Time averaging for the strongly confined nonlinear Schrödinger equation, using almost periodicity.

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Introduction

• **Asymptotic behavior** of a **nonlinear** gas of quantum particles, evolving in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \ni (x, z)$,
strongly confined along the vertical z direction

$\longrightarrow$ asymptotic dynamics should occur along the **remaining** x **plane**.
$\longrightarrow$ Can one describe the limiting dynamics ?

• In other words :

$$i\partial_t \Psi^\varepsilon(t, x, z) = H_x \Psi^\varepsilon + \frac{1}{\varepsilon} H_z \Psi^\varepsilon + F\left(|\Psi^\varepsilon|^2\right) \Psi^\varepsilon,$$

$$\Psi^\varepsilon(t = 0, x, z) = \text{"smooth"},$$

where the Hamiltonians H_x and H_z (in the x and z directions) are

$$H_x = -\Delta_x + V(x) \quad \text{with } V(x) \text{ "arbitrary"} ,$$

$$H_z = -\partial_z^2 + V_{\text{c}}(z) \quad \text{with } V_{\text{c}}(z) \quad \textbf{confining.}$$

(H_z has discrete spectrum).

Question : $\Psi^\varepsilon(t, x, z) \underset{\varepsilon \to 0}{\longrightarrow}$? ? ?

Formal analysis

- **Eigenenergies/functions of** H_z :

$$H_z\chi_p(z) = E_p\,\chi_p(z), \qquad \text{where} \qquad E_0 \le E_1 \le \cdots \le E_p \underset{p\to\infty}{\longrightarrow} +\infty.$$

- **Projecting** the equations on the χ_p's produces (say $F(|\Psi^\varepsilon|^2) = |\Psi^\varepsilon|^2$) :

$$\psi_p^\varepsilon(t,x) = \langle \Psi^\varepsilon, \chi_p\rangle(t,x),$$

$$i\partial_t\psi_p^\varepsilon(t,x) = H_x\psi_p^\varepsilon + \frac{E_p}{\varepsilon}\psi_p^\varepsilon + \sum_{q,r,s}\langle\chi_q\chi_r,\chi_s\chi_p\rangle\psi_q^\varepsilon\,\psi_r^\varepsilon\,\psi_s^\varepsilon.$$

- **Filtering out** the oscillations (Schochet, Grenier) produces :

$$\phi_p^\varepsilon(t,x) = \exp\left(+it\frac{E_p}{\varepsilon}\right)\,\psi_p^\varepsilon(t,x),$$

$$i\partial_t\phi_p^\varepsilon(t,x) = H_x\phi_p^\varepsilon + \sum_{q,r,s}\exp\left(+it\frac{E_q - E_r + E_s - E_p}{\varepsilon}\right)\langle\chi_q\chi_r,\chi_s\chi_p\rangle\phi_q^\varepsilon\,\phi_r^\varepsilon\,\phi_s^\varepsilon$$
$$= O(1).$$

Now : $\qquad \phi_p^\varepsilon \longrightarrow\ ???$

Formal analysis (bis)

• System of **nonlinear**, **coupled** ODE's, of the form,

$$\partial_t u_\varepsilon = A u_\varepsilon + B\left(\frac{t}{\varepsilon}, u_\varepsilon\right).$$

→ provided $B(\tau, .)$ possesses some **ergodicity** in τ, should go to

$$\partial_t u = A u + B_{\text{av}}(u),$$
$$B_{\text{av}}(u) := \lim_{T\to\infty} \frac{1}{T}\int_0^T B(\tau, u)\, d\tau.$$

• In our case, $B(\tau, .)$ has **countably many** frequencies
→ B is **almost-periodic**, hence **ergodic**, and, **formally**

$$\phi_p^\varepsilon \to \phi_p(t,x),$$

with : $$i\partial_t \phi_p = H_x \phi_p + \sum_{q,r,s} \mathbf{1}[E_q - E_r + E_s - E_p = 0]\langle \chi_q \chi_r, \chi_s \chi_p\rangle \phi_q\, \phi_r\, \phi_s,$$

$$\left(\mathbf{1}[E_q - E_r + E_s - E_p = 0] = \lim_{T\to\infty}\frac{1}{T}\int_0^T \exp\left(+i\tau[E_q - E_r + E_s - E_p]\right) d\tau\right).$$

Difficulties

- **control of the coefficients**

$$\langle \chi_q \chi_r, \chi_s \chi_p \rangle$$

and of the series

$$\sum_{q,r,s} \langle \chi_q \chi_r, \chi_s \chi_p \rangle \cdots$$

$\rightarrow$ **out of reach** even in the case $H_z =$ harmonic oscillator !

