Probability & Geometry on Configuration Spaces

Berlin, July '23

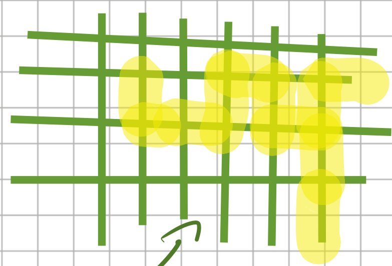
1) Basics in discrete percolation
2) Recent results on
   k-nearest-neighbor percolation
3) Basics in continuum percolation
4) Continuum percolation in random environment (dynamics)

Section I

References
1) Grimmett: Percolation, 99
2) Bollobás, Riordan: -"- , 06
3) Duminil-Copin: Lecture notes 18

and many more

Consider iid Ber(p)-distributed RV $(e_i)_{i \in B}$
B bonds of $\mathbb{Z}^d$, $p \in [0,1]$
If $e_i = 1$ we call bond open and if
$e_i = 0$ -"- closed.



real. of config. of open bonds in $\mathbb{Z}^2$

Percolation-probability

$$\Theta(p) := \mathbb{P}_p\left(o \underset{\text{bonds}}{\overset{\text{via open}}{\longleftrightarrow}} \infty \right)$$

Critical parameter

$$p_c := \inf\{ p > 0 : \Theta(p) > 0 \}$$

Thm 1 (Nontriviality of Bernoulli-bond-percolation)
For $d \geq 2$ we have $0 < p_c < 1$

Remark: For $p<p_c$ the system is <u>subcritical</u>.
For $p>p_c$ — " — is <u>supercritical</u>.

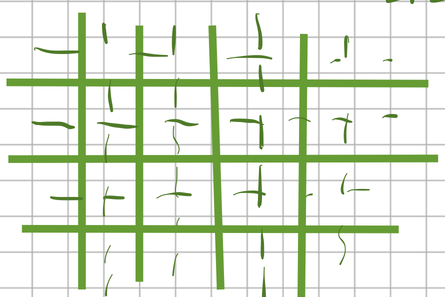
Proof: "$0<p_c$" via 1-st moment bound.

$$\Theta(p) \overset{\forall n}{\le} \mathbb{P}_p(\exists \text{ selfavoiding open path of length } n \text{ starting at } o)$$

$$\overset{\text{Markov}}{\le} \sum_{\text{such path}} p^n \le (2dp)^n \longrightarrow 0 \quad \text{for } p<\frac{1}{2d} \Rightarrow p_c \ge \frac{1}{2d}$$

"$p_c<1$" via Peierls' argument for $d=2$ (perc. in $d=2 \Rightarrow$ perc in $d>2$)

$$\mathbb{Z}'^d := \mathbb{Z}^d + \left(\tfrac{1}{2},\tfrac{1}{2}\right)$$



e closed $\iff e'$ closed

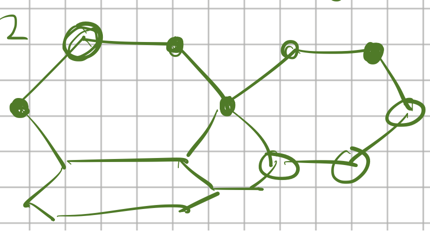
$$1-\Theta(p) \le \mathbb{P}_p(\exists \text{ closed interface in } \mathbb{Z}'^2 \text{ surrounding the origin})$$

$$\le \sum_{n\ge 0} \mathbb{P}_p(\exists \text{ closed path in } \mathbb{Z}'^2 \text{ of length } 2n+4 \text{ passing through } (n+\tfrac{1}{2},0))$$

$$\le \sum_{n\ge 0} (\underset{\uparrow 2d}{4}(1-p))^{2n+4} < 1 \quad \text{for } p \text{ suff. close to } 1.$$

Rem: 1) $\mathbb{E}[\text{degree}(0)] = p\,2d \Rightarrow$ for $p<\frac{1}{2d}$ $\mathbb{E}[\text{degree}(0)]<1$ and people call $\frac{1}{2d}$ the <u>branching bound</u>. The criterion holds for a large class of graphs. In $\mathbb{Z}^d$ $p_c(d)\cdot 2d \xrightarrow[d\uparrow\infty]{} 1$.

2) Similar arguments lead to $0<p_c^{site}<1$ for <u>Bernoulli-site-percolation</u>. For example for triangular-lattice $\mathbb{T}$ $p_c^{site}=\frac{1}{2}$ in $d=2$



3) In $d=1$ $p_c=1$

Note that $\Theta(p) > 0 \iff \mathbb{P}_p(\exists$ infinite connected open component $) > 0$ (cluster)

Indeed: $\Theta(p) > 0 \Rightarrow \mathbb{P}_p(\text{-"-}) > 0$

and if $\Theta(p) = 0$ then $\mathbb{P}_p(\text{-"-}) = \mathbb{P}_p(\exists x \in \mathbb{Z}^d : x \leftrightarrow \infty)$

$$\leq \sum_{x \in \mathbb{Z}^d} \underbrace{\mathbb{P}_p(x \leftrightarrow \infty)}_{\Theta(p)} = 0 \quad \text{(translation invariance)}$$

Is it possible that $\mathbb{P}_p(\exists \text{ inf. cluster}) \in (0,1)$?

Answer: No, by ergodicity. Consider

translation-invariant sets: $\mathcal{A} \subset \{0,1\}^{\mathcal{B}}$ with $\mathcal{A} + x = \mathcal{A} \ \forall x \in \mathbb{Z}^d$

ex: $\{\exists \text{ inf. cluster}\} = \mathcal{A}$

translation-invariant measure: $\mu(\mathcal{A}+x) = \mu(\mathcal{A}) \ \forall$ events $\mathcal{A}$, $x \in \mathbb{Z}^d$

ergodic measure $\quad \mu(\mathcal{A}) \in \{0,1\} \ \forall$ transl. inv. sets $\mathcal{A}$

Lemma 2 (Ergodicity) $\mathbb{P}_p$ is ergodic.

proof: It suffices to prove $\mathbb{P}_p(\mathcal{A}) \leq \mathbb{P}_p(\mathcal{A})^2 \ \forall$ transl.-inv. sets $\mathcal{A}$.

Note that $\forall \varepsilon > 0 \ \exists$ event B depending on only finitely many bonds and such that $\mathbb{P}(\mathcal{A} \Delta B) < \varepsilon$. Since B dep. on only finitely many bonds E, $\exists x \in \mathbb{Z}^d$ such that

$E \cap (E+x) = \emptyset \Rightarrow \mathbb{P}(B \cap (B+x)) = \mathbb{P}_p(B)^2$

by transl.-invariance. $\Rightarrow \mathbb{P}_p(\mathcal{A}) = \mathbb{P}_p(\mathcal{A} \cap \mathcal{A}) = \mathbb{P}_p(\mathcal{A} \cap (\mathcal{A}+x))$

$$\leq \mathbb{P}_p(B \cap (B+x)) + 2\varepsilon = \mathbb{P}_p(B)^2 + 2\varepsilon$$

$$\leq \mathbb{P}_p(\mathcal{A})^2 + 4\varepsilon \xrightarrow[\varepsilon \to 0]{} \mathbb{P}_p(\mathcal{A})^2 \qquad \square$$

This implies that for $p > p_c$ $\mathbb{P}_p(\exists \text{ inf. cluster}) = 1$

What about uniqueness?

Thm 3 For $p \in [0,1]$ with $\Theta(p) > 0$: $\mathbb{P}_p(\exists \text{ a unique inf. cluster}) = 1$

Section II (Recent results on k-nearest-neighbor percolation)

Ref: 1) Percolation in lattice k-nearest-neighbor graphs (J. Köppl, A. Tóbiás, B. Lodewijks, B.J. 23)

2) Oriented percolation in $d \geq 4$ (Cox & Durrett, 83)

3) Percolation (Grimmett)

We can consider a high variety of lattice perc. models. For example

1) Site percolation

2) Mixed percolation in $d=2$: open horizontal bonds with prob. p_h
open vertical —"— p_v

3) AB percolation: iid field of occupied | unoccupied sites
draw edge whenever ○—●

4) Long-range percolation.

5) First-passage percolation. $(\tau_i)_{i \in B}$ iid passage times on edges
wet site at time t

$$W_t := \{ x \in \mathbb{Z}^d : \exists \text{ path } \gamma : 0 \to x \text{ with } \sum_{i \in \gamma} \tau_i \leq t \}$$

6) Directed percolation:

e.g. $d=2$ iid vertex x has an edge facing north with prob. p (and south otherwise) and and edge facing west —"— . (and east otherwise)

$\mathbb{P}_p(0 \leftrightarrow \infty) = 0$ for $p = \frac{1}{2}$

$\mathbb{P}_p(0 \leftrightarrow \infty) > 0$ for $p \neq \frac{1}{2}$?

Consider k-neighbor percolation: each vertex chooses precisely k out of its $2d$ nearest neighbors uniformly at random and we open an edge towards them.

This gives rise to (at least) three models

i) directed k-nG (k-DnG) as the resulting random graph of directed open edges

ii) undirected k-nG (k-UnG) forget arrows $k=2=d$

iii) bidirectional k-nG (k-BnG) open

We are interested in $\Theta^{*}(k,d) = \mathbb{P}(0 \to \infty \text{ in } k\text{-}{*}\text{nG in } \mathbb{Z}^d)$, $* \in \{B, D, U\}$

$$\Theta^B(k,d) \leq \Theta^D(k,d) \leq \Theta^U(k,d)$$

Sec II.1 (k-DnG)

Prop. 4. $\Theta^D(1,d) = 0$ and $\Theta^D(k,d) = 1$ for $k \geq d+1$

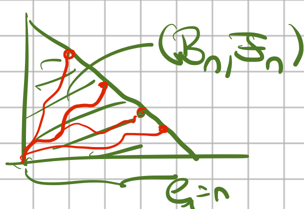
proof: $\mathbb{P}(\exists \text{ path starting at } o \text{ of length } n) \leq (2d)^{-n} \xrightarrow[n\to\infty]{} 0$.
Further, for $k \geq d+1$

$$\max_{x\in G_n} \|x\|_1 < \max_{x \in G_{n+1}} \|x_1\| \text{ a.s. where}$$

$$G_n := \{x \in \mathbb{Z}^d : \exists \text{ open path of length } n \text{ from } 0 \to x\} \qquad \square$$

Thm 5 $\Theta^D(k,d) > 0$ for $d,k \geq 4$ or $d \geq 5$, $k=3$.

proof: We use Martingale argument. Consider 1st quadrant, and let $\mathcal{F}_n$ be the σ-algebra up to ℓ_1-distance n. Let



$N_n :=$ # open paths in $\mathcal{R}_n$ ← set of directed paths $0 \rightsquigarrow \delta\mathcal{B}_n$

Note that $\mathbb{E}[N_{n+1} \mid \mathcal{F}_n] = \frac{k}{2} N_n$

$\Rightarrow$ $W_n := \left(\frac{2}{K}\right)^n N_n$ is a $\mathcal{F}_n$-Martingal. Since $\mathbb{E}[W_1]=1$

↳ Expected # of outwards open edges.

$$W_n \xrightarrow[n\uparrow\infty]{} W \quad \mathbb{P}\text{-a.s. and } \mathbb{P}(W>0)>0 \text{ if}$$

$$\overline{\lim_{n\uparrow\infty}}\, \mathbb{E}[W_n^2] < \infty$$

But $\mathbb{E}[N_n^2] = \sum_{s,t\in R_n} \mathbb{P}(s,t \text{ open}) \le \sum_{s,t\in R_n} p^{2n-K(s,t)}$

by negative correlation

$K(s,t)$: # joint edges in s,t.

$p = \frac{K}{2d}$

$$= (dp)^{2n}\, \mathbb{E}[p^{-K(n)}]$$

$K(n)$: # joint edges of two independent random walks $(S_m, S'_m)_{m\le n}$

$$\Rightarrow \mathbb{E}[W_n^2] \xrightarrow[n\uparrow\infty]{} \mathbb{E}[p^{-K}]$$

K: total # of joint rw edges.

$$\mathbb{P}(K=k) = \mathfrak{z}_d^k(1-\mathfrak{z}_d) \quad \text{strong}$$

$\mathfrak{z}_d$: $\mathbb{P}(\exists m: S_m = S'_m,\ S_{m+1} = S'_{m+1})$

$$\Rightarrow \mathbb{E}[p^{-K}] < \infty \iff \frac{K}{2d} > \mathfrak{z}_d$$

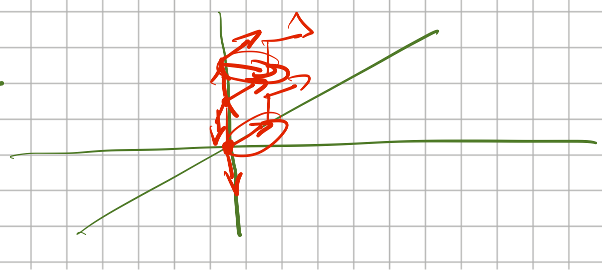
precise evaluation yields the result □

<u>Open problem</u>: $\forall d \ge 1$ $\Theta^D(2,d) > 0$ overwhelming numerical evidence!!

<u>Thm 6</u> $\Theta^D(k+1, d+1) \ge \Theta^D(k,d)$

<u>Proof idea</u>: We use coupling arguments. $k,d=2$

<u>algorithm</u>: Go up/down until you find two non-vertical edges.



$$\mathbb{P}_{2,2}(\underset{\{0 \to \partial B_n \text{ in } \mathbb{Z}^2\}}{A_n^{(2)}}) = \mathbb{E}_{3,3}[\underset{\text{coupling measure}}{\nu}(A_n^{(2)}|\cdot)]$$

$$= \mathbb{E}_{3,3}[\underbrace{\nu(A_n^{(2)}|\cdot)}_{\le 1}\mathbb{1}_{A_n^{(3)}}] + \cancel{\mathbb{E}_{3,3}[\nu(A_n^{(2)}|\cdot)\mathbb{1}_{A_n^{(3)c}}]}$$

$$\le \mathbb{P}_{3,3}(A_n^{(3)})$$

□

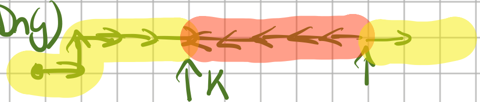
Sec II.2 (k-UnG)

Thm 7 $\Theta^u(1,d)=0 \ \forall d\ge 1$ but $\Theta^u(2,2), \Theta^u(3,3)>0$

Proof: 1) Assume $\mathbb{P}(A)>0$ with $A=\{0\to\infty$ in 1-UnG$\}$
and $0=x_1, x_2, \ldots \to \infty$ in A

Let $K:=\inf\{k\ge 0 : (x_k, x_{k+1})$ not in 1-DnG$\}$

$L=\inf\{k>K : (x_k, x_{k+1})$ in 1-DnG$\}$



Since $\Theta^D(1,d)=0$ $\quad \mathbb{P}(K=\infty)=0 \quad \mathbb{P}(L<\infty)=1$

Spatially, let $\kappa:=\inf\{k>0 : C_{\text{directed}}(0)\subset B_k\}$
(directed: outgoing directed cluster of 0)

$\mathbb{P}(\kappa=k_0|A)\ge\varepsilon$ for some k_0
since $\mathbb{P}(\kappa=2)=1$

$$\Rightarrow \varepsilon\,\mathbb{P}(A)\le\mathbb{P}(\kappa=k_0, A)$$

$$\overset{k_n}{\le} \mathbb{P}(\exists \text{ selfavoiding path of length } n \text{ starting from } \partial B_n \text{ in 1-DnG}) \xrightarrow[n\uparrow\infty]{} 0 \text{ by Prop 4}$$

Section III (Continuum percolation)

Ref: - Cont. Perc. Meester & Roy '96
1) Gouéré '08

Consider point cloud in $\mathbb{R}^d$ without interaction, the Poisson point process (PPP)

$X^\lambda = (X_i)_{i\in I}$ with intensity λdx i.e.

1) $\mathbb{P}(X(A)=n) = \mathrm{Pois}_{\lambda|A|}(n)$ $A \in \mathcal{B}(\mathbb{R}^d)$

↑ Lebesgue volume of A

2) Total independence $\mathbb{E}[f(X^\lambda_A)\, g(X^\lambda_B)]$

$$= \mathbb{E}[f(X^\lambda_A)]\,\mathbb{E}[g(X^\lambda_B)]$$

$\forall f,g \geq 0$ mb. $A,B \in \mathcal{B}(\mathbb{R}^d)$ $A\cap B = \emptyset$.

