} - k^2)^{-1}$ is given by R^k_Γ

Spectrum of $H_{\alpha,\Gamma}$

Let us first look at the essential spectrum:

Proposition: $\sigma_{\rm ess}(H_{\alpha,\Gamma}) = \sigma_{\rm ess}(H_{\alpha,\Sigma}) = [-\frac{1}{4}\alpha^2, \infty)$

Proof: Check that $B^k := \mathrm{R}^k_{\Sigma,\nu}(\Theta^k)^{-1}(\mathrm{R}^k_{\Sigma,\nu})^*$ is compact for some $k \in \mathbb{C}^+$. We know that $(\Theta^{i\kappa})^{-1} \in \mathcal{B}(\mathrm{h})$ and $(\mathrm{R}^{i\kappa}_{\Sigma,\nu})^*$ is bounded if κ is large enough. By [BEKŠ'94] we have $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |G^{i\kappa}(x-y)|^2\,\nu_j(\mathrm{d}y)\,\mathrm{d}x < \infty$, and for $\kappa > \frac{1}{2}\alpha$ and $j = 0, \dots, N$ the second component ξ^k of $G^{i\kappa}_\Sigma$ satisfies

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\xi^k(x,y)|^2\,\nu_j(\mathrm{d}y)\,\mathrm{d}x < CL_j \int_{\mathbb{R}^2} \frac{\mathrm{d}p}{(p^2+\kappa)^2} < \infty\,,$$

where C is a constant and L_j denote the length of Λ_j. This yields compactness of $\mathrm{R}^k_{\Sigma,\nu}$, and thus the same for B^k. $\square$

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Remark: $\sigma_{\rm disc}(H_{\alpha,\Gamma})$ given by singularities of Θ^k is often non-empty – see above – but it is not our concern here

Wave operators

The existence and completeness of wave operators for the pair $(H_{\alpha,\Gamma}, H_{\alpha,\Sigma})$ follows from the standard trace-class criterion, conventionally called Birman-Kuroda theorem. Specifically, we have

Theorem [E.-Kondej, 2005]: $B^{i\kappa}$ is a trace class operator for κ sufficiently large

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Proof is inspired by [Brasche-Teta'92]. We use the estimate $(\Theta^{i\kappa})^{-1} \le C'(\Theta^{i\kappa,+})^{-1}$, where $\Theta^{i\kappa,+} := \alpha^{-1}\mathbb{I} + \mathrm{R}^{i\kappa}_{\Sigma,\nu\nu}$ and $\mathbb{I}$ is the $(N+1)\times(N+1)$ unit matrix, for some $C'>0$ and all κ sufficiently large; it is clear that $(\Theta^{i\kappa,+})^{-1}$ is positive and bounded. This gives

$$B^{i\kappa} \le C' B^{i\kappa,+}\,, \qquad B^{i\kappa,+} := \mathrm{R}^{i\kappa}_{\Sigma,\nu}(\Theta^{i\kappa,+})^{-1}(\mathrm{R}^{i\kappa}_{\Sigma,\nu})^*$$

Proof, continued

Define $B_\delta^{i\kappa,+}$ as integral operator with the kernel

$$B_\delta^{i\kappa,+}(x,y) = \chi_\delta(x) B^{i\kappa,+}(x,y) \chi_\delta(y)\,,$$

where χ_δ stands for the indicator function of the ball $\mathcal{B}(0,\delta)$; one has $B_\delta^{i\kappa,+} \to B^{i\kappa,+}$ as $\delta \to \infty$ in the weak sense.

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$$\int_{\mathbb{R}^2} B_\delta^{i\kappa,+}(x,x)\mathrm{d}x = \int_{\mathbb{R}^2} (G_\Sigma^{i\kappa}(\cdot,x)\chi_\delta(x), (\Theta^{i\kappa,+})^{-1} G_\Sigma^{i\kappa}(\cdot,x)\chi_\delta(x))_\mathrm{h}\,\mathrm{d}x$$

$$\leq \|(\Theta^{i\kappa,+})^{-1}\| \int_{\mathbb{R}^2} \|G_\Sigma^{i\kappa}(\cdot,x)\chi_\delta(x)\|_\mathrm{h}^2\,\mathrm{d}x \leq C\|(\Theta^{i\kappa,+})^{-1}\|\,,$$

hence $B_\delta^{i\kappa,+}$ is trace class for any $\delta > 0$, and the same is true for the limiting operator.

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$$B_\delta^{i\kappa,+}(x,y) = \chi_\delta(x) B^{i\kappa,+}(x,y)\chi_\delta(y)\,,$$

where χ_δ stands for the indicator function of the ball $\mathcal{B}(0,\delta)$; one has $B_\delta^{i\kappa,+} \to B^{i\kappa,+}$ as $\delta \to \infty$ in the weak sense. Then

$$\int_{\mathbb{R}^2} B_\delta^{i\kappa,+}(x,x)\mathrm{d}x = \int_{\mathbb{R}^2} (G_\Sigma^{i\kappa}(\cdot,x)\chi_\delta(x), (\Theta^{i\kappa,+})^{-1} G_\Sigma^{i\kappa}(\cdot,x)\chi_\delta(x))_{\mathrm{h}}\,\mathrm{d}x$$

$$\le \|(\Theta^{i\kappa,+})^{-1}\| \int_{\mathbb{R}^2} \|G_\Sigma^{i\kappa}(\cdot,x)\chi_\delta(x)\|_{\mathrm{h}}^2\,\mathrm{d}x \le C\|(\Theta^{i\kappa,+})^{-1}\|\,,$$

hence $B_\delta^{i\kappa,+}$ is trace class for any $\delta > 0$, and the same is true for the limiting operator.