- **control of the small denominators**

$$\frac{1}{T}\int_0^T \exp\left(+i\tau(E_q - E_r + E_s - E_p)\right) d\tau \sim \frac{\mathbf{1}[E_q - E_r + E_s - E_p \neq 0]}{E_q - E_r + E_s - E_p}$$

$\rightarrow$ **out of reach** in general (the E_p's are arbitrary).

How to make the analysis rigorous

1- Assume $\Psi^\varepsilon(t=0) = \psi_0^\varepsilon(t=0,x) \times \chi_0(z)$ (**lowest energy**).
Then, energy estimate provides, for any time t,
$\Psi^\varepsilon(t) = \psi_0^\varepsilon(t,x) \times \chi_0(z) +$ small.

All the above systems are then **scalar** !
(Ben Abdallah-Méhats, Ben Abdallah-Méhats-Schmeiser-Weishäupl, see also Ben Abdallah-Méhats-Pinaud).

2- **Formally truncate** the system for finitely many modes (Bao-Markowich-Schmeiser-Weishäupl).

3- In this talk :
we provide a clean procedure to tackle the general case.
we also justify the formal truncated problems studied previously.

Key idea :
avoid projecting the equations,
exploit almost-periodicity.

A clean procedure

1- Define

$$\Phi^{\varepsilon}(t,x,z) = \exp\left(+it\frac{H_z}{\varepsilon}\right)\ \Psi^{\varepsilon}.$$

We have

$$i\partial_t\Phi^{\varepsilon} = H_x\Phi^{\varepsilon} + \exp\left(+it\frac{H_z}{\varepsilon}\right)\ F\left(\left|\exp\left(+it\frac{H_z}{\varepsilon}\right)\ \Phi^{\varepsilon}\right|^2\right)\exp\left(+it\frac{H_z}{\varepsilon}\right)\ \Phi^{\varepsilon}.$$

In other words

$$i\partial_t\Phi^{\varepsilon} = H_x\Phi^{\varepsilon} + G\left(\frac{t}{\varepsilon},\Phi^{\varepsilon}\right),$$

where,

$$G\left(\tau,u\right) = \exp\left(+i\tau H_z\right)\ F\left(\left|\exp\left(-i\tau H_z\right)\ u\right|^2\right)\ \exp\left(-i\tau H_z\right)\ u.$$

A clean procedure (bis)

2- Observe that for any $\ell \geq 0$,

$$\|\Psi^\varepsilon\|_{L^2}, \quad \|H_x^\ell \Psi^\varepsilon\|_{L^2}, \quad \|H_z^\ell \Psi^\varepsilon\|_{L^2},$$

are **bounded, uniformly in** ε.

This is because H_x and H_z commute with $H_x + \dfrac{H_z}{\varepsilon}$.

Similar bounds for Φ^ε.

$\longrightarrow$ Define the Sobolev space

$$B_\ell = \left\{ u \in L^2 \text{ s.t. } H_x^{\ell/2} u \in L^2, \quad H_z^{\ell/2} u \in L^2 \right\}.$$

A clean procedure (ter)

Main result B_ℓ is an **algebra**, whenever $\ell > 3/2$.
The B_ℓ norm **identifies** with

$$\|u\|_{H^\ell} + \|V(x)^{\ell/2} u\|_{L^2} + \|V_{\mathsf{c}}(z)^{\ell/2} u\|_{L^2}$$

For any given $u \in B_\ell$, the function

$$\tau \mapsto G(\tau, u) = \exp\left(+i\tau H_z\right)\, F\left(|\exp\left(-i\tau H_z\right)\, u|^2\right) \exp\left(-i\tau H_z\right)\, u.$$

is **almost periodic, with values in** B_ℓ.

Corollary Φ^ε goes to Φ, solution to

$$i\partial_t \Phi = H_x \Phi + G_{\mathsf{av}}(\Phi),$$

where

$$G_{\mathsf{av}}(\Phi) := \lim_{T\to\infty} \frac{1}{T} \int_0^T G(\tau, \Phi)\, d\tau.$$

Comments

• This result allows to recover all known results.

• Also allows to recover the **explicit value** of G_{av} when at hand.

• One can **project a posteriori** the limiting equation and recover the formal model obtained at the begining.

• Allows to circumvent the difficulties linked with **small denominators** and with **convergence of series** $\sum_{q,r,s} \cdots$ at once.

• Allows to describe how the various modes are switched on, starting from an initial datum carrying given modes.