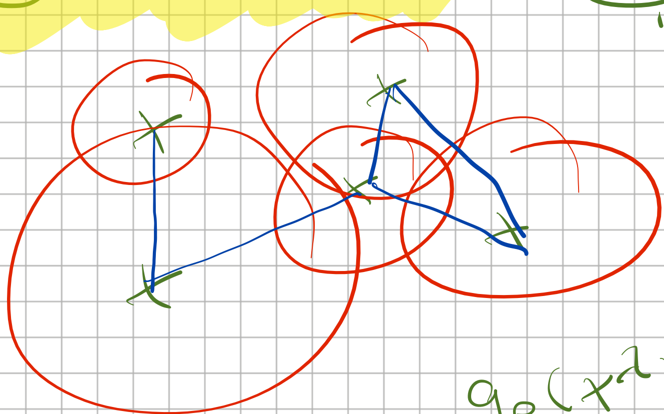
X^λ is stationary & ergodic wrt shifts in $\mathbb{R}^d$

Attache iid marks $(\rho_i)_{i\in I}$ with $\rho \geq 0$

$\mathbb{P}(\rho>0)>0$ and consider

Boolean model $\quad C = \bigcup_{i\in I} B_{\rho_i}(x_i)$

↑ balls with radius ρ_i centered at x_i



Associated random graph

$g_\rho(X^\lambda)$ with vertices X^λ and

edges $(x_i, x_j) \iff B_{\rho_i}(x_i)\cap B_{\rho_j}(x_j) \neq \emptyset$.

1) $\rho \equiv r$: $g_r(X^\lambda)$ is called

Gilbert graph;

Percolation: $\lambda_c(r) := \inf\{\lambda>0 : \mathbb{P}(g_r(X^\lambda)$ contains an unbounded component$)>0\}$

<u>Observe</u>: i) $\{g_r(X^\lambda) \text{ perc.}\}$ is shift inv., hence

$$\mathbb{P}(g_r(X^\lambda) \text{ perc.}) > 0 \iff \mathbb{P}(g_r(X^\lambda) \text{ perc.}) = 1$$

ii) Scale invariance

$$\mathbb{P}(g_r(X^\lambda) \text{ perc.}) = \mathbb{P}(g_1(X^{r^d \lambda}) \text{ perc.})$$

$$\left[\text{degree: } \mathbb{E}[X^\lambda(B_r(0))] = \lambda v_d r^d \right]$$

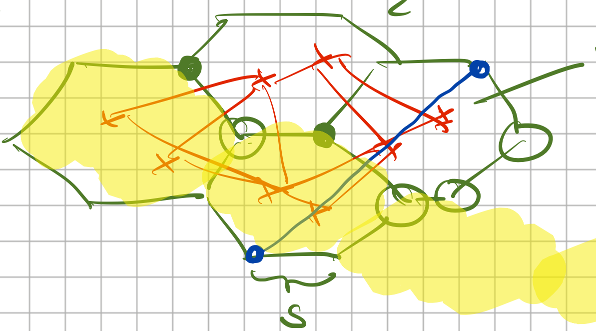
Is $0 < \lambda_c(1) < \infty$?

$\uparrow$ subcr. $\quad \uparrow$ supercr.

Sec III.1 Supercritical phase

Thm 10 For $d \geq 2$ $\lambda_c(1) < \infty$

proof: via discretization: for $d=2$:



Face percolation in hexagons = site percolation in triangular lattice $p_c = \frac{1}{2}$

Call face A open $\iff X(A) > 0$

Note: 1) $|A| = cs^2$ $\quad (c = \frac{3\sqrt{3}}{2})$

2) $\mathbb{P}(X(A) = 0) = e^{-\lambda |A|}$

3) If $1 - e^{-\lambda|A|} > \frac{1}{2}$ we see face percolation.

Now $|x_i - x_j| < c's$ $\forall x_i, x_j$ in neighb. faces

$\implies$ If $c's < 1$ points x_i, x_j in -"- are conn. in $g_1(X^\lambda)$

$\implies$ Choose $s < \frac{1}{c'}$ and $\lambda > \frac{\log 2}{cs^2}$ we see face perc. implying perc. in $g_r(x^2)$ $\square$

This can be extended to random radii ρ_i.

What about $d=1$?

Note $\mathbb{P}(\exists \text{ inf. cluster}) > 0 \iff \mathbb{P}^0(0 \to \infty) > 0$

since "$\Leftarrow$" clear and "$\Rightarrow$" $\exists\, n$ s.t.

$$0 < \mathbb{P}(\exists \text{ inf. cluster } C : C \cap X_{B_n} \neq \emptyset)$$

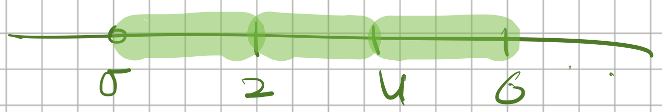
$$\le \mathbb{E}\Big[\sum_{x \in B_n} \mathbb{1}\{x_i \in C\}\Big] \underset{\text{Mecke}}{=} \int_{B_n} dx\, \mathbb{P}(x \in C \cup \{x\})$$

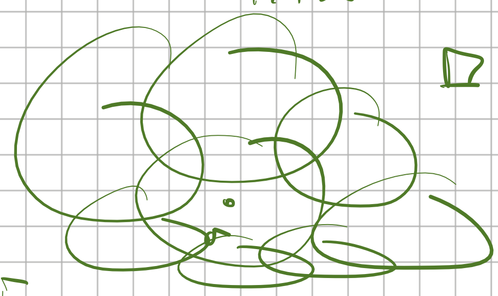
$$= |B_n|\, \mathbb{P}^0(0 \to \infty)$$

Thm 11 For $d=1$ $\lambda_c(1) = \infty$

proof: Note that $\theta_n := \mathbb{P}^0(0 \leadsto \delta B_n)$

$$\le (1 - e^{-2\lambda})^{n/2} \xrightarrow[n\uparrow\infty]{} 0$$





$\square$

Sec III.2 Subcritical phase

Problem: $\mathbb{E}^0[\text{degree}(0)] = \mathbb{E}^0[\#\{X_i : |X_i| < \rho_0 + \rho_i\}]$

$$= \mathbb{E}_{\rho_0,\rho_1}[\lambda |B_{\rho_0+\rho_1}|] = \lambda \kappa_d\, \mathbb{E}[(\rho_0+\rho_1)^d]$$

$$\in (0, \infty]$$

Indeed if $\mathbb{E}[\rho^d] = \infty$

$\Rightarrow \mathbb{P}(C = \mathbb{R}^d) = 1$ full coverage

true for any stationary point process

But even if $\mathbb{E}[\varrho^d]<\infty$, do we have a subcritical regime?

<u>Thm 12</u> For any $d\geq 1$ if $\mathbb{E}[\varrho^d]<\infty$

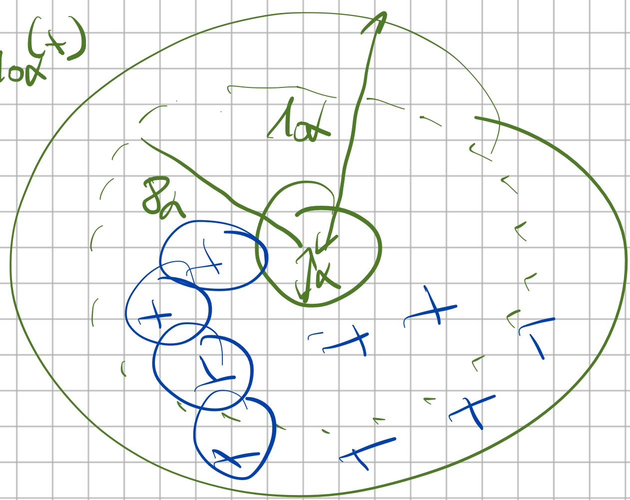
$$\lambda_c := \inf\{\lambda>0 : \mathbb{P}^0(C_0 \text{ is unbounded})>0\}>0$$

C_0: conn. com. of origin in C

<u>proof:</u> via multiscale argument (Gouéré '08)

$$\mathcal{G}(x,\alpha) := \{ C_\alpha(x) \not\subset B_{8\alpha}(x)\}$$

$C_\alpha(x)$: connected component attached to $B_\alpha(x)$ using points only in $X|_{B_{10\alpha}(x)}$



Two error events

$\mathcal{H}(\alpha) := \{\exists\, x_i \in B_{10\alpha}^c : B_{\varrho}(x_i)\cap B_{8\alpha} \neq \emptyset\}$

(help from outside)

$\mathcal{F}(\alpha) := \{\exists\, x_i\in B_{10\alpha} : \varrho_i \geq \alpha\}$

(large balls inside)

$M := \sup_{x\in C_0}\|x\|$ radius of C_0 then

$$\{M\geq 9\alpha\} \subset \mathcal{G}(0,\alpha)\cup\mathcal{H}(\alpha)$$

where

$$\mathbb{P}(H(\alpha)) \le \mathbb{E}\Big[\sum_{x_i \in B^c_{10\alpha}} \mu\{B_g(x_i)\cap B_{g\alpha} \neq \emptyset\}\Big]$$

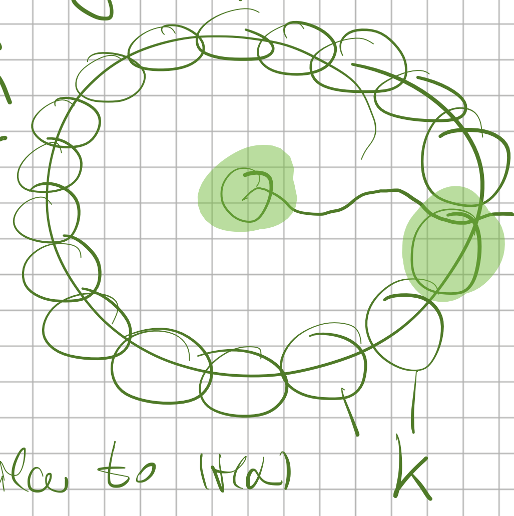
$$= \lambda \int dx \int \mathbb{P}(dr)\, \mu\{x \in B_{g\alpha+r}\setminus B_{10\alpha}\}$$

$$\le \lambda \int_\alpha^\infty \mathbb{P}(dr)\,|B_{10r}| = \lambda c' \int_\alpha^\infty r^d\, \mathbb{P}(dr) \xrightarrow[\alpha\uparrow\infty]{} 0$$

also

$$G(0,10\alpha)\setminus F(\alpha)$$

$$\subset G(0,\alpha) \cap \bigcup_{k\in K} G(\alpha k,\alpha)$$



$$\Rightarrow \mathbb{P}(G(0,10\alpha))$$

$$\le C'' \,\mathbb{P}(G(0,\alpha))^2$$

$$+ \underbrace{\mathbb{P}(F(\alpha))}_{\xrightarrow[\alpha\uparrow\infty]{} 0}$$ similar to $H(\alpha)$

Hence we have an inequality of the form

$$f(\alpha) \le c\, f\!\left(\frac{\alpha}{10}\right)^2 + g(\alpha) \qquad (*)$$

$\nearrow$ $\xrightarrow[\alpha]{} 0$ very fast $g(\alpha) \to 0$

Input: Gouerée for $f \le \frac{1}{2}$ on $[1,10]$ $g \le \frac{1}{4}$ on $[1,\infty)$ and $(*)$.

If $g(\alpha) \xrightarrow[\alpha\uparrow\infty]{} 0$ then $f(\alpha) \xrightarrow[\alpha\uparrow\infty]{} 0$.

But $g(\alpha) = \lambda C \int_{\alpha}^{\infty} r^d \, \mathbb{P}(dr)$ small for λ small

$f(\alpha) \leq \mathbb{P}(\exists \, x_i \in B_{10\alpha})$ small for λ small

$h \in [1, 10]$

$\square$

Continuum Percolation in Random Environments

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Stoch. Proc. Appl. (2019)
Braz. J. Probab. Stat. (2022)
Journ. Appl. Probab. (2023)
IEEE Info. Theory (2023)

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Setting: Boolean models for Cox point processes

- $X = (X_i)_{i \in I}$ = stationary Cox point process in $\mathbb{R}^d$, i.e., Poisson point process (PPP) with stationary random intensity measure Λ where $\mathbb{E}[\Lambda([0,1]^d)] < \infty$
- ϱ_i = i.i.d. interaction radius of X_i with $\mathbb{P}(\varrho_i > 0) > 0$
- $\mathcal{C} = \bigcup_{i \in I} B_{\varrho_i}(X_i)$ = Cox–Boolean model, where $B_r(x)$ = ball with radius $r \geq 0$ centered at $x \in \mathbb{R}^d$

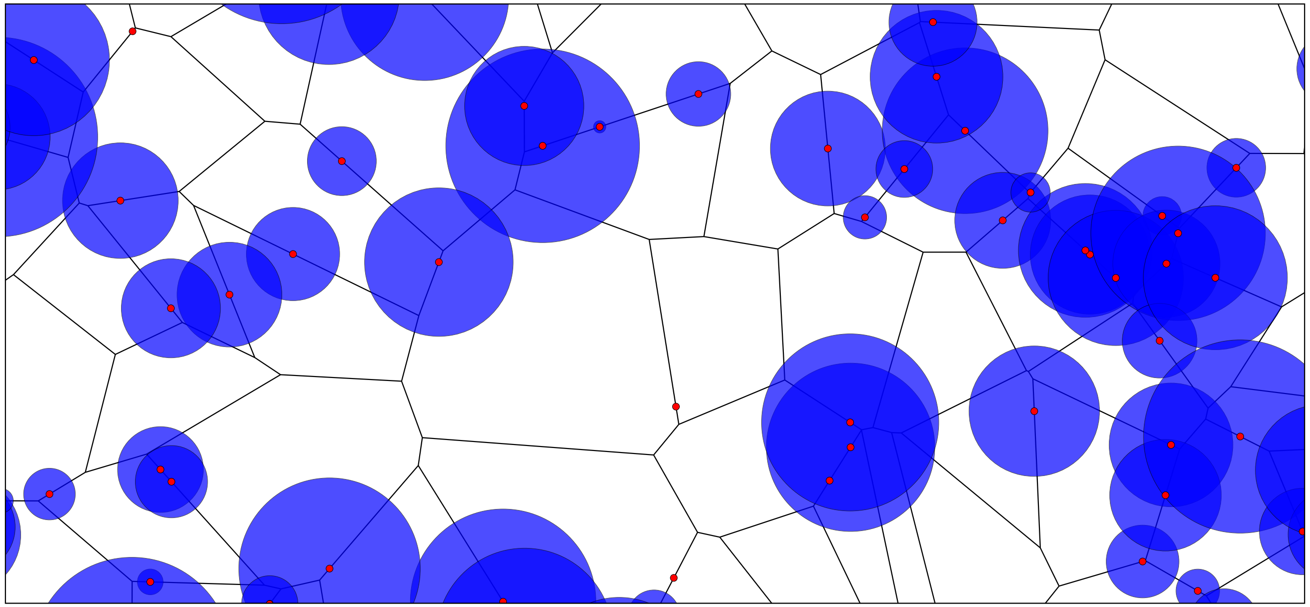


Figure: Realization of a Cox–Boolean model (blue) based on a Cox point process (red) with directing measure given by a realization of a Poisson–Voronoi tessellation (black).

Examples: Singular random environments

- Singular random intensity measure, e.g.,
  - $\Lambda(\mathrm{d}x) = \nu_1(S \cap \mathrm{d}x)$, where ν_1 is one-dimensional Hausdorff measure and $S \subset \mathbb{R}^d$ = stationary random segment process such as
    - Poisson–Voronoi tessellation (PVT),
    - Poisson–Delaunay tessellation (PDT),
    - Poisson line tessellation (PLT),
    - Manhattan grid (MG), etc.