Similarly one finds a Hermitian trace class operator $B^{i\kappa,-}$ which provides an estimate from below, $B^{i\kappa,-} \le B^{i\kappa}$; this means that $B^{i\kappa}$ is a trace class operator too. $\square$

Generalized eigenfunctions

We want to find the S-matrix, $S\psi_\lambda^- = \psi_\lambda^+$, for scattering in the *negative part of the spectrum* with a fixed energy $\lambda \in (-\frac{1}{4}\alpha^2, 0)$ corresponding to the effective momentum $k_\alpha(\lambda) := (\lambda + \alpha^2/4)^{1/2}$. We employ generalized ef's of $H_{\alpha,\Sigma}$,

$$\omega_\lambda(x_1, x_2) = \mathrm{e}^{i(\lambda+\alpha^2/4)^{1/2}x_1}\mathrm{e}^{-\alpha|x_2|/2},$$

their analogues ω_z for complex energies and regularizations $\omega_z^\delta(x) = \mathrm{e}^{-\delta x_1^2}\omega_z(x)$ for $z \in \rho(H_{\alpha,\Sigma})$, belonging to $D(H_{\alpha,\Sigma})$.

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$$\psi_\lambda^\delta = \omega_\lambda^\delta + \mathrm{R}_{\Sigma,\nu}^{k_\alpha(\lambda)}(\Theta^{k_\alpha(\lambda)})^{-1}I_\Lambda\omega_\lambda^\delta$$

Generalized eigenfunctions, continued

Here $\mathrm{R}_{\Sigma,\nu}^{k_\alpha(\lambda)}$ is integral operator on the Hilbert space $\mathfrak{h}$ with the kernel $G_\Sigma^{k_\alpha(\lambda)}(x-y) := \lim_{\varepsilon\to 0} G_\Sigma^{k_\alpha(\lambda+i\varepsilon)}(x-y)$ and $\Theta^{k_\alpha(\lambda)} := -\alpha^{-1}\check{\mathbb{I}} - \mathrm{R}_{\Sigma,\nu\nu}^{k_\alpha(\lambda)}$ are the operators on $\mathfrak{h}$ with $\mathrm{R}_{\Sigma,\nu\nu}^{k_\alpha(\lambda)}$ being the natural embedding . By a direct computation, the kernel is found to be

$$G_\Sigma^{k_\alpha(\lambda)}(x-y) = K_0(i\sqrt{\lambda}|x-y|)$$
$$+\mathcal{P}\int_0^\infty \frac{\mu_0(t;x,y)}{t-\lambda-\alpha^2/4}\,\mathrm{d}t + \frac{i\alpha}{8k_\alpha(\lambda)}\,\mathrm{e}^{ik_\alpha(\lambda)|x_1-y_1|}\,\mathrm{e}^{-\alpha/2(|x_2|+|y_2|)},$$

where

$$\mu_0(t;x,y) := -\frac{i\alpha}{2^5\pi}\,\frac{\mathrm{e}^{it^{1/2}(x_1-y_1)}\,\mathrm{e}^{-(t-\lambda)^{1/2}(|x_2|+|y_2|)^{1/2}}}{t^{1/2}((t-\lambda)^{1/2})}\,.$$

Generalized eigenfunctions, continued

Of course, the pointwise limits $\psi_\lambda = \lim_{\delta\to 0} \psi_\lambda^\delta$ cease to L^2, however, they still belong to L^2 locally and provide us with the generalized eigenfunction of $H_{\alpha,\Gamma}$ in the form

$$\psi_\lambda = \omega_\lambda + \mathrm{R}_{\Sigma,\nu}^{k_\alpha(\lambda)} (\Theta^{k_\alpha(\lambda)})^{-1} J_\Lambda \omega_\lambda \,,$$

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To find the S-matrix we have to investigate the behavior of ψ_λ for $|x_1| \to \infty$. By a direct computation, we find that for y of a compact $M \subset \mathbb{R}^2$ and $|x_1| \to \infty$ we have

$$G_\Sigma^{k_\alpha(\lambda)}(x-y) \approx \frac{i\alpha}{8k_\alpha(\lambda)}\, \mathrm{e}^{ik_\alpha(\lambda)|x_1-y_1|}\, e^{-\alpha/2(|x_2|+|y_2|)}$$

S-matrix at negative energy

Using this asymptotics we find the sought on-shell S-matrix:

Theorem [E.-Kondej, 2005]: For a fixed $\lambda \in (-\frac{1}{4}\alpha^2, 0)$ the generalized eigenfunctions behave asymptotically as

$$\psi_\lambda(x) \approx \begin{cases} \mathcal{T}(\lambda)\, \mathrm{e}^{ik_\alpha(\lambda)x_1}\, \mathrm{e}^{-\alpha|x_2|/2} & \text{for} \quad x_1 \to \infty \\ \mathrm{e}^{ik_\alpha(\lambda)x_1} \mathrm{e}^{-\alpha|x_2|/2} + \mathcal{R}(\lambda)\, \mathrm{e}^{-ik_\alpha(\lambda)x_1} \mathrm{e}^{-\alpha|x_2|/2} & \text{for} \quad x_1 \to -\infty \end{cases}$$

where $k_\alpha(\lambda) := (\lambda + \alpha^2/4)^{1/2}$ and the *transmission and reflection amplitudes* $\mathcal{T}(\lambda)\,, \mathcal{R}(\lambda)$ are given respectively by

$$\mathcal{T}(\lambda) = 1 - \frac{i\alpha}{8k_\alpha(\lambda)} \left((\Theta^{k_\alpha(\lambda)})^{-1} J_\Lambda \omega_\lambda, J_\Lambda \omega_\lambda \right)_{\mathrm{h}}$$

and

$$\mathcal{R}(\lambda) = \frac{i\alpha}{8k_\alpha(\lambda)} \left((\Theta^{k_\alpha(\lambda)})^{-1} J_\Lambda \omega_\lambda, J_\Lambda \bar{\omega}_\lambda \right)_{\mathrm{h}}$$

Strong coupling: a conjecture

Consider Γ which is a C^4-*smooth* local deformation of a line. In analogy with the spectral result of [E.-Yoshitomi'01] quoted above one expects that in *strong coupling* case the scattering will be determined in the leading order by the *local geometry* of Γ through the same comparison operator, namely $K_\Gamma := -\frac{\mathrm{d}}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$ on $L^2(\mathbb{R})$.