• Justifies the truncated problems, which are shown to converge towards the untruncated ones as the truncation parameter goes to infinity.

• Works exploiting a similar parallel with ODE's of the form $\partial_t u_\varepsilon = Au_\varepsilon + B(t/\varepsilon, u_\varepsilon)$, in relation with the **"ergodicity"** of B : Schochet, Métivier-Schochet, Grenier, Lannes, Bidégaray-Castella-Degond, Castella-Goudon-Degond, ...

Idea of proof

- H_x^ℓ resp. H_z^ℓ have **"symbol"** $\Big(\xi^2 + V(x)\Big)^\ell$ resp. $\Big(\zeta^2 + V_{\mathsf{C}}(z)\Big)^\ell$.

Since

$$\Big(\xi^2 + V(x)\Big)^\ell \sim \xi^{2\ell} + V(x)^\ell \qquad \text{resp.} \qquad \Big(\zeta^2 + V_{\mathsf{C}}(z)\Big)^\ell \sim \zeta^{2\ell} + V_{\mathsf{C}}(z)^\ell,$$

we recover the equivalences :

$$\|u\|_{L^2}^2 + \|H_x^{\ell/2}\,u\|_{L^2}^2$$
$$\sim \|u\|_{L^2}^2 + \|(-\Delta_x)^{\ell/2}\,u\|_{L^2}^2 + \|V(x)^{\ell/2}\,u\|_{L^2}^2$$

and

$$\|u\|_{L^2}^2 + \|H_z^{\ell/2}\,u\|_{L^2}^2$$
$$\sim \|u\|_{L^2}^2 + \|(-\partial_z^2)^{\ell/2}\,u\|_{L^2}^2 + \|V_{\mathsf{C}}(z)^{\ell/2}\,u\|_{L^2}^2.$$

Idea of proof (bis)

• **Difficulty :** define the notion of symbol, i.e. how to count powers of $|\xi|$, $|\zeta|$ **AND** powers of $V(x)$, $V_{\mathsf{C}}(z)$.

Case when V, $V_{\mathsf{C}} =$ harmonic oscillators : Helffer.
General case : Weyl-Hörmander calculus of Bony-Chemin, and recent adaptations by Helffer-Nier.
Here : adaptation of Helffer-Nier.

Idea of proof (ter)

• where does almost-periodicity come from ?

Idea :

given u, $\tau \mapsto \exp(i\tau H_z)u$ has countably many frequencies :
it actually is the uniform limit of trigonometric polynomials.

Next, for any smooth function f, the function $\tau \mapsto f(\exp(i\tau H_z)u)$ has countably many frequencies as well :
it is the uniform limit of trigonometric polynomials.

Idea of proof (last)

- Last argument : given a large $T(\varepsilon) \ll 1/\varepsilon$, introduce

$$\begin{aligned} i\partial_t \Phi^\varepsilon &= H_x \Phi^\varepsilon + G\left(\frac{t}{\varepsilon}, \Phi^\varepsilon\right), \\ i\partial_t \Phi &= H_x \Phi + G_{\mathsf{av}}(\Phi), \\ i\partial_t \widetilde{\Phi^\varepsilon} &= H_x \widetilde{\Phi^\varepsilon} + \widetilde{G_\varepsilon}\left(\frac{t}{\varepsilon}, \widetilde{\Phi^\varepsilon}\right), \end{aligned}$$

where

$$G_{\mathsf{av}}(\Phi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T G(\tau,\Phi)\, d\tau, \qquad \widetilde{G_\varepsilon}\left(\frac{t}{\varepsilon}, \widetilde{\Phi^\varepsilon}\right) = \frac{1}{T(\varepsilon)}\int_0^{T(\varepsilon)} G\left(\frac{t}{\varepsilon}+\tau, \widetilde{\Phi^\varepsilon}\right) d\tau.$$

Then,

$$\begin{aligned} G - \widetilde{G_\varepsilon} &\sim \varepsilon\, T(\varepsilon) \quad (\text{"integration by parts"}) \\ \widetilde{G_\varepsilon} - G_{\mathsf{av}} &\sim \frac{\delta(\varepsilon)}{\varepsilon\, T(\varepsilon)} \quad (\delta(\varepsilon) \to 0 \text{ due to the definition of } G_{\mathsf{av}}). \end{aligned}$$

$\to$ chose $T(\varepsilon) = \sqrt{\delta(\varepsilon)}/\varepsilon$.
(Ideas borrowed from the ODE context : Sanders-Verhulst, Lochak-Meunier, ...)