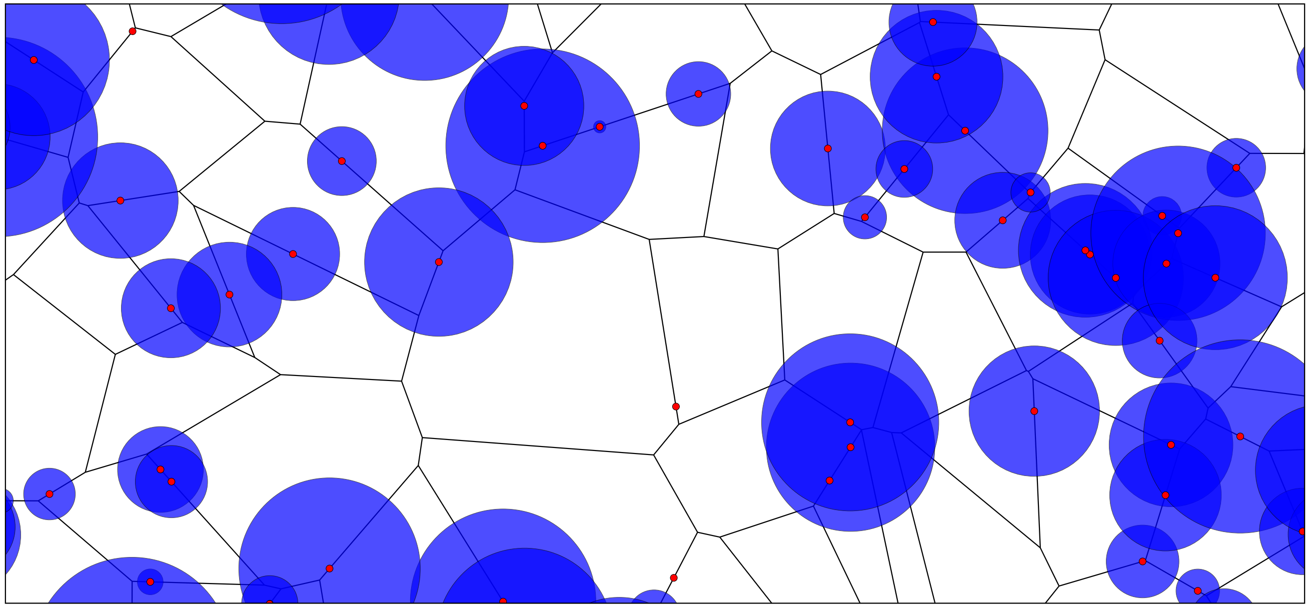


Figure: Realization of a Cox–Boolean model (blue) based on a Cox point process (red) with directing measure given by a realization of a Poisson–Voronoi tessellation (black).

Examples: Absolutely continuous random environments

- $\Lambda(\mathrm{d}x) = \ell_x \mathrm{d}x =$ absolutely continuous random intensity measure, e.g.,
  - $\ell_x = \lambda \mathbb{1}\{x \in \Xi\} + \lambda' \mathbb{1}\{x \notin \Xi\}$, where $\Xi \subset \mathbb{R}^d$ stationary random closed set and $\lambda, \lambda' \geq 0$
  - $\ell_x = \sum_{i \in I} \kappa(Y_i - x) =$ shot-noise field, where $(Y_i)_{i \in I}$ stationary PPP and $\kappa \colon \mathbb{R}^d \to [0, \infty)$ integrable
  - $\ell_x = \lambda \sum_{i \in I} \mathbb{1}\{|Y_i - x| < \rho_i\}$ where $(Y_i)_{i \in I}$ stationary PPP with i.i.d. marks ρ_i and $\lambda > 0$

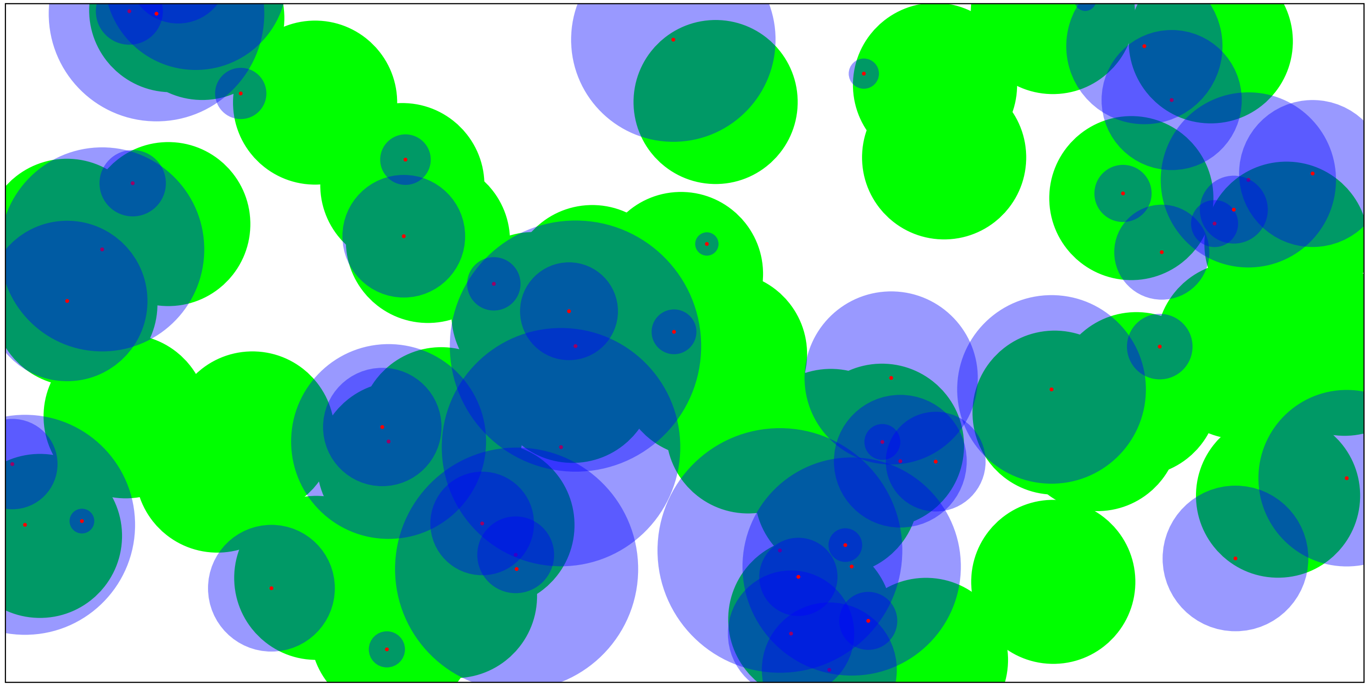


Figure: Realization of a Cox–Boolean model (blue) based on a Cox point process (red) with directing measure given by an independent Poisson–Boolean model with fixed radii (green).

Complete coverage and one-dimensional triviality

Proposition (No complete coverage)

For all $d \geq 1$, $\mathbb{E}[\varrho^d] < \infty$ implies $\mathbb{P}(\mathcal{C} = \mathbb{R}^d) < 1$. If additionally Λ is ergodic, then $\mathbb{P}(\mathcal{C} = \mathbb{R}^d) = 0$.

Proposition (One-dimensional triviality)

Let $d = 1$ and Λ ergodic, then $\mathbb{E}[\varrho] < \infty$ implies $\mathbb{P}(\mathcal{C}$ contains unbounded connected component$) = 0$.

- Stationary point processes: $\mathbb{E}[\varrho^d] = \infty \Rightarrow \mathbb{P}(\mathcal{C} = \mathbb{R}^d) = 1$

  Meester & Roy '96: *Continuum Percolation*, Cambridge University Press
- Stationary PPP, $d = 1$: $\mathbb{E}[\varrho] < \infty \Rightarrow \mathbb{P}(\mathcal{C}$ contains unbounded component$) = 0$

  Meester & Roy '96: *Continuum Percolation*, Cambridge University Press
- Rectangular lilypond models: Hirsch '16: *On the absence of percolation in a line-segment based lilypond model*, Ann. Henri Poincaré 52
- 2-degree random graphs: Tóbiás & J '22: *Absence of percolation in graphs based on stationary point processes with degrees bounded by two*, Random Structures and Algorithms

Essentially connected environments

Definition (Essential r-connectedness)

Let $r > 0$ and $Q_\alpha = [-\alpha, \alpha]^d$. The stationary random measure Λ is essentially r-connected, if there exists a random field of connectivity radii $R = \{R_x\}_{x\in\mathbb{R}^d}$, defined on the same probability space as Λ, such that

1. (Λ, R) are jointly stationary,
2. $\lim_{\alpha\uparrow\infty} \mathbb{P}(\sup_{y\in Q_\alpha\cap\mathbb{Q}^d} R_y \geq \alpha) = 0$, and
3. for all $\alpha \geq 1$, whenever $\sup_{y\in Q_{2\alpha}\cap\mathbb{Q}^d} R_y < \alpha/2$, we have that for all $x, y \in \mathrm{supp}(\Lambda_{Q_\alpha})$ there exists a finite sequence of points $(x_1, \dots, x_l) \subset \mathrm{supp}(\Lambda_{Q_{2\alpha}})$ such that $|x_i - x_{i+1}| < r$ for all $i \in \{0, 1, \dots, l+1\}$ where $x = x_0$ and $y = x_{l+1}$.

- $\ell_x = \lambda_1 \mathbb{1}\{x \in \Xi\} + \lambda_2 \mathbb{1}\{x \notin \Xi\}$ with $\lambda_1, \lambda_2 > 0$ and $\Lambda(\mathrm{d}x) = \nu_1(S \cap \mathrm{d}x)$ for S a PVT or PDT or MG, are essentially r-connected for any $r > 0$
- Shot-noise fields and $\ell_x = \lambda \sum_{i\in I} \mathbb{1}\{|Y_i - x| < \rho_i\}$ are in general not essentially r-connected

Uniqueness of infinite cluster

Theorem (Uniqueness)

Let $d \geq 1$ and Λ ergodic. If $r =$ esssup$(\varrho) < \infty$ and Λ is essentially r-connected, then $\mathbb{P}(\mathcal{C}$ contains at most one unbounded connected component$) = 1$.

- Proof: adaptation of Burton–Keane argument and FKG inequality

- Stationary point processes: esssup$(\varrho) = \infty$ or $\Lambda \equiv 1 \Rightarrow$ uniqueness
  Meester & Roy '96: *Continuum Percolation*, Cambridge University Press
- Nearest neighbor graphs: Häggström & Meester '96: *Nearest neighbor and hard sphere models in continuum percolation*, Random Struct. Algorithms 167
- Insertion-tolerant point processes: Holroyd & Soo '13: *Insertion and deletion tolerance of point processes*, Electron. J. Probab. 18
- Levelset percolation: Broman & Meester '17: *Phase transition and uniqueness of levelset percolation*, J. Stat. Phys. 167

Critical behavior

- $X = (X_i)_{i\in I}$ = Cox point process in $\mathbb{R}^d$ with stationary random intensity measure $\lambda\Lambda$, where $\mathbb{E}[\Lambda([0,1]^d)] = 1$ and $\lambda \geq 0$
- $\mathcal{C} \supset C_o$ = connected component containing the origin
- For $A \subset \mathbb{R}^d$, $|A|$ = Lebesgue volume; diam(A) = diameter; $X(A)$ = number of Cox points
- Critical values:

$$\lambda_{\rm vp} = \inf\{\lambda > 0 \colon \mathbb{P}(|C_o| = \infty) > 0\}$$
$$\lambda_{\rm dp} = \inf\{\lambda > 0 \colon \mathbb{P}(\text{diam}(C_o) = \infty) > 0\}$$
$$\lambda_{\rm np} = \inf\{\lambda > 0 \colon \mathbb{P}(X(C_o) = \infty) > 0\}$$
$$\lambda_{\rm v}(s) = \inf\{\lambda > 0 \colon \mathbb{E}[|C_o|^s] = \infty\}, \text{ for } s > 0$$
$$\lambda_{\rm d}(s) = \inf\{\lambda > 0 \colon \mathbb{E}[\text{diam}(C_o)^s] = \infty\}, \text{ for } s > 0$$
$$\lambda_{\rm n}(s) = \inf\{\lambda > 0 \colon \mathbb{E}[X(C_o)^s] = \infty\}, \text{ for } s > 0$$

Pathological percolation behavior for Cox point processes

- Insufficiently connected environments:
  for all $s > 0$, $\lambda_{\mathrm{vp}} = \lambda_{\mathrm{dp}} = \lambda_{\mathrm{np}} = \lambda_{\mathrm{v}}(s) = \lambda_{\mathrm{d}}(s) = \lambda_{\mathrm{n}}(s) = \infty$
- Unbounded environments: $Z\mathrm{d}x$ with esssup$(Z) = \infty$
  for all $s > 0$, $\lambda_{\mathrm{vp}} = \lambda_{\mathrm{dp}} = \lambda_{\mathrm{np}} = \lambda_{\mathrm{v}}(s) = \lambda_{\mathrm{d}}(s) = \lambda_{\mathrm{n}}(s) = 0$
  (non-ergodic, mixed PPP)

- Attractive point processes: Blaszczyszyn & Yogeshwaran '13: *Clustering and percolation of point processes*, Electron. J. Probab.
- SINR percolation I: Dousse, Baccelli & Thiran '05: *Impact of interferences on connectivity in ad hoc networks*, IEEE Trans. Netw.
- SINR percolation II: Dousse, Franceschetti, Macris, Meester & Thiran '06: *Percolation in the signal to interference ratio graph*, J. Appl. Probab.
- Cox-SINR percolation I: Tóbiás '20: *Signal to interference ratio percolation for Cox point processes*, Lat. Am. J. Probab. Math. Stat.
- Cox-SINR percolation II: Tóbiás & J '22: *SINR percolation for Cox point processes with random powers*, Adv. Appl. Probab.

Stabilizing environments

Definition (Stabilization)

The stationary random measure Λ is φ-stabilizing if there exists a random field of stabilization radii $R = \{R_x\}_{x\in\mathbb{R}^d}$, defined on the same probability space as Λ, with

1. (Λ, R) are jointly stationary,
2. $\lim_{\alpha\uparrow\infty} \varphi(\alpha) = 0$, where $\varphi(\alpha) = \mathbb{P}(\sup_{y\in Q_\alpha\cap\mathbb{Q}^d} R_y \geq \alpha)$, and
3. for all $\alpha \geq 1$, the random variables
$$\Big(f(\Lambda_{Q_\alpha(x)})\mathbb{1}\{\sup_{y\in Q_\alpha(x)\cap\mathbb{Q}^d} R_y < \alpha\}\Big)_{x\in\psi}$$
are independent for all non-negative bounded measurable functions f and finite $\psi \subset \mathbb{R}^d$, as long as dist$(x, \psi \setminus x) > 3\alpha$ for all $x \in \psi$.

- φ-stabilization implies ergodicity
- Λ b-dependent if $\varphi(\alpha) = 0$ for all $\alpha > b$
- Λ exponentially stabilizing if $\varphi(\alpha) \leq \exp(-c\alpha)$

Examples: stabilizing environments

- Shot-noise fields are b-dependent for compactly supported kernels; kernels with unbounded support lead to absence of φ-stabilization
- $\Lambda(\mathrm{d}x) = \nu_1(S \cap \mathrm{d}x)$ for S a PVT or PDT, are exponentially stabilizing, $R_x = \min\{|x - X_i|: X_i \in \mathcal{X}|\}$

  Hirsch, Cali & J '19: *Continuum percolation for Cox point processes*, Stoch. Proc. Appl.
- $\Lambda(\mathrm{d}x) = \nu_1(S \cap \mathrm{d}x)$ for S a PLT or MG are not φ-stabilizing
- $\ell_x = \lambda \sum_{i \in I} \mathbb{1}\{|Y_i - x| < \rho_i\}$ with $b = \mathrm{esssup}(\rho) < \infty$ are b-dependent
- $\ell_x = \lambda \sum_{i \in I} \mathbb{1}\{|Y_i - x| < \rho_i\}$ with $\mathrm{esssup}(\rho) = \infty$ and $\mathbb{E}[\rho^{d+s}] < \infty$ for $s > 0$, are φ-stabilizing with $\int_0^\infty \alpha^{s-1}\varphi(\alpha)\mathrm{d}\alpha < \infty$

Phase transitions for Cox–Boolean models

Theorem (Phase transitions)

Let $d \geq 2$, $s > 0$ and Λ be stationary.