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Let $\mathcal{T}_K(k)$, $\mathcal{R}_K(k)$ be the corresponding transmission and reflection amplitudes at a fixed momentum k. Denote by $\mathbf{S}_{\Gamma,\alpha}(\lambda)$ and $\mathbf{S}_K(\lambda)$ the on-shell $S-$matrixes of $H_{\alpha,\Gamma}$ and K at energy λ, respectively.

Conjecture: For a fixed $k \neq 0$ and $\alpha \to \infty$ we have the relation

$$\mathbf{S}_{\Gamma,\alpha}\left(k^2 - \frac{1}{4}\alpha^2\right) \to \mathbf{S}_K(k^2)$$

How to find the spectrum?

To say something about resonances, let us return to the spectral problem. The general results do not tell us how to find the spectrum for a particular Γ. The options:

- *Direct solution* of the PDE problem $H_{\alpha,\Gamma}\psi = \lambda\psi$ is feasible in a few simple examples only

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- *Direct solution* of the PDE problem $H_{\alpha,\Gamma}\psi = \lambda\psi$ is feasible in a few simple examples only
- Using trace maps of $R^k \equiv (-\Delta - k^2)^{-1}$ and the *generalized BS principle*

$$R^k := R_0^k + \alpha R_{\mathrm{d}x,m}^k [I - \alpha R_{m,m}^k]^{-1} R_{m,\mathrm{d}x}^k \,,$$

where m is δ measure on Γ, we pass to a 1D integral operator problem, $\alpha R_{m,m}^k \psi = \psi$

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  where m is δ measure on Γ, we pass to a 1D integral operator problem, $\alpha R_{m,m}^k \psi = \psi$
- *discretization* of the latter which amounts to a point-interaction approximations to $H_{\alpha,\Gamma}$

2D point interactions

Such an interaction at the point a with the "coupling constant" α is defined by b.c. which change *locally* the domain of $-\Delta$: the functions behave as

$$\psi(x) = -\frac{1}{2\pi} \log|x-a|\, L_0(\psi,a) + L_1(\psi,a) + \mathcal{O}(|x-a|)\,,$$

where the generalized b.v. $L_0(\psi,a)$ and $L_1(\psi,a)$ satisfy

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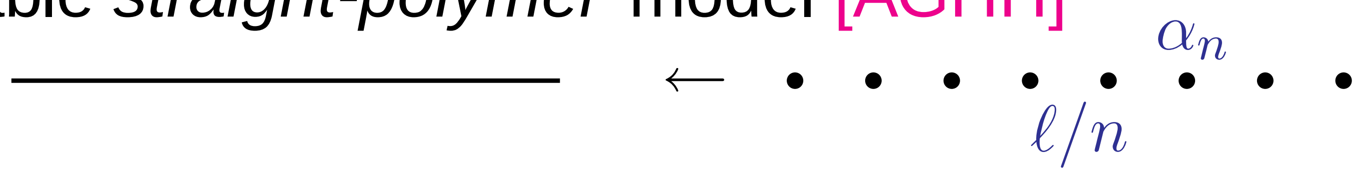
For our purpose, the coupling should depend on the set Y approximating Γ. To see how compare a line Γ with the solvable *straight-polymer* model [AGHH]



2D point-interaction approximation

Spectral threshold convergence requires $\alpha_n = \alpha n$ which means that individual point interactions get *weaker*. Hence we approximate $H_{\alpha,\Gamma}$ by point-interaction Hamiltonians H_{α_n,Y_n} with $\alpha_n = \alpha|Y_n|$, where $|Y_n| := \sharp Y_n$.

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Theorem [E.-Němcová, 2003]: Let a family $\{Y_n\}$ of finite sets $Y_n \subset \Gamma \subset \mathbb{R}^2$ be such that

$$\frac{1}{|Y_n|} \sum_{y \in Y_n} f(y) \to \int_\Gamma f \, \mathrm{d}m$$

holds for any bounded continuous function $f : \Gamma \to \mathbb{C}$, *together with technical conditions*, then $H_{\alpha_n,Y_n} \to H_{\alpha,\Gamma}$ in the strong resolvent sense as $n \to \infty$.

Comments on the approximation

- A more general result is valid: Γ need not be a graph and the coupling may be non-constant

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- A more general result is valid: Γ need not be a graph and the coupling may be non-constant
- The result applies to finite graphs, however, an infinite Γ can be approximated in strong resolvent sense by a *family of cut-off graphs*
- The idea is due to Brasche, Figari and Teta, 1998, who analyzed point-interaction approximations of measure perturbations with $\operatorname{codim}\Gamma = 1$ in $\mathbb{R}^3$. There are differences, however, for instance in the 2D case we can approximate *attractive* interactions only

Scheme of the proof

Resolvent of H_{α_n,Y_n} is given *Krein's formula*. Given $k^2 \in \rho(H_{\alpha_n,Y_n})$ define $|Y_n| \times |Y_n|$ matrix by

$$\Lambda_{\alpha_n,Y_n}(k^2;x,y) = \frac{1}{2\pi}\left[2\pi|Y_n|\alpha + \ln\left(\frac{ik}{2}\right) + \gamma_E\right]\delta_{xy} - G_k(x-y)\,(1-\delta_{xy})$$

for $x, y \in Y_n$, where γ_E is *Euler' constant*.