1. Λ *φ-stabilizing with sufficiently large esssup(ϱ) implies* $\lambda_{\mathrm{vp}}, \lambda_{\mathrm{dp}}, \lambda_{\mathrm{np}} < \infty$.
2. $\mathbb{E}[\varrho^d] < \infty$ *and* Λ *φ-stabilizing implies* $\lambda_{\mathrm{vp}}, \lambda_{\mathrm{dp}}, \lambda_{\mathrm{np}} > 0$.
3. $\mathbb{E}[\varrho^{d+s}] < \infty$ *and* Λ *φ-stabilizing with* $\int_0^\infty \alpha^{s-1}\varphi(\alpha)\mathrm{d}\alpha < \infty$ *implies* $\lambda_{\mathrm{v}}(s/d), \lambda_{\mathrm{d}}(s) > 0$. $\lambda_{\mathrm{n}}(s/d) > 0$ *if additionally*
$$\int_0^\infty \alpha^{s-1}\mathbb{P}(\Lambda(B_\alpha) \geq c\alpha^d)\mathrm{d}\alpha < \infty \quad \textit{for some } c > 0. \tag{1}$$
4. $\mathbb{E}[\varrho^{d+s}] = \infty$ *and* Λ *ergodic implies* $\lambda_{\mathrm{v}}(s/d) = \lambda_{\mathrm{d}}(s) = \lambda_{\mathrm{n}}(s/d) = 0$.

- Proof via multi-scale and branching-process arguments (see below)
- Overshoot-Condition (1) satisfied by canonical examples (see below)

Related work

- $0 < \lambda_{\rm vp} < \infty$ for fixed radii

  Hirsch, Cali & J '19: *Continuum percolation for Cox point processes*, Stoch. Proc. Appl.

  Tóbiás '20: *Signal to interference ratio percolation for Cox point processes*, Lat. Am. J. Probab. Math. Stat.

- $\lambda_{\rm vp} > 0, \lambda_{\rm dp} > 0, \lambda_{\rm v}(s/d) > 0$ and $\lambda_{\rm d}(s) > 0$ alternatively via

  Gouéré, '09: *Subcritical regimes in some models of continuum percolation*, Ann. Appl. Probab.

- Version for Poisson point processes

  Meester & Roy '96: *Continuum Percolation*, Cambridge University Press

  Gouéré '08: *Subcritical regimes in the Poisson Boolean model of continuum percolation*, Ann. Probab.

  Gouéré & Théret '18: *Equivalence of some subcritical properties in continuum percolation*, Bernoulli

- Percolation for Gibbs point processes with fixed radii

  Mürmann '75: *Equilibrium distributions of physical clusters*, Comm. Math. Phys.

  Stucki '13: *Continuum percolation for Gibbs point processes*, Electron. Commun. Probab.

  Jansen '16: *Continuum percolation for Gibbsian point processes with attractive interactions*, Electron. J. Probab.

  Magazinov '18: *On percolation of two-dimensional hard disks*, Comm. Math. Phys.

Related work

- Percolation for repelling point processes in $\mathbb{R}^2$ include Ginibre ensembles and Gaussian zero processes with fixed radii

  Ghosh, Krishnapur & Peres '16: *Continuum percolation for Gaussian zeroes and Ginibre eigenvalues*, Ann. Probab.

- Percolation for negatively associated point processes, including determinantal point processes and some perturbed lattices

  Georgii & Yoo '05: *Conditional intensity and Gibbsianness of determinantal point processes*, J. Stat. Phys.

  Blaszczyszyn & Yogeshwaran '13: *Clustering and percolation of point processes*, Electron. J. Probab.

Criteria for Condition (1)

Lemma

Condition (1) *holds if any of the following two conditions holds,*

$$\limsup_{\alpha\uparrow\infty} \alpha^{-d} \log \mathbb{E}[\exp(\beta\Lambda(B_\alpha))] < \infty \qquad \textit{for some } \beta > 0,$$

$$\limsup_{\alpha\uparrow\infty} \alpha^{s-d\beta+\varepsilon} \mathbb{E}\big[\big|\Lambda(B_\alpha) - |B_\alpha|\big|^\beta\big] < \infty \qquad \textit{for some } \beta \geq 1 \textit{ and } \varepsilon > 0.$$

Applicable:

- Shot-noise field with compactly supported kernel κ
- $\Lambda(\mathrm{d}x) = \ell_x \mathrm{d}x$ with $\ell_x = \lambda \sum_{i\in I} \mathbb{1}\{|Y_i - x| < \rho_i\}$ and $\mathbb{E}[\rho^{d+2s+\varepsilon}] < \infty$
- $\Lambda(\mathrm{d}x) = \nu_1(S \cap \mathrm{d}x)$ with S edge set of PDT in $\mathbb{R}^2$

Ideas of proof

- Part 1: Existence of supercritical percolation regimes for sufficiently-large radii via coupling with percolating Cox–Boolean model with large fixed radii
- Part 2 & 3: Existence of subcritical percolation regimes via multi-scale argument; main idea, let $f(\alpha) = \mathbb{P}\big(o \leftrightsquigarrow B^{\mathrm{c}}_\alpha(o)\big)$ and $g(\alpha) = \lambda\mathbb{E}[\varrho^d \mathbb{1}\{\varrho \geq \alpha\}] + 2\varphi(10\alpha)$, then there exists $c = c(d) < \infty$ such that

$$f(10\alpha) \leq c\big(f(\alpha)^2 + g(\alpha)\big)$$

  [Gouéré, '08]: $\lim_{\alpha\uparrow\infty} g(\alpha) = 0$ implies $\lim_{\alpha\uparrow\infty} f(\alpha) = 0$ and $\int_1^\infty \alpha^s g(\alpha)\mathrm{d}\alpha < \infty$ implies $\int_1^\infty \alpha^s f(\alpha)\mathrm{d}\alpha < \infty$
- Part 4: Absence of subcritical percolation regimes via branching-process arguments

Percolation probabilities and Palm calculus

- Fix $r > 0$ and consider Cox–Gilbert graph $g_r(X)$
- Assume normalization $\mathbb{E}[\Lambda(Q_1)] = 1$ with $Q_r = [-r/2, r/2]^d$
- Define Palm version X^* of X via

$$\mathbb{E}[f(X^*)] = \frac{1}{\lambda}\mathbb{E}\Big[\sum_{X_i \in Q_1} f(X - X_i)\Big]$$

- Define percolation probability by

$$\theta(\lambda, r) = \mathbb{P}(o \rightsquigarrow \infty \text{ in } g_r(X^*))$$

- Define Palm version Λ^* of Λ via

$$\mathbb{E}[f(\Lambda^*)] = \mathbb{E}\Big[\int_{Q_1} \Lambda(\mathrm{d}x) f(\Lambda - x)\Big]$$

Large-radius limit of the percolation probability

- For Poisson–Gilbert graph: $\lim_{r\uparrow\infty} r^{-d}\log(1-\theta(\lambda,r)) = -|B_1(o)|\lambda$

  Penrose ’91: *On a continuum percolation model*, Adv. Appl. Probab.
- Large-deviations rate function given by isolation probability
- Remains true for b-dependent Cox processes with Λ-dependent rate:

Theorem

If $\lambda > 0$, then

$$\liminf_{r\uparrow\infty} r^{-d}\log(1-\theta(\lambda,r)) \geq \liminf_{r\uparrow\infty} r^{-d}\log \mathbb{E}[\exp(-\lambda\Lambda^*(B_r(o)))].$$

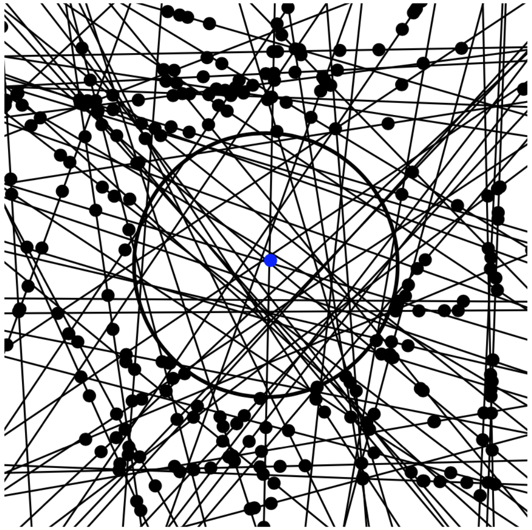
If, additionally, Λ is b-dependent and $\Lambda([0,1]^d)$ has all exponential moments, then the limit

$$I^* = -\lim_{r\uparrow\infty} r^{-d}\log \mathbb{E}[\exp(-\lambda\Lambda^*(Q_r))]$$

exists and

$$\lim_{r\uparrow\infty} r^{-d}\log(1-\theta(\lambda,r)) = -|B_1(o)|I^*.$$

Large-radius limit of the percolation probability



- Global lower bound given by isolation probability
- Upper bound applicable for example for environment based on Ξ or shot-noise fields
- Limiting Laplace transform computable for example for shot-noise fields,

$$\lim_{r\uparrow\infty} r^{-d} \log \mathbb{E}[\exp(-\lambda\Lambda(Q_r))] = \lambda_{\rm s}(e^{-\lambda K} - 1)$$

where $K = \int \mathrm{d}x\, k(x)$

Large-intensity limit of the percolation probability

- For Poisson–Gilbert graph, due to scale invariance, large-intensity statements can be obtained from large-radius statements
- In general not true for Cox–Gilbert graphs
- Cox case: connectivity structure of the support of Λ^* becomes prominent
- Define r-boundary of set $A \subset \mathbb{R}^d$ by $\partial_r A = \{x \in \mathbb{R}^d : \text{dist}(x, A) < r\} \setminus A$
- Define $\mathcal{R}_r =$ set of all compact sets that contain the origin and are r-connected

Theorem

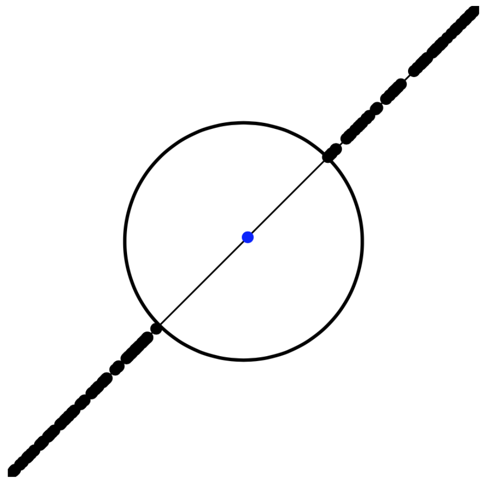
Let $r > 0$. Then,

$$\liminf_{\lambda \uparrow \infty} \lambda^{-1} \log(1 - \theta(\lambda, r)) \geq - \inf_{A \in \mathcal{R}_r} \operatorname{essinf}(\Lambda^*(\partial_r A)).$$

If, additionally, Λ is b-dependent and $\operatorname{essinf}(\Lambda(Q_\delta)) > 0$ for every $\delta > 0$, then

$$\limsup_{\lambda \uparrow \infty} \lambda^{-1} \log(1 - \theta(\lambda, r)) \leq - \lim_{\varepsilon \downarrow 0} \inf_{A \in \mathcal{R}_{r+\varepsilon}} \operatorname{essinf}(\Lambda^*(\partial_{r-\varepsilon} A)).$$

Large-intensity limit of the percolation probability



- Global lower bound not necessarily given by isolation probability
- For example for environment based on Ξ with $\lambda_1, \lambda_2 > 0$, r.h.s. optimal for $A = \{o\}$
- For PVT r.h.s. of the lower bound can be computed to be $-2r$
- Condition $\mathrm{essinf}(\Lambda(Q_\delta)) > 0$ not satisfied for shot-noise field and singular examples

Sharp thresholds for finite-range Cox–Boolean models

- Environments with uniformly bounded dependencies and bounded intensity, e.g., PVT superposed with sparse grids and thickened edges or capped intensities
- Finite-volume percolation probability

$$\theta_n(\lambda) = \mathbb{P}\big(o \leftrightsquigarrow [-n/2, n/2]^{\mathrm{c}} \text{ in } g_1(X^\lambda)\big)$$

Theorem

1. $\limsup_{n\to\infty} n^{-1} \log \theta_n(\lambda) < 0$ *holds for every* $\lambda < \lambda_{\mathrm{c}}$*, and*
2. $\liminf_{\lambda\downarrow\lambda_{\mathrm{c}}} \theta(\lambda)/(\lambda - \lambda_{\mathrm{c}}) > 0$*.*

- Proof based on OSSS inequality and coarse-graining construction

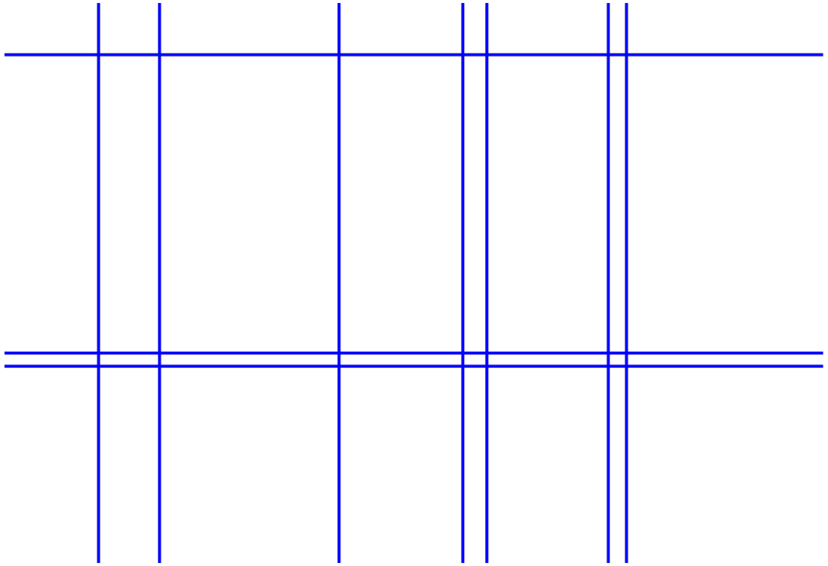
Hirsch, Muirhead & J ’22: *Sharp phase transition for Cox percolation*, Elec. Comm. Probab.

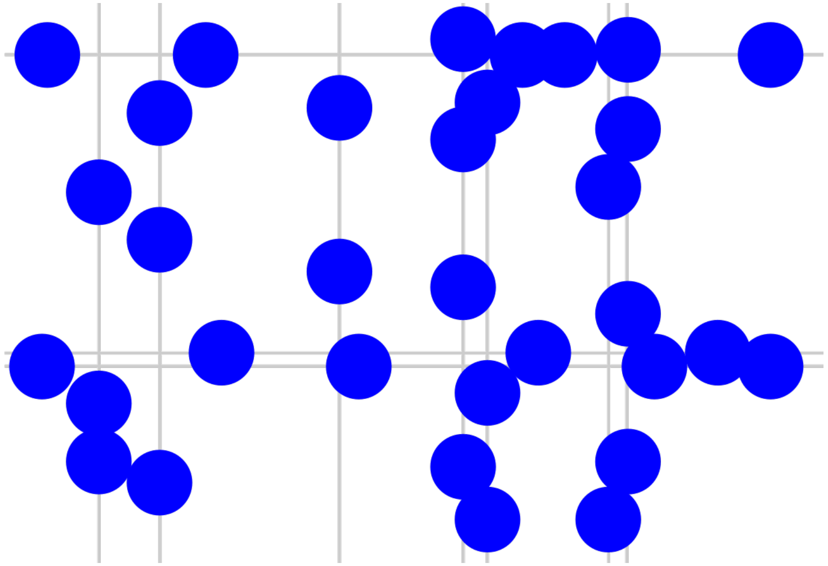
Duminil-Copin, Raoufi & Tassion’19: *Sharp phase transition for the random-cluster and Potts models via decision trees*, Ann. of Math.

Beyond stabilization: Manhattan–Boolean models

- Absence of stabilization features long (infinite)-range dependencies
- Manhattan grid based on two independent homogeneous Poisson point processes $\Phi^{\mathrm{x}}, \Phi^{\mathrm{y}} \subset \mathbb{R}$ with intensities μ_{x} and μ_{y}

$$\Lambda(A) := \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}\{(x, y) \in A\} \mathrm{d}y \, \Phi^{\mathrm{x}}(\mathrm{d}x) + \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}\{(x, y) \in A\} \mathrm{d}x \, \Phi^{\mathrm{y}}(\mathrm{d}y)$$





No sharp thresholds in Manhattan–Boolean models

- $g_r(X^\lambda)$ Gilbert graph based on Poisson point process with intensity $\lambda\Lambda$
- Accumulation of streets is probabilistically cheap, leading to subexponential decay

Vu, Jhawar & J '22: *Continuum percolation in a nonstabilizing environment*, arXiv:2205.15366

Proposition

For all $r > \sqrt{2}$ *and* $\mu_{\mathrm{x}}, \mu_{\mathrm{y}}, \lambda > 0$ *we have that*

$$\liminf_{n\to\infty} \lfloor \lambda^{-1} \log n \rfloor !\, n^{\lambda^{-1} \log(\mu_{\mathrm{x}} \wedge \mu_{\mathrm{y}})} \theta_n(r, \mu_{\mathrm{x}}, \mu_{\mathrm{y}}, \lambda) > 0\,,$$

in particular, for every $\varepsilon > 0$

$$\liminf_{n\to\infty} n^{(1+\varepsilon)\lambda^{-1} \log(\log n)} \theta_n(r, \mu_{\mathrm{x}}, \mu_{\mathrm{y}}, \lambda) = \infty\,.$$

Supercritical regimes in Manhattan–Boolean models

Theorem

Let $r > 0$ be arbitrary.

1. *For every $\lambda > 0$, there exists $\mu_{\mathrm{c}}(r, \lambda) > 0$, such that for all $\mu_{\mathrm{x}},\ \mu_{\mathrm{y}} \geq \mu_{\mathrm{c}}(r, \lambda)$: $\mathcal{C}(r, \mu_{\mathrm{x}}, \mu_{\mathrm{y}}, \lambda)$ percolates almost surely.*
2. *For every $\mu_{\mathrm{x}},\ \mu_{\mathrm{y}} > 0$, there exists $\lambda_c(r, \mu_{\mathrm{x}}, \mu_{\mathrm{y}}) > 0$, such that for all $\lambda \geq \lambda_c(r, \mu_x, \mu_y)$: $\mathcal{C}(r, \mu_{\mathrm{x}}, \mu_{\mathrm{y}}, \lambda)$ percolates almost surely.*
3. *For every $\lambda,\ \mu_{\mathrm{x}} > 0$, there exists $\mu_{\mathrm{y},c}(r, \mu_{\mathrm{x}}, \lambda) > 0$, such that for all $\mu_{\mathrm{y}} \geq \mu_{\mathrm{y,c}}(r, \mu_x, \lambda)$: $\mathcal{C}(r, \mu_{\mathrm{x}}, \mu_{\mathrm{y}}, \lambda)$ percolates almost surely.*

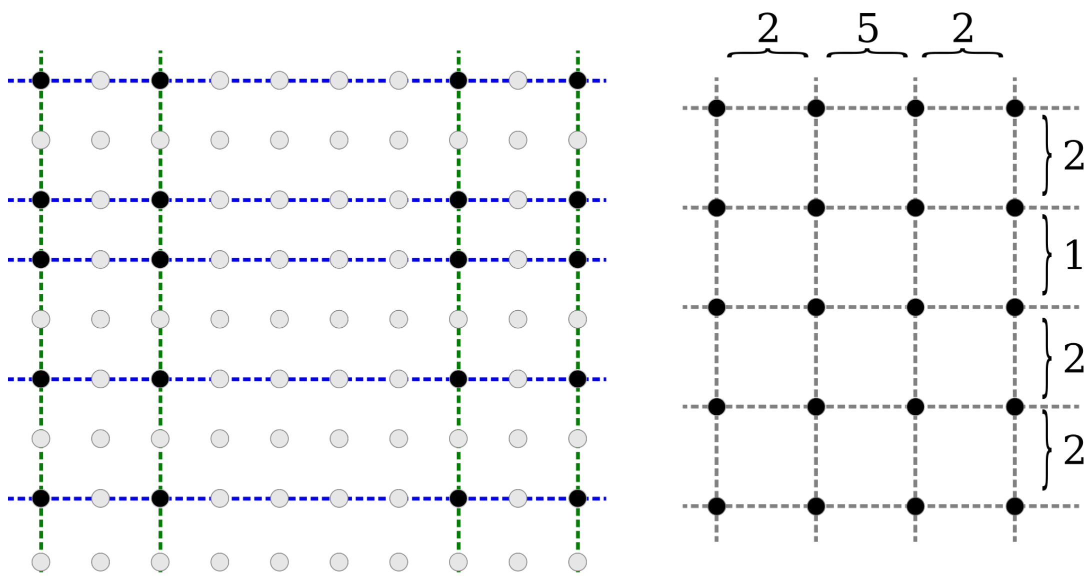
- Proof by comparison to supercritical bond percolation in randomly stretched lattice

Hoffman ’05: *Phase transition in dependent percolation*, Comm. Math. Phys.

Randomly stretched lattice (RSL)

- $N^{(x)} := (N_i^{(x)})_{i\in\mathbb{Z}}$ and $N^{(y)} := (N_j^{(y)})_{j\in\mathbb{Z}}$ = families of mutually independent positive random variables and $p \in (0,1)$

$$\mathbb{P}\big((i,j) \leftrightarrow (i+1,j) \text{ is open} \,|\, N^{(x)},\, N^{(y)}\big) := p^{N_i^{(x)}}$$



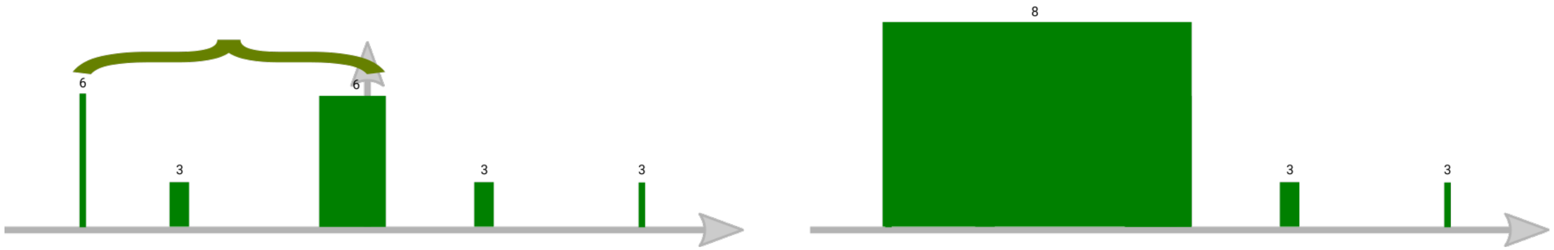
- Hoffman: If $\mathbb{P}(N_i^{(x)} \geq l), P(N_j^{(y)} \geq l) \leq 2^{-1000l}$, then there exists $p_\text{c} \in (0,1)$ such that RSL percolates almost surely for all $p \geq p_\text{c}$.

Subcritical regimes in Manhattan–Boolean models

Theorem

For every $r > 0$ and every $\mu_{\mathrm{x}}, \mu_{\mathrm{y}} \geq 0$, there exists a $\lambda_{\mathrm{c}}(r, \mu_{\mathrm{x}}, \mu_{\mathrm{y}}) > 0$ such that for all $\lambda \leq \lambda_{\mathrm{c}}(r, \mu_{\mathrm{x}}, \mu_{\mathrm{y}})$: $\mathcal{C}(r, \mu_{\mathrm{x}}, \mu_{\mathrm{y}}, \lambda)$ does not percolate almost surely.

- Proof significantly more tedious via Peierls' argument and existence of closed circuits in dual of subcritical RSL
- Discretization of streets into merging bands and labels mimicking and extending the strategy of Hoffman



Setting: Device locations

- Devices at time zero: Stationary Cox point process $X^\lambda = (X_i)_{i\in I}$ on random segment processes S with intensity measure $\Lambda_S(\mathrm{d}x) = \lambda|S \cap \mathrm{d}x|$, where $\lambda > 0$ and $\mathbb{E}[\Lambda_S([0,1]^d)] = 1$

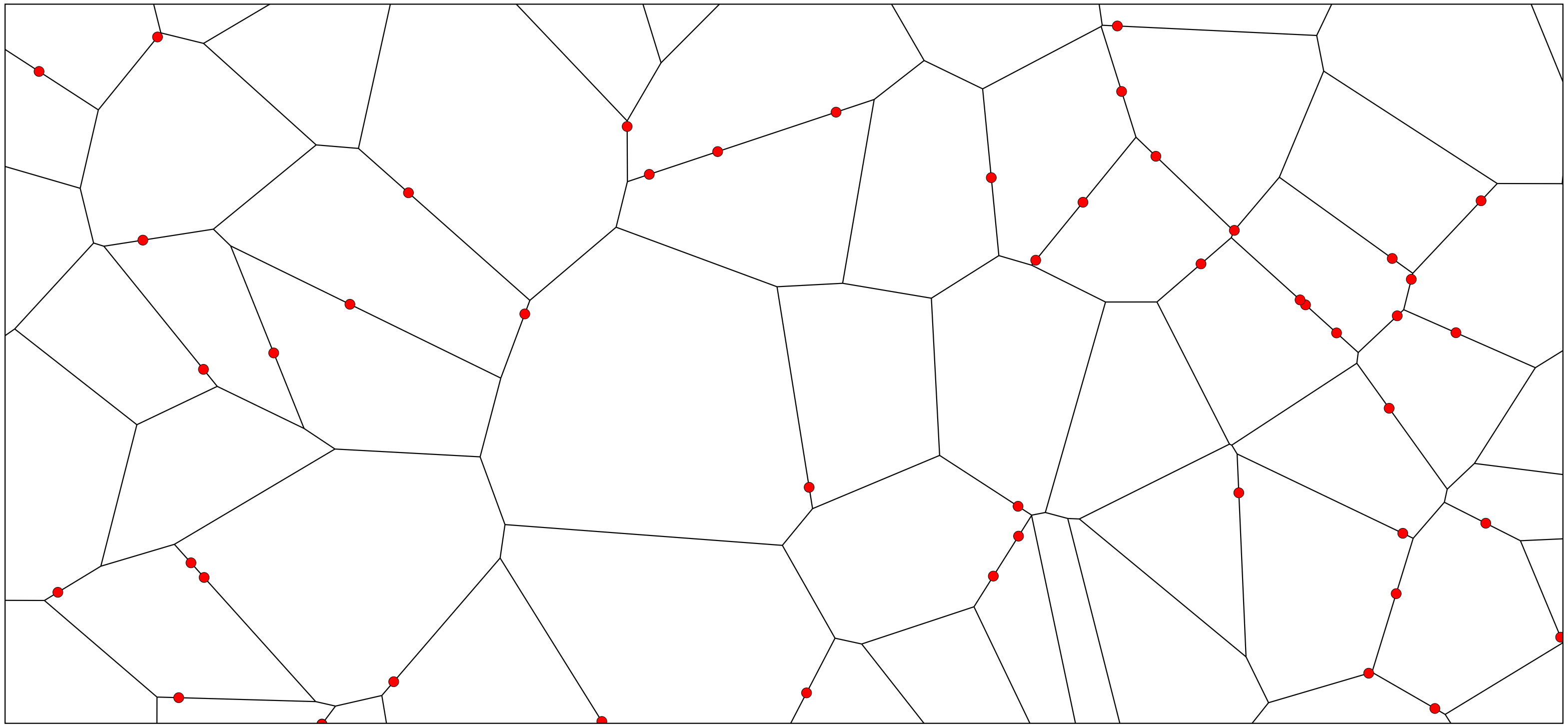


Figure: Realization of a Cox–Boolean model (blue) based on a Cox point process (red) with directing measure given by a realization of a Poisson–Voronoi tessellation (black).

Stabilizing environments

Definition (Stabilization)

A stationary random segment process S is called stabilizing if there exists a random field of stabilization radii $R = \{R_x\}_{x\in\mathbb{R}^2}$, defined on the same probability space as S, with

1. (S, R) are jointly stationary,
2. $\lim_{n\uparrow\infty} \mathbb{P}(\sup_{y\in Q_n\cap\mathbb{Q}^2} R_y < n) = 1$ and
3. for all $n \geq 1$, the random variables

$$\Big(f(S_{Q_n(x)})\mathbb{1}\{\sup_{y\in Q_n(x)\cap\mathbb{Q}^2} R_y < n\}\Big)_{x\in\psi}$$

are independent for all non-negative bounded measurable functions f and finite $\psi \subset \mathbb{R}^2$, as long as dist$(x, \psi \setminus x) > 3n$ for all $x \in \psi$.

- stabilization implies ergodicity
- S b-dependent if $\mathbb{P}(\sup_{y\in Q_n\cap\mathbb{Q}^2} R_y \geq n) = 0$ for all $n > b$
- S exponentially stabilizing if $\mathbb{P}(\sup_{y\in Q_n\cap\mathbb{Q}^2} R_y \geq n) \leq C\exp(-cn)$

Asymptotic essential connected environments

- Λ_S for S a PVT or PDT are exponentially stabilizing, $R_x = \min\{|x - X_i|: X_i\}$

  Hirsch, Cali & J '19: *Continuum percolation for Cox point processes*, Stochastic Processes and Applications

- Λ_S for S a PLT or MG are not stabilizing

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Definition (Asymptotic essential connectedness)

A stabilizing random segment process S with stabilization radii R is asymptotically essentially connected if, for all sufficiently large $n \geq 1$, whenever $\sup_{y \in Q_{2n} \cap \mathbb{Q}^2} R_y < n/2$, we have that

1. $|S \cap Q_n| > 0$ and
2. $S \cap Q_n$ is contained in one of the connected components of $S \cap Q_{2n}$.

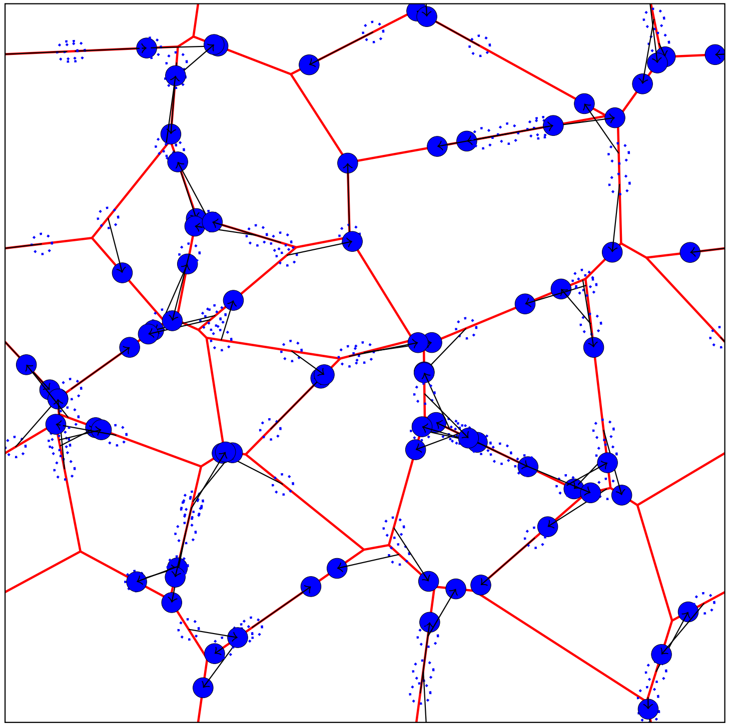
- Λ_S for S a PVT or PDT are asymptotically essentially connected

Mobility in urban device-to-device networks

- Waypoint kernel: device X_i picks target location Y_i independently via stationary kernel