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$$-G_k(x-y)\,(1-\delta_{xy})$$

for $x, y \in Y_n$, where γ_E is *Euler' constant*. Then

$$(H_{\alpha_n,Y_n} - k^2)^{-1}(x,y) = G_k(x-y)$$
$$+ \sum_{x',y' \in Y_n} \left[\Lambda_{\alpha_n,Y_n}(k^2)\right]^{-1}(x',y') G_k(x-x')\, G_k(y-y')$$

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Remarks:

- Spectral condition in the n-th approximation, i.e. $\det \Lambda_{\alpha_n,Y_n}(k^2) = 0$, is a discretization of the integral equation coming from the generalized BS principle
- A solution to $\Lambda_{\alpha_n,Y_n}(k^2)\eta = 0$ determines the approximating ef by $\psi(x) = \sum_{y_j \in Y_n} \eta_j G_k(x - y_j)$
- A *match with solvable models* illustrates the convergence and shows that it is *not fast,* slower than n^{-1} in the eigenvalues. This comes from singular "spikes" in the approximating functions

Finally, the resonances

Consider infinite curves Γ, straight outside a compact, and ask for examples of resonances. Recall the L^2-approach: in 1D potential scattering one explores *spectral properties* of the problem cut to a finite length L. It is time-honored trick that scattering resonances are manifested as avoided crossings in L dependence of the spectrum – for a recent proof see Hagedorn-Meller, 2000. Try the same here:

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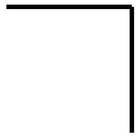
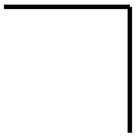
- *Broken line:* absence of "intrinsic" resonances due lack of higher transverse thresholds
- *Z-shaped* Γ*:* if a single bend has a significant reflection, a double band should exhibit resonances
- *Bottleneck curve:* a good candidate to demonstrate tunneling resonances

Broken line



$\alpha = 1$

Broken line



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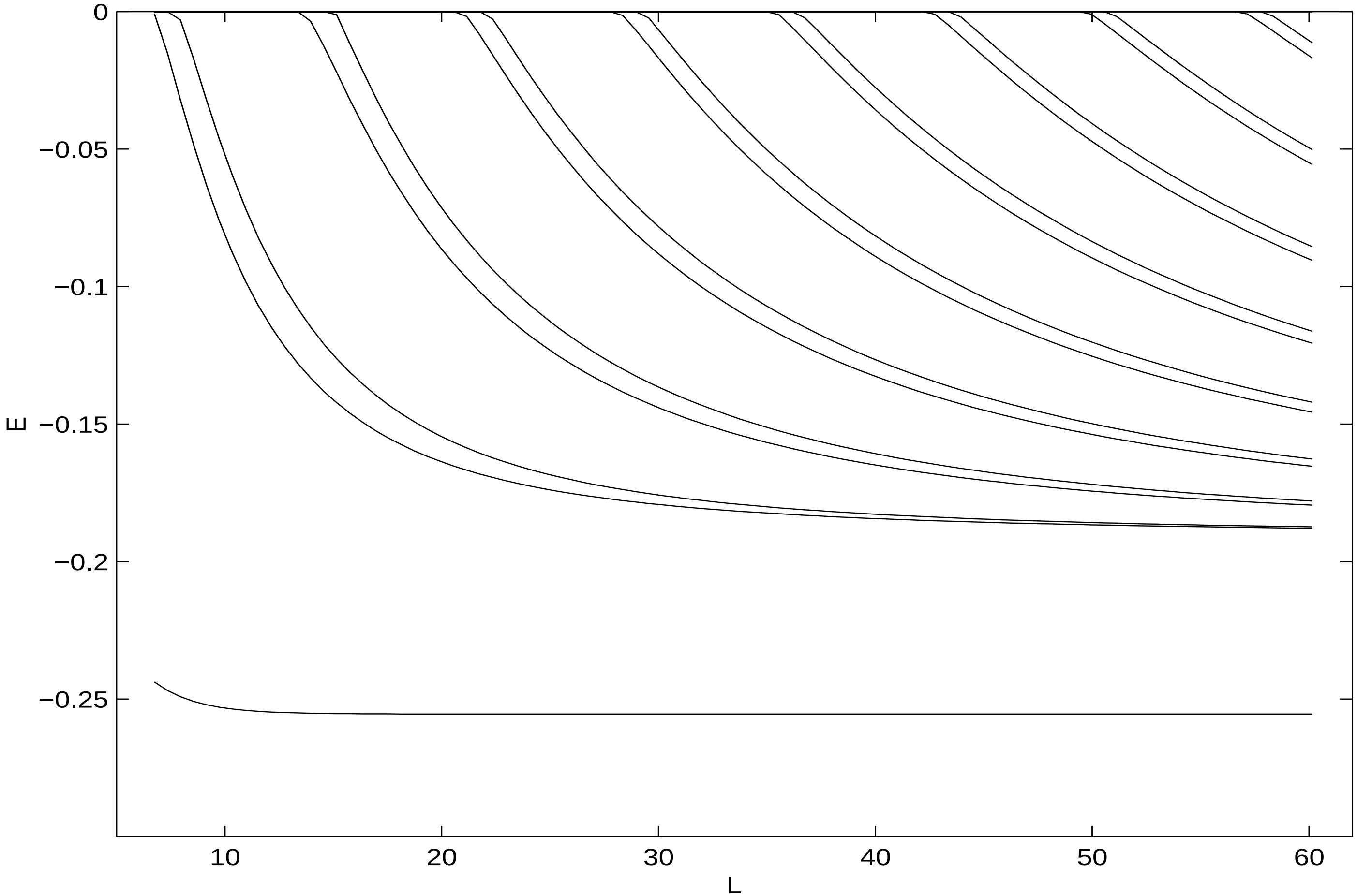
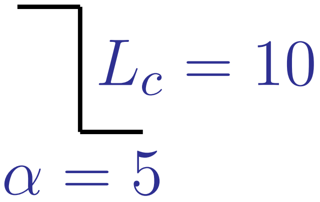