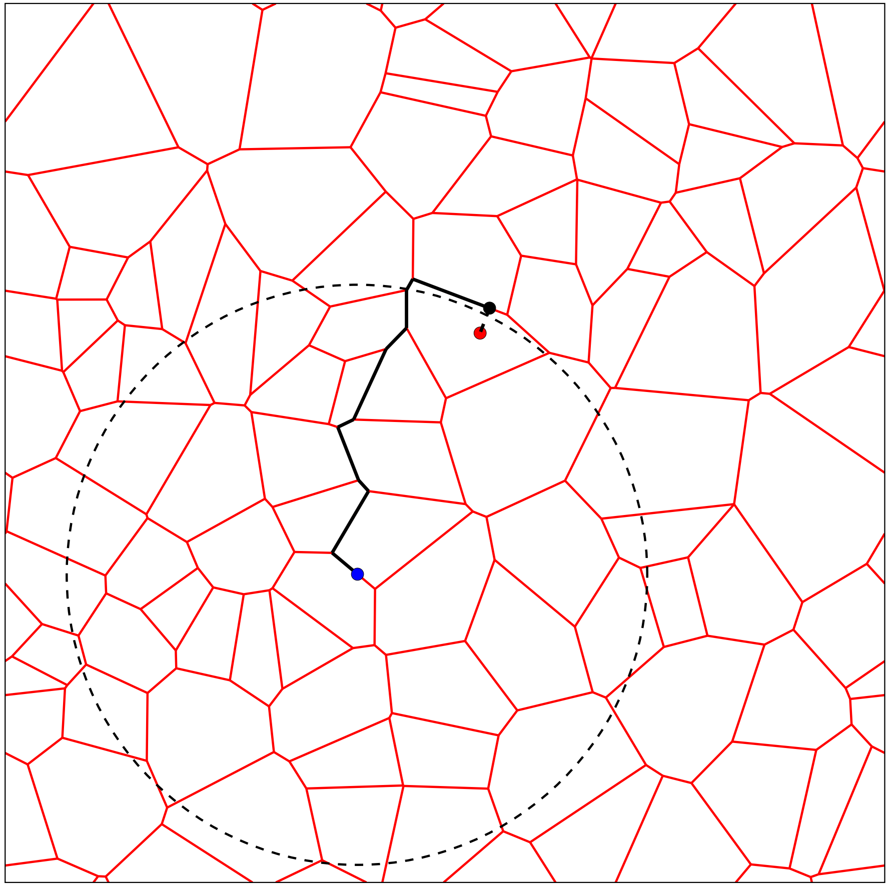
$$\kappa^S(X_i, \mathrm{d}y)$$

- Example: $\kappa^S(x, \mathrm{d}y) = |S \cap B_R(x)|^{-1} |\mathrm{d}y \cap S \cap B_R(x)|$



Mobility

- Movement: device X_i moves to Y_i and back with iid velocity V_i (governed by μ_{v}, with $\mathrm{supp}(\mu_{\mathrm{v}}) = [v_{\min}, v_{\max}] \subset (0, \infty)$) to waypoint via shortest path on S



Part 1: Connectivity in mobile device-to-device networks

- Contact times: device X_i and X_j with trajectories $(T_{i,t})_{t\geq 0}$ and $(T_{j,t})_{t\geq 0}$ have contact at times

$$Z(X_i, X_j) = \{0 \leq t \leq T \colon |T_{i,t} - T_{j,t}| < r \text{ and } T_{i,t}, T_{j,t} \text{ are on same street}\}$$

- Connection: device X_i and X_j are connected if for some $t \geq 0$,

$$[t, t + \rho] \subset Z(X_i, X_j)$$

where $\rho = \rho_o + \rho_1$ with initialization time $\rho_o \geq 0$ and transmission time $\rho_1 > 0$

Analysis of clustering behavior of graph

$$g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$$

with respect to system parameters.

Illustration of connectivity graph $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$

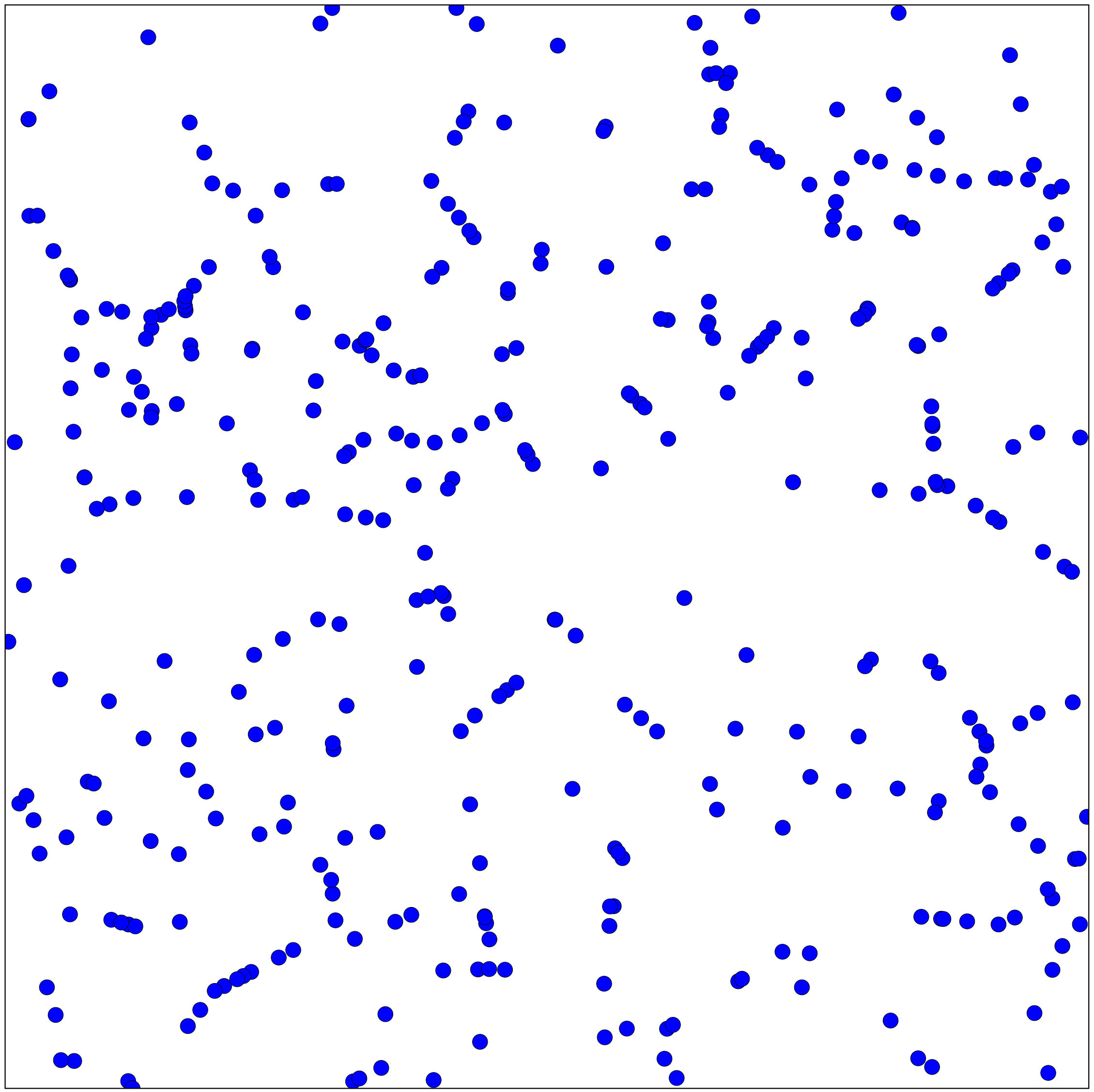


Figure: Realization of $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$ with $T = 0$. The street system is suppressed.

Illustration of connectivity graph $g_{T,\mu_{\mathrm{v}},\rho,r}(X^{\lambda})$

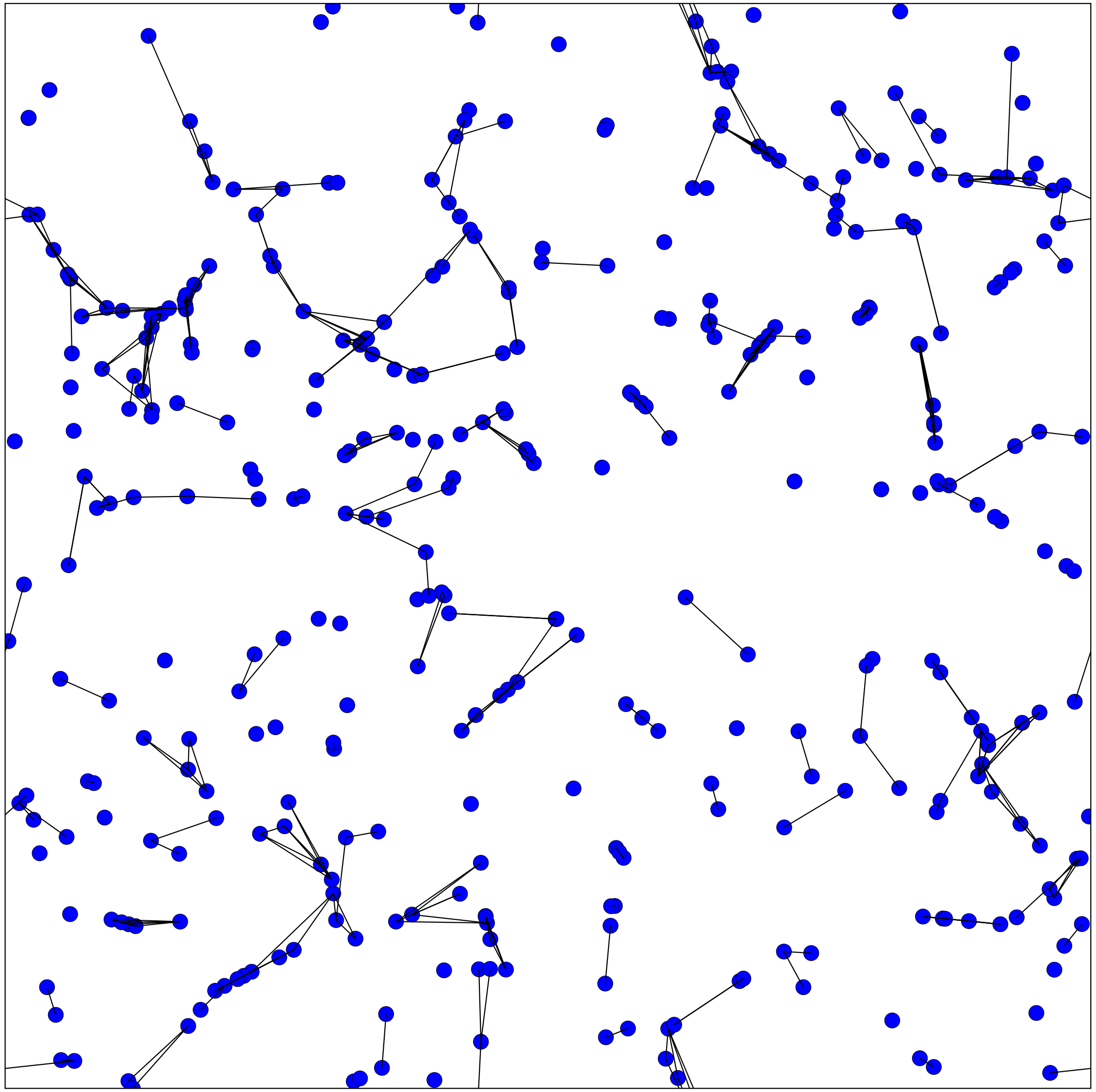


Figure: Realization of $g_{T,\mu_{\mathrm{v}},\rho,r}(X^{\lambda})$ with $T = 1$. The street system is suppressed.

Illustration of connectivity graph $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$

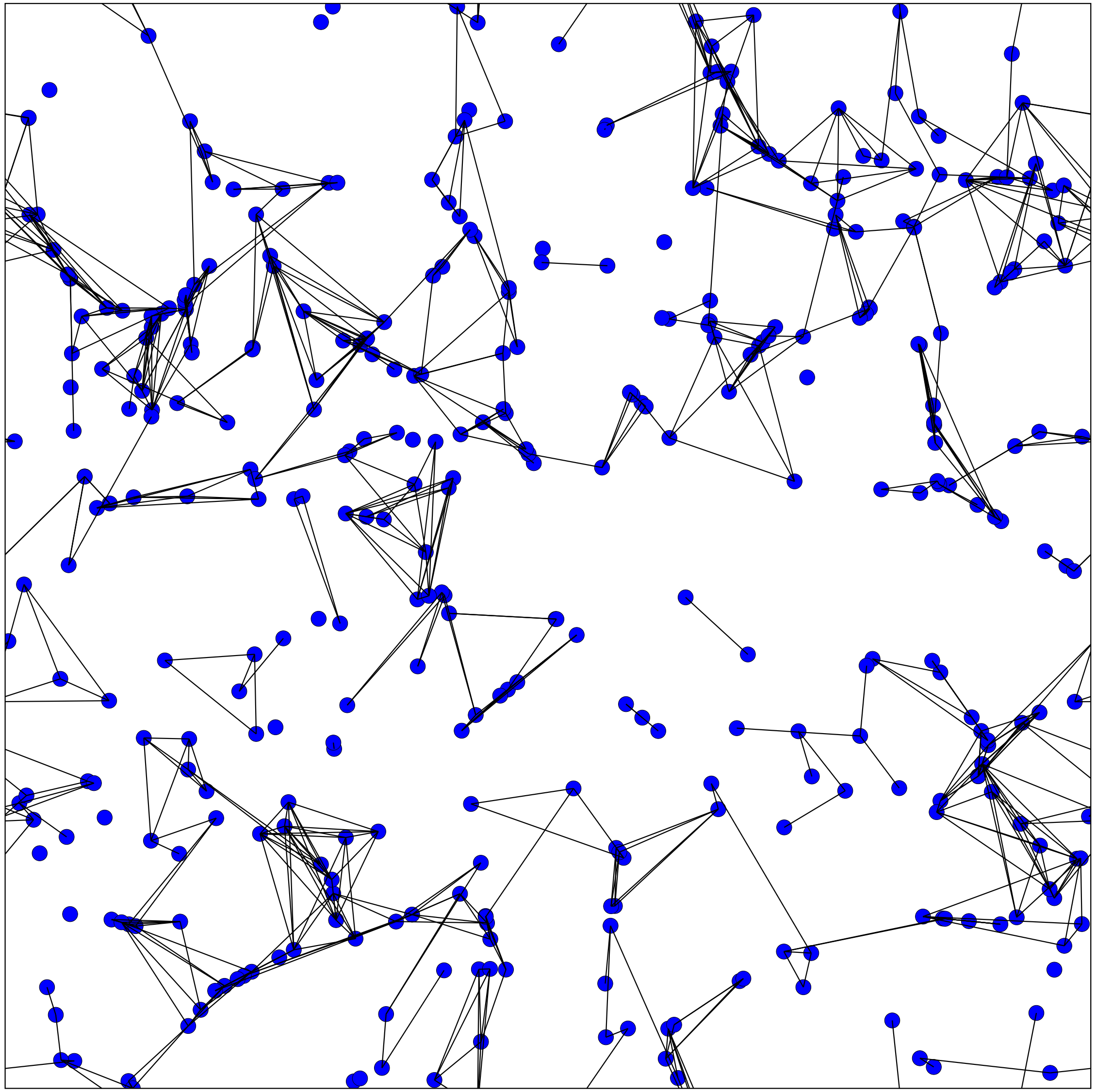


Figure: Realization of $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$ with $T = 2$. The street system is suppressed.

Illustration of connectivity graph $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$

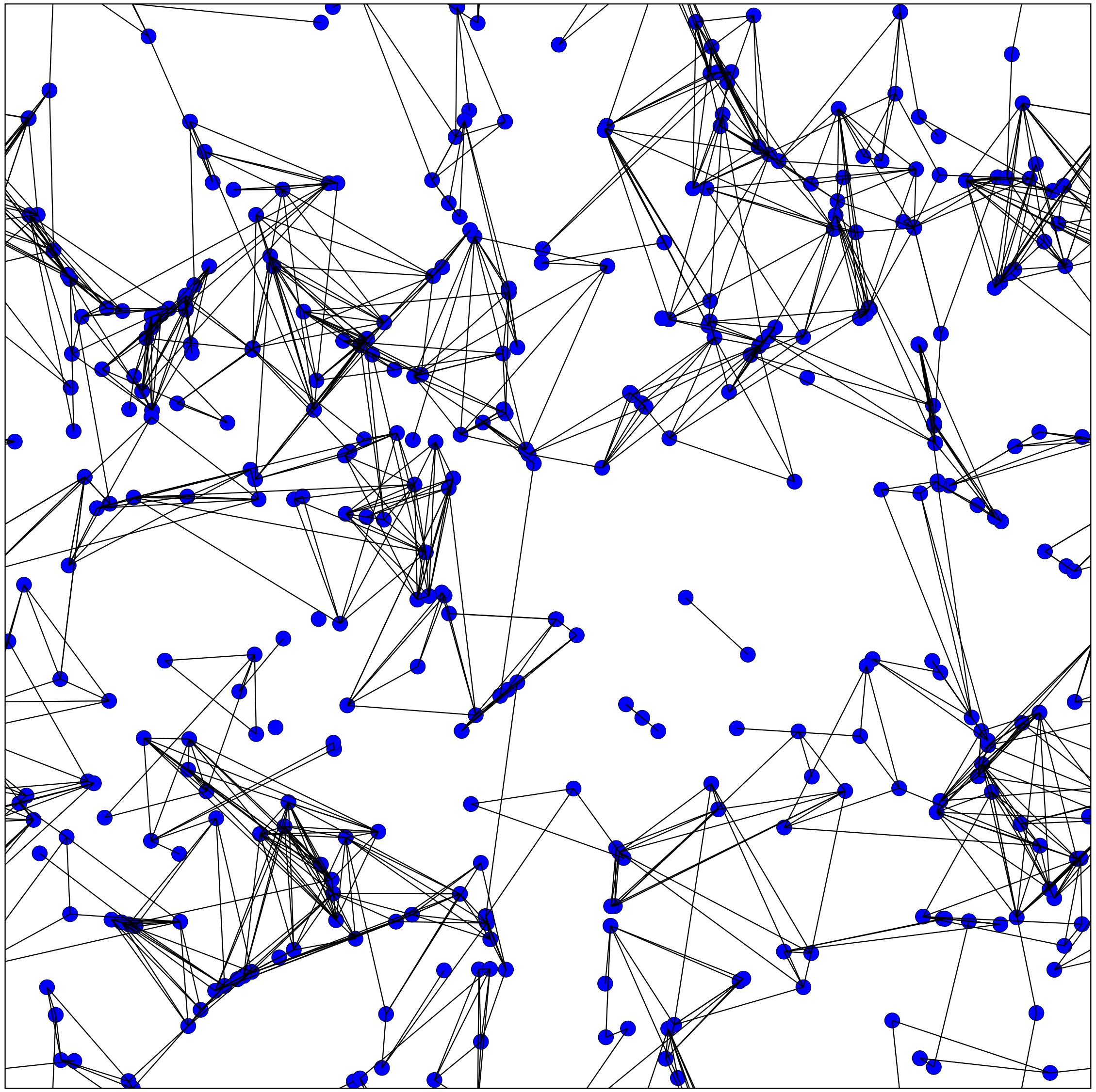


Figure: Realization of $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$ with $T = 3$. The street system is suppressed.

Illustration of connectivity graph $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$

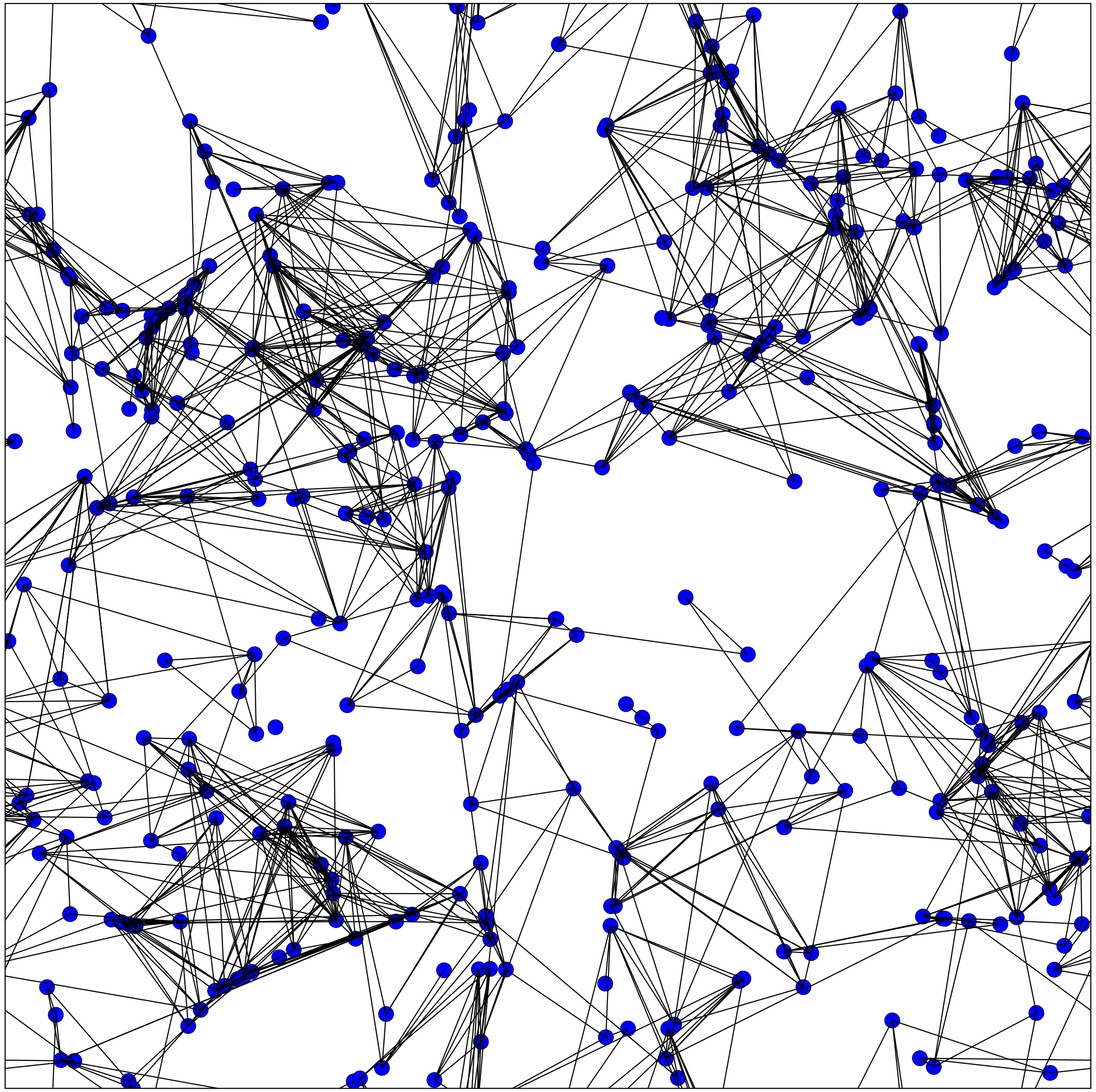


Figure: Realization of $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$ with $T = 4$. The street system is suppressed.

Illustration of connectivity graph $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$

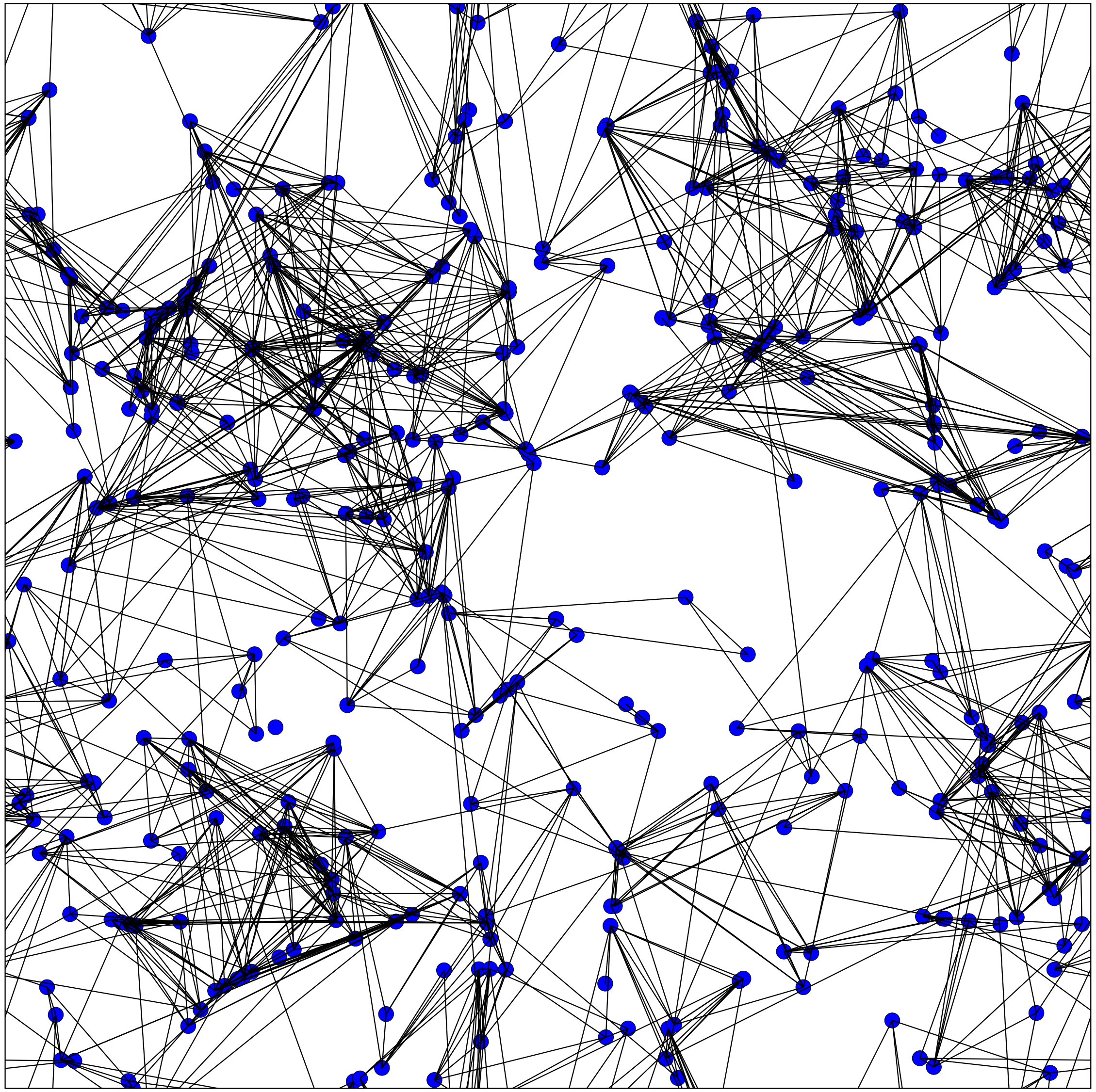


Figure: Realization of $g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda)$ with $T = 5$. The street system is suppressed

Critical parameters for percolation

- Define critical parameters for percolation:

$$
\begin{aligned}
T_{\mathrm{c}} =& T_{\mathrm{c}}(\lambda, \mu_{\mathrm{v}}, \rho, r) := \inf\{T > 0\colon \mathbb{P}(g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda) \text{ percolates}) > 0\}\\
\lambda_{\mathrm{c}} =& \lambda_{\mathrm{c}}(T, \mu_{\mathrm{v}}, \rho, r) := \inf\{\lambda > 0\colon \mathbb{P}(g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda) \text{ percolates}) > 0\}\\
\rho_{\mathrm{c}} =& \rho_{\mathrm{c}}(T, \lambda, \mu_{\mathrm{v}}, r) := \sup\{\rho > 0\colon \mathbb{P}(g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda) \text{ percolates}) > 0\}\\
r_{\mathrm{c}} =& r_{\mathrm{c}}(T, \lambda, \mu_{\mathrm{v}}, \rho) := \inf\{r > 0\colon \mathbb{P}(g_{T,\mu_{\mathrm{v}},\rho,r}(X^\lambda) \text{ percolates}) > 0\}\\
v_{\mathrm{c}} =& v_{\mathrm{c}}(T, \lambda, \rho, r) := \inf\{a v_{\max} > 0\colon \mathbb{P}(g_{T,\mu_{\mathrm{v}}^a,\rho,r}(X^\lambda) \text{ percolates}) > 0\}\\
v^{\mathrm{c}} =& v^{\mathrm{c}}(T, \lambda, \rho, r) := \sup\{a v_{\min} > 0\colon \mathbb{P}(g_{T,\mu_{\mathrm{v}}^a,\rho,r}(X^\lambda) \text{ percolates}) > 0\}
\end{aligned}
$$

with $\mu_{\mathrm{v}}^a(\mathrm{d}v) := \mu_{\mathrm{v}}(\mathrm{d}v/a)$

Critical parameters for percolation

- Define critical parameters for percolation:

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T_c =& T_c(\lambda, \mu_v, \rho, r) := \inf\{T > 0 \colon \mathbb{P}(g_{T,\mu_v,\rho,r}(X^\lambda) \text{ percolates}) > 0\} \\
\lambda_c =& \lambda_c(T, \mu_v, \rho, r) := \inf\{\lambda > 0 \colon \mathbb{P}(g_{T,\mu_v,\rho,r}(X^\lambda) \text{ percolates}) > 0\} \\
\rho_c =& \rho_c(T, \lambda, \mu_v, r) := \sup\{\rho > 0 \colon \mathbb{P}(g_{T,\mu_v,\rho,r}(X^\lambda) \text{ percolates}) > 0\} \\
r_c =& r_c(T, \lambda, \mu_v, \rho) := \inf\{r > 0 \colon \mathbb{P}(g_{T,\mu_v,\rho,r}(X^\lambda) \text{ percolates}) > 0\} \\
v_c =& v_c(T, \lambda, \rho, r) := \inf\{a v_{\max} > 0 \colon \mathbb{P}(g_{T,\mu_v^a,\rho,r}(X^\lambda) \text{ percolates}) > 0\} \\
v^c =& v^c(T, \lambda, \rho, r) := \sup\{a v_{\min} > 0 \colon \mathbb{P}(g_{T,\mu_v^a,\rho,r}(X^\lambda) \text{ percolates}) > 0\}
\end{aligned}
$$

with $\mu_v^a(\mathrm{d}v) := \mu_v(\mathrm{d}v/a)$

- Partial monotonicities:
  - $T \mapsto \mathbb{P}(g_{T,\mu_v,\rho,r}(X^\lambda) \text{ percolates})$ increasing
  - $\lambda \mapsto \mathbb{P}(g_{T,\mu_v,\rho,r}(X^\lambda) \text{ percolates})$ increasing
  - $r \mapsto \mathbb{P}(g_{T,\mu_v,\rho,r}(X^\lambda) \text{ percolates})$ increasing
  - $\rho \mapsto \mathbb{P}(g_{T,\mu_v,\rho,r}(X^\lambda) \text{ percolates})$ decreasing
  - $a \mapsto \mathbb{P}(g_{T,\mu_v^a,\rho,r}(X^\lambda) \text{ percolates})$ non monotone

Absence of percolation I

Theorem ($T_{\mathrm{c}}, \lambda_{\mathrm{c}}, v_{\mathrm{c}} > 0$)

For any κ and all $r, \rho \geq 0$ the following holds:

(1) *For all $\lambda \geq 0$ and μ_{v} we have that $T_{\mathrm{c}}(\lambda, \mu_{\mathrm{v}}, \rho, r) > 2\rho$,*

(2) *for all $T \geq 0$ and μ_{v} we have that $\lambda_{\mathrm{c}}(T, \mu_{\mathrm{v}}, \rho, r) > 0$ and*

(3) *for all $T, \lambda \geq 0$ we have that $v_{\mathrm{c}}(T, \lambda, \rho, r) > 0$.*

- Ideas for the proof: Comparison to Cox–Boolean models and discretization

Absence of percolation I

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- Ideas for the proof: Comparison to Cox–Boolean models and discretization

Theorem ($v_{\mathrm{c}} < \infty$)

Let S be a Poisson–Voronoi or Poisson–Delaunay tessellation. For any κ with bounded support and all $T, \lambda, r \geq 0$ and all $\rho > 0$ we have that $v^{\mathrm{c}}(T, \lambda, \rho, r) < \infty$.

- Ideas for the proof: Multiscale argument for geostatistical markings

Absence of percolation I

Take-home message

In-and-out of percolation for $v \mapsto g_{T,\delta_v,\rho,r}(X^\lambda)$.