Z shape with $\theta = \frac{\pi}{2}$



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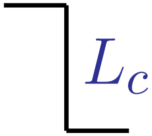
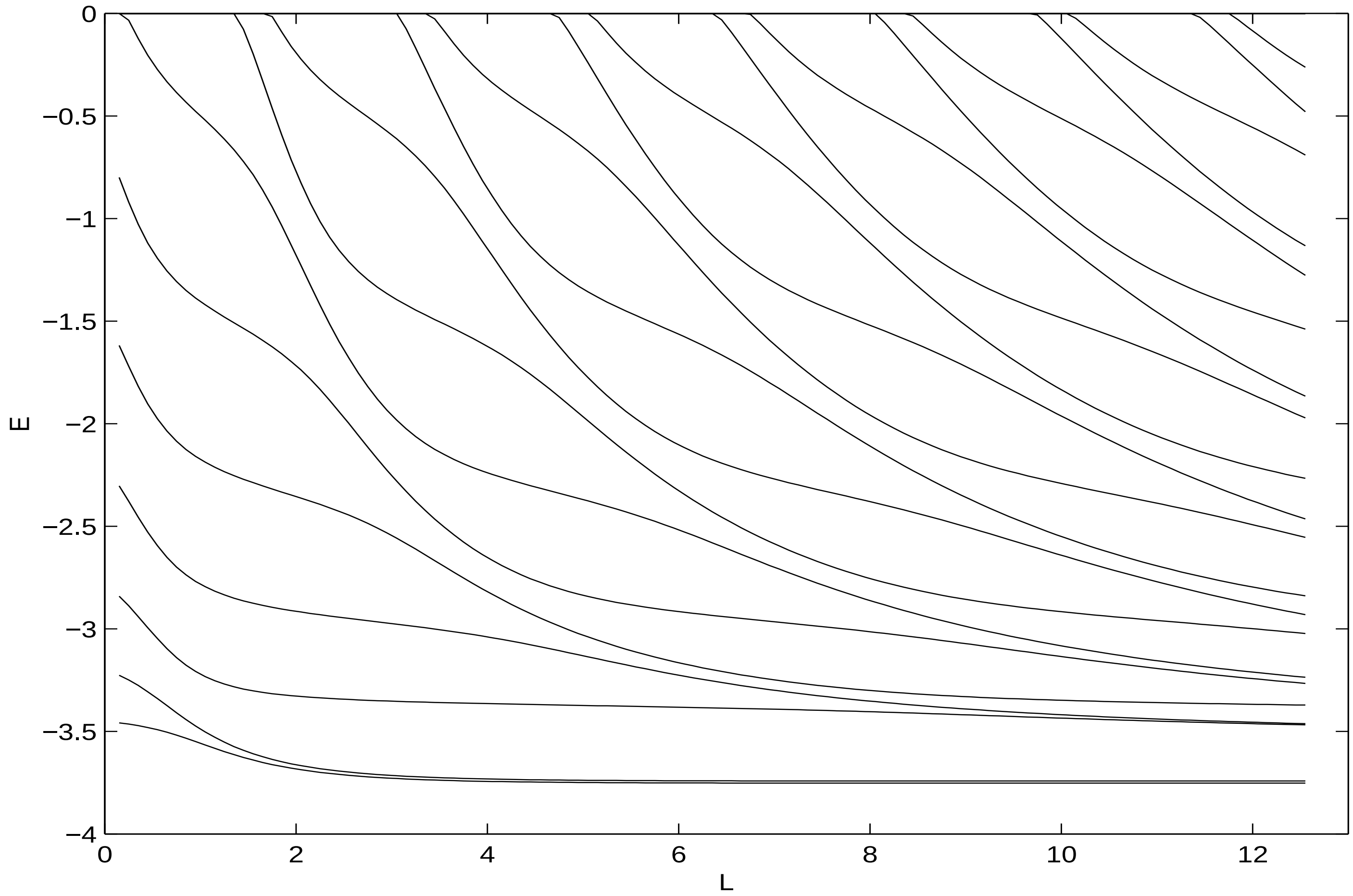