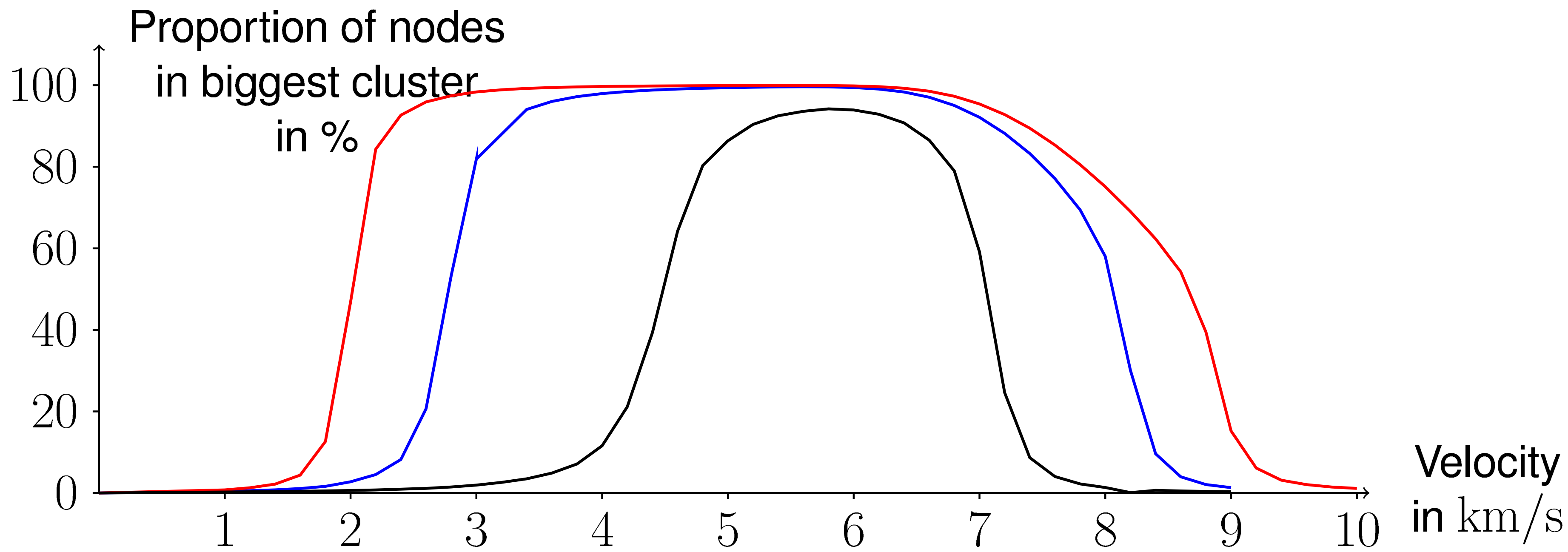

Figure: In-and-out of percolation: Simulation of the largest connected cluster at different times $T = 3\text{min}$ (black), $T = 4.5\text{min}$ (blue), and $T = 6\text{min}$ (red). Furthermore, we used $\rho = 10\text{sec}, r = 20\text{m}$, $\lambda = 20\text{devices}/\text{km}$ and a street intensity of $20\text{km}/\text{km}^2$. For the velocities we chose a normal distribution, conditioned to be positive $\mu_{\text{v}} = \mathcal{N}^+(v, v/5)$.

Absence of percolation II

Theorem ($r = 0$: $\rho_c = 0$)

For any κ, μ_v and all $T, \gamma, \lambda \geq 0$ we have that $\rho_c(T, \lambda, \mu_v, 0) = 0$.

Theorem ($r > 0$: $\rho_c < \rho_c'(T, \mu_v)$)

For any κ, μ_v and $T, \lambda, r \geq 0$: $\rho_c(T, \lambda, \mu_v, r) \leq \rho_c'(T, \mu_v) \wedge T/2$.

Here $\rho_c'(T, \delta_v) := \sup\{\rho > 0\colon \mathbb{P}(S^{v\rho, v(T-2\rho)}$ percolates$) > 0\}$, with $S^{a,b}$ given by edges $\ell \subset S$ with $|\ell| \geq a$ and edges between endpoints of ℓ at distance $\leq b$.

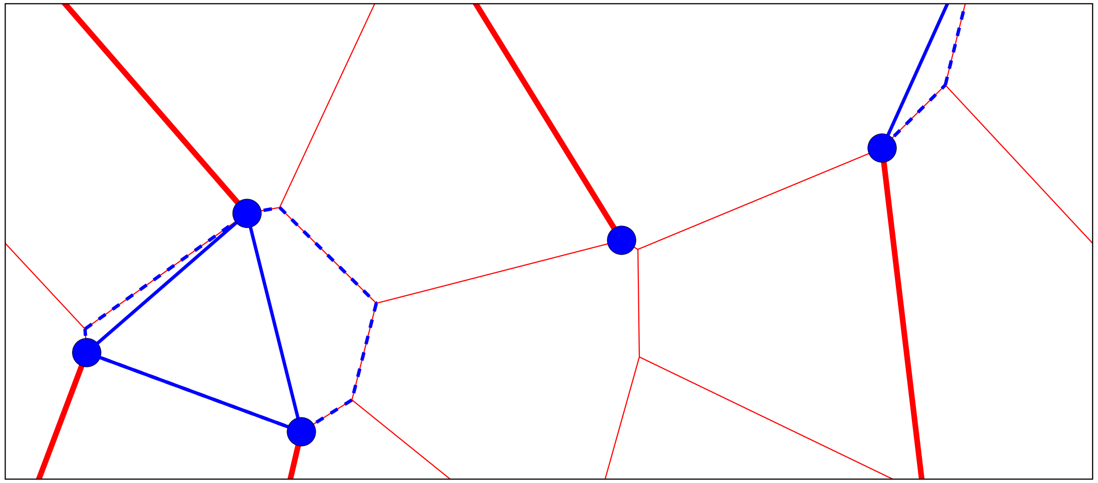

Figure: Realization of $S^{v\rho, v(T-2\rho)}$ based on S.

Thinned street systems

- Consider $a_c := \lim_{b \downarrow 0} a_c(b)$ where

$$a_c(b) := \sup\{a > 0\colon \mathbb{P}(S^{a,b} \text{ percolates}) > 0\}$$

Theorem

We have that $a_c < \infty$. Further, for any κ and $T \geq 0$ as well as $v > 4a_c/T$ it holds that $\rho_c'(T, \delta_v) < T/2$.

- Proof via stabilization arguments

Existence of percolation

- κ is c-well behaved if for almost-all S and all $x \in S$

$$B_c(x) \cap S \subset \text{supp}\big(\kappa^S(x, \mathrm{d}y)\big)$$

κ is well behaved if κ is c-well behaved for some $c > 0$

Theorem ($\rho = 0$: $\lambda_\mathrm{c} < \infty$)

For κ well behaved and $T > 0$ then $\lambda_\mathrm{c}(T, \mu_\mathrm{v}, 0, r) < \infty$.

Theorem ($\rho > 0$: $\lambda_\mathrm{c} < \infty$)

We have that $0 < a_\mathrm{c}(0)$. Let κ be c-well behaved for some $c > 0$, $r, \rho > 0$ and $T > 2\rho$. Then, for all sufficiently small $v_{\min}$ we have $\lambda_\mathrm{c}(T, \mu_\mathrm{v}, \rho, r) < \infty$.

- Proof via coupling to system of good and bad streets and stabilization arguments

Part 2: Chase escape in mobile device-to-device networks

- Contact times: device X_i and X_j with trajectories $(T_{i,t})_{t\ge 0}$ and $(T_{j,t})_{t\ge 0}$ have contact at times

$$Z(X_i, X_j) = \{0 \le t \le T \colon |T_{i,t} - T_{j,t}| < r \text{ and } T_{i,t}, T_{j,t} \text{ are on same street}\}$$

- Generalized connections: device X_i and X_j are connected if for some $t \ge 0$,

$$[t - \rho(X_i, X_j), t] \subset Z(X_i, X_j)$$

with $\rho(X_i, X_j)$ iid infection time with distribution ϱ

Finite degrees

- Shortest-path lengths: $\ell_S(x) = \sup\{|\ell_S(x,y)|\colon y \in \text{supp}(\kappa^S(x,\mathrm{d}y))\} + r/2$ where $\ell_S(x,y)$ shortest path between x and y on S
- Degree: $\deg(X_i) = \#\{X_j \in X^\lambda \setminus X_i\colon |X_i - X_j| \le \ell_S(X_i) + \ell_S(X_j)\}$
- Local connectedness assumption: $\mathbb{P}(\exists X_i \in X^\lambda$ such that $\deg(X_i) = \infty) = 0$

Proposition (Local connectedness)

If κ has bounded support, S is exponentially stabilizing and $|S \cap Q_1|$ has exponential moments, then the network is locally connected.

- Proof via first-moment method for typical device
- Satisfied for standard examples PVT and PDT

Devices and white knights

- Initial infected device: X_o, typical device at origin under Palm measure
- Initial susceptible devices: Cox process X^λ on S
- Initial white knights: Cox process Y^{λ_W} on S
- Infection time distribution: ϱ_{I} and patching time distribution: ϱ_{W}
- Minimal times: $\varrho_{\mathrm{I,min}} = \inf\{x \geq 0 \colon x \in \mathrm{supp}(\varrho_{\mathrm{I}})\}$, analogously for $\varrho_{\mathrm{W,min}}$

- Transmission mechanism:
  - Infection: Infected X_i transmits to susceptible X_j after completion of $\rho_{\mathrm{I}}(X_i, X_j)$
  - Patching: White knight X_i transmits to infected X_j after completion of $\rho_{\mathrm{W}}(X_i, X_j)$
- Process of susceptible and infected devices and white knights: $(S_t, I_t, W_t)_{t \geq 0}$

Simulations

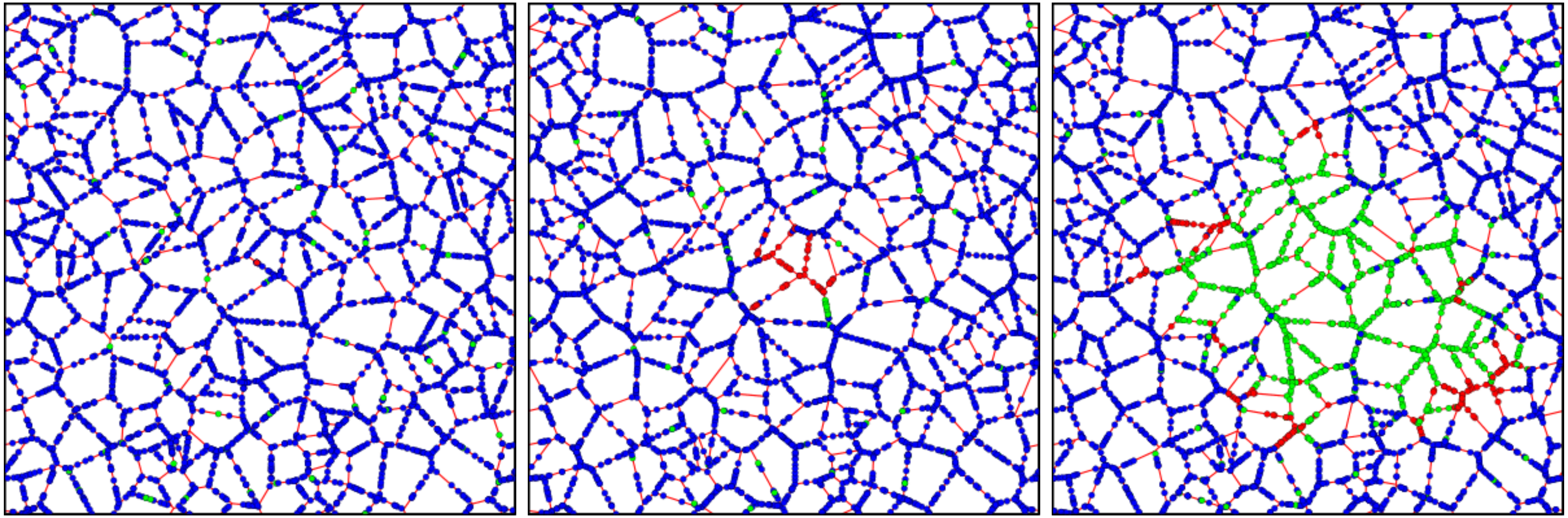

Figure: Propagation of infected devices (red) on a street system given by a Poisson–Voronoi tessellation. In the initial state (left) there is exactly one infected device present in the center and the remaining devices are either susceptible (blue) or white knights (green). At some small positive time (middle) further devices in the vicinity of the initially infected device have become infected and have started to make contact to white knights. At some later time (right) infected devices are only present along the boundary of the set of white knights in the center region.

Global survival

- Waypoint continuity: κ is c-continuous if $\inf_{y \in B_c(x)} \kappa^S(x,y) > 0$ where $\kappa^S(x,y)$ Lebesgue density defined for almost-all S and $x \in S$
- Street connectedness: $S^a = \{s \in S \colon |s| \geq a\}$ and R^a_x distance to furthest point from x reachable without crossing S^a. Call graph R^a-connected if

$$\lim_{n \uparrow \infty} \mathbb{P}\Big(\sup_{x \in Q_n \cap \mathbb{Q}^2} R^a_x < n \Big) = 1$$

  Let $a_c := \sup\{a > 0 \colon S^a \text{ is } R^a\text{-connected}\}$
- Global survival: $\big|\bigcup_{t \geq 0} I_t\big| = \infty$

Theorem (Global survival)

Let κ be c-continuous for some $c > 0$ and $\varrho_{\mathrm{W,min}} > \varrho_{\mathrm{I,min}} > 0$. Then, for all sufficiently small $v_{\min}$ such that $0 < v_{\min}\varrho_{\mathrm{I,min}} < \min(a_c/2, r, c/2)$, there exists $\lambda_c > 0$ such that for all $\lambda > \lambda_c$ and all $\lambda_{\mathrm{W}} \geq 0$,

$$\mathbb{P}(\textit{infection survives globally}) > 0.$$

Extinction

- Punchline: If $\varrho_{\mathrm{W,min}} > \varrho_{\mathrm{I,min}} > 0$, then survival guaranteed for all $\lambda_{\mathrm{W}} \geq 0$
- Proof idea: Existence of infinite cluster of good streets on which infected wins against patching

Extinction

- Punchline: If $\varrho_{\mathrm{W,min}} > \varrho_{\mathrm{I,min}} > 0$, then survival guaranteed for all $\lambda_{\mathrm{W}} \geq 0$
- Proof idea: Existence of infinite cluster of good streets on which infected wins against patching

Theorem (Extinction)

Let S be a Poisson–Voronoi or Poisson–Delaunay tessellation. Let κ be well behaved with bounded support and assume that $r > v_{\max}\varrho_{\mathrm{W,min}}$ and $\varrho_{\mathrm{I,min}} > \varrho_{\mathrm{W,min}} > 0$. Then, for all $\lambda \geq 0$, there exists $\lambda_{\mathrm{W,c}} > 0$ such that for all $\lambda_{\mathrm{W}} > \lambda_{\mathrm{W,c}}$ we have that

$$\mathbb{P}(\textit{infection survives globally}) = 0.$$

- Proof idea: Multiscale argument

Speed-dependent survival and extinction

Theorem (Speed-dependent survival and extinction)

Let S be a Poisson–Voronoi or Poisson–Delaunay tessellation. Let κ have bounded support and be c-well behaved for some $c > 0$. Assume further that $\varrho_{\mathrm{W}} = \delta_{\rho_{\mathrm{W}}}$ and $\varrho_{\mathrm{I}} = \delta_{\rho_{\mathrm{I}}}$ with $\rho_{\mathrm{W}} > \rho_{\mathrm{I}}$, and let $0 < v_o < \min(a_{\mathrm{c}}/2, r, c/2)/\rho_{\mathrm{I}}$. Then, there exists $\lambda_{\mathrm{c}} > 0$ and $\lambda_{\mathrm{W,c}} > 0$ (independent of λ_{c} and v_o), such that for all $\lambda > \lambda_{\mathrm{c}}$ and all $\lambda_{\mathrm{W}} > \lambda_{\mathrm{W,c}}$ we have that

(1) *there exists $v_o > v_{\mathrm{c}}(\lambda, \lambda_{\mathrm{W}}) > 0$ such that for all $\mu_{\mathrm{v}} = \delta_v$ with $v < v_{\mathrm{c}}(\lambda, \lambda_{\mathrm{W}})$ we have*

$$\mathbb{P}(\textit{infection survives globally}) = 0,$$

(2) *for $\mu_{\mathrm{v}} = \delta_{v_o}$ we have $\mathbb{P}(\textit{infection survives globally}) > 0$, and*

(3) *there exists $\infty > v^{\mathrm{c}} > v_o$ (independent of λ and λ_{W}), such that for all $\mu_{\mathrm{v}} = \delta_v$ with $v > v^{\mathrm{c}}$ we have*

$$\mathbb{P}(\textit{infection survives globally}) = 0.$$

Thank you for your attention

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