$L_c = 10$

$\alpha = 5$



Z shape with $\theta = 0.32\pi$

$L_c = 10$

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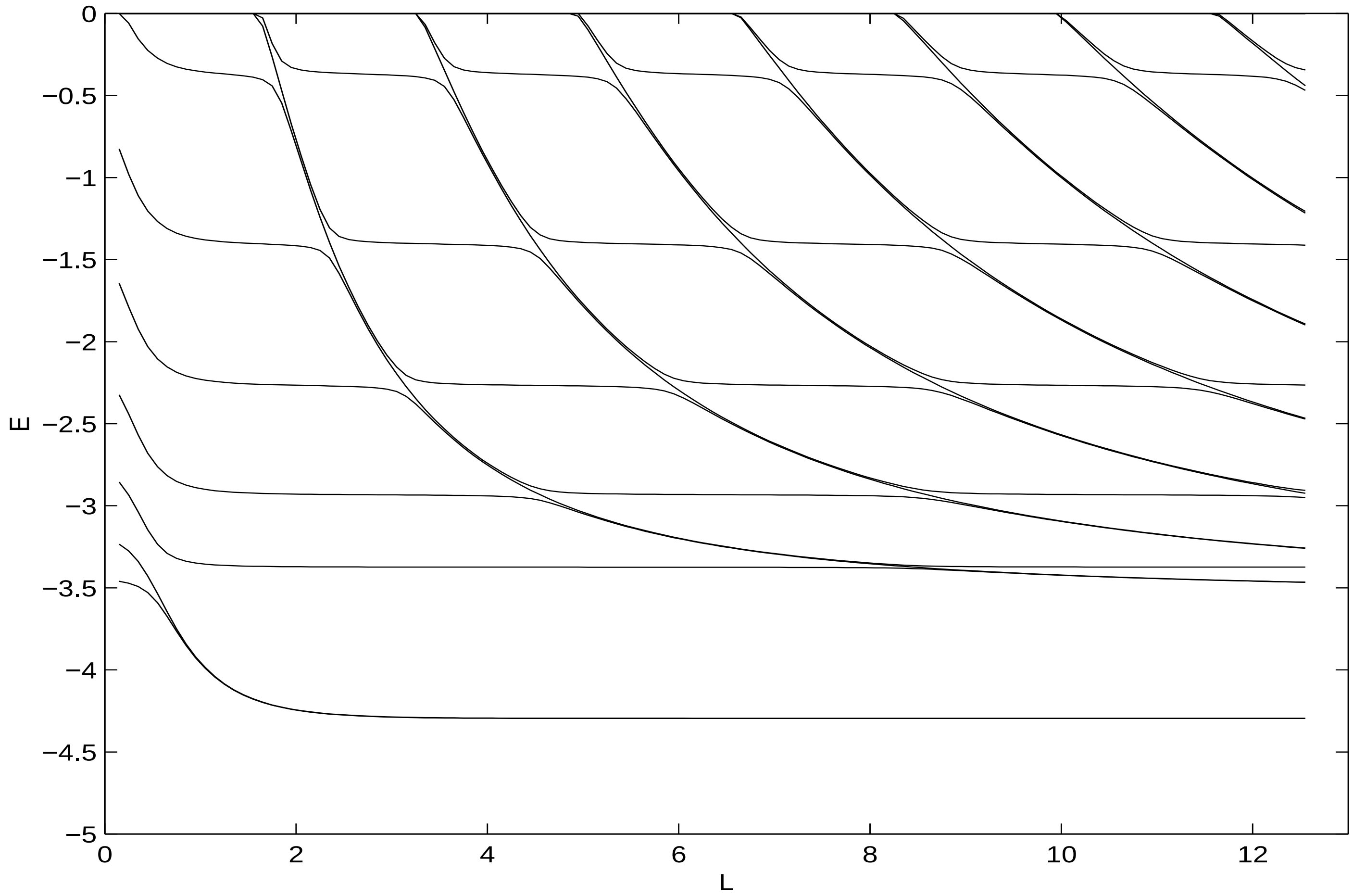
Z shape with $\theta = 0.32\pi$

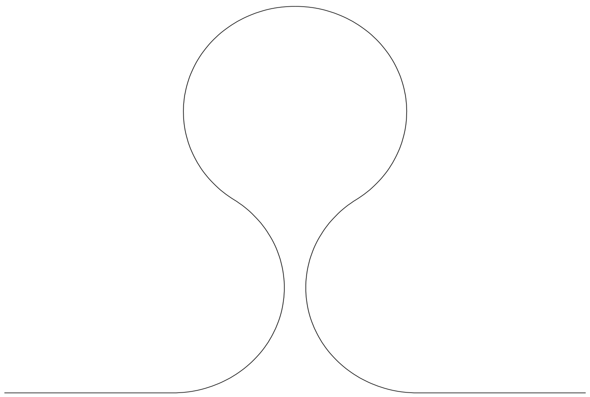
$\sum L_c = 10$

$\alpha = 5$



A bottleneck curve

Consider a straight line deformation which shaped as an open loop with a bottleneck the width a of which we will vary



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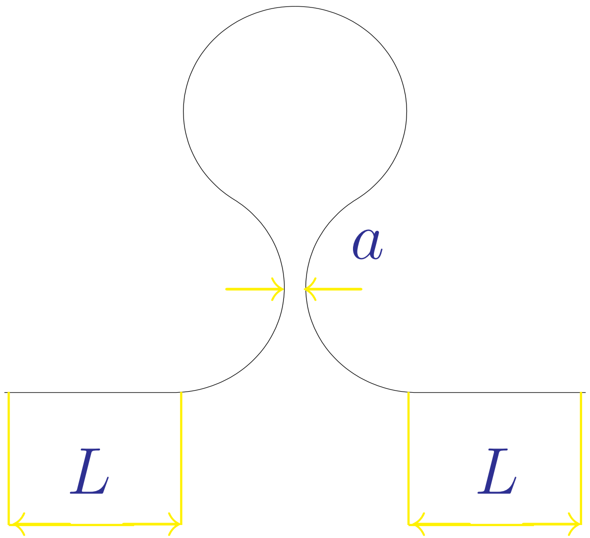
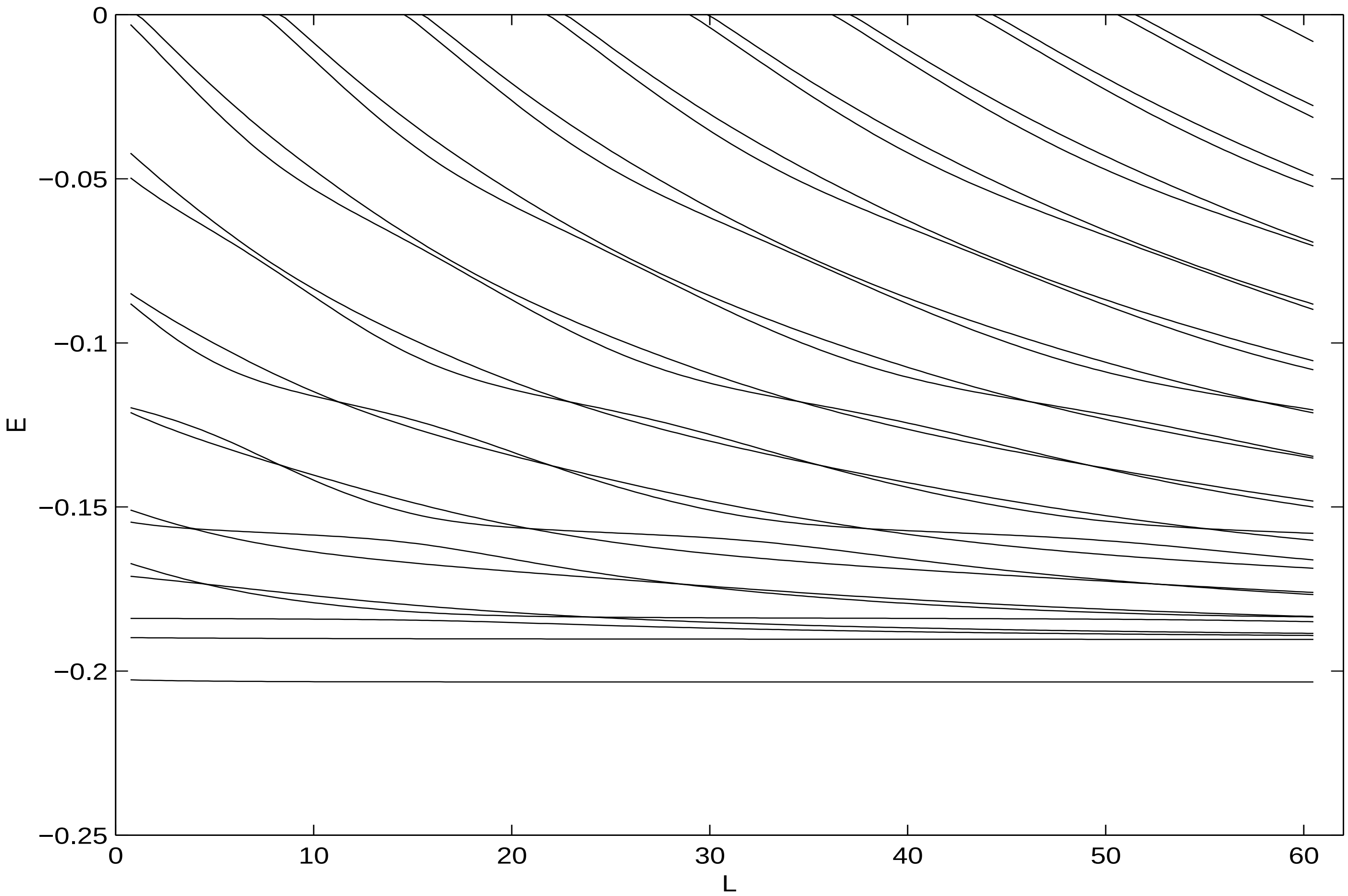
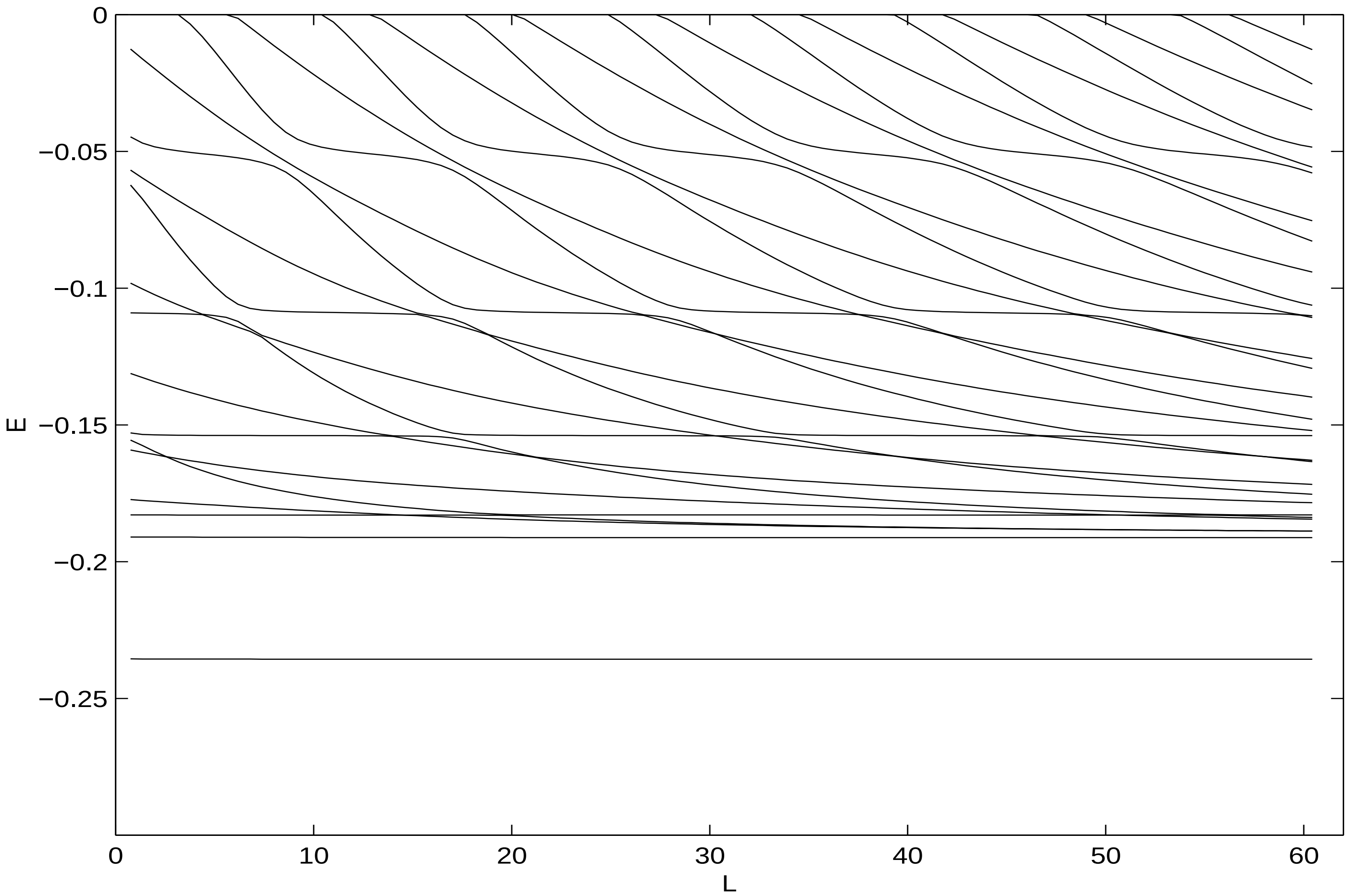


If Γ is a straight line, the transverse eigenfunction is $\mathrm{e}^{-\alpha|y|/2}$, hence the distance at which tunneling becomes significant is $\approx 4\alpha^{-1}$. In the example, we choose $\alpha = 1$

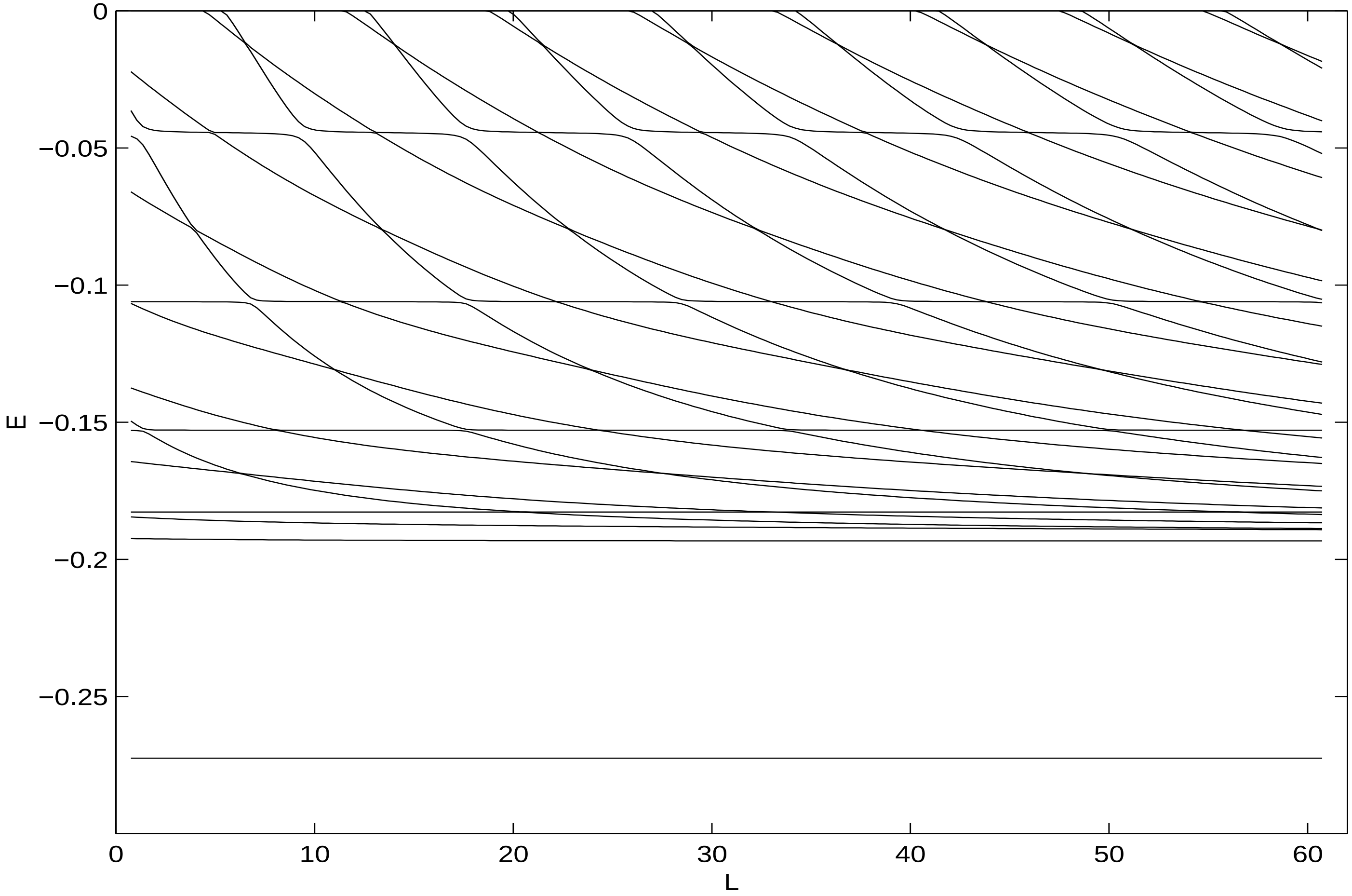
Bottleneck with $a = 5.2$



Bottleneck with $a = 2.9$



Bottleneck with $a = 1.9$



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- etc., etc.

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