SINR percolation and k-nearest neighbour graphs

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Continuum percolation

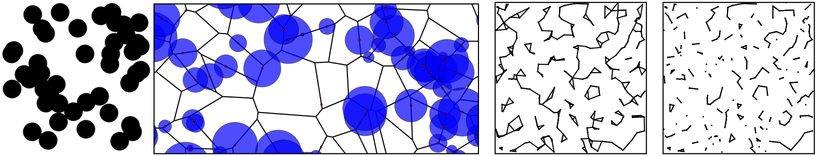
Continuum percolation: introduced by Gilbert (1961), context of communications already mentioned there.

Users of network situated according to a (simple, stationary) point process in $\mathbb{R}^d$, edges drawn between them according to certain rules.

A cluster is a maximal connected component in the graph.
Main question: does the graph percolate, i.e., does it have an infinite (unbounded) connected cluster?

Two basic variants of continuum percolation models:

1. distance-based (Gilbert graph/Boolean model, possibly with *random connection radii*, references see below),
2. k-nearest neighbour (undirected version introduced by Häggström and Meester (1996), bidirectional by Balister and Bollobás (2008)). Here $k = 2$.



Signal-to-Interference plus Noise Ratio (SINR) graphs

SINR graphs are an infinite-range dependent variant of continuum percolation, with particular applications in telecommunications. They feature both

- distance-based connection thresholds and
- bounded degrees.

Case of a homogeneous Poisson point process (PPP) in $\mathbb{R}^2$: percolation properties first studied by Dousse, Baccelli, and Thiran (2005).
Percolation in an infinite-range dependent setting proven by Dousse, Franceschetti, Macris, Meester, and Thiran (2006).

Idea of SINR: a transmission between two users is successful $\Leftrightarrow$ measured at the receiver, the signal power of the transmitter is strong enough compared to the interference coming from all the other users.

Notion of SINR known in the mathematical literature since the seminal paper by Gupta and Kumar (2000).

Definition of SINR

Setting: $X^\lambda = (X_i)_{i\in I}$: simple, stationary point process in $\mathbb{R}^d$, with $\lambda = \mathbb{E}[X^\lambda([0,1]^d)] \in (0,\infty)$.
$(P_i)_{i\in I}$: sequence of i.i.d. nonnegative random variables.
P_i: signal power of user X_i. $\mathbf{X}^\lambda = (X_i, P_i)_{i\in I}$: marked point process.

Definition

Choose a path-loss function $\ell\colon [0,\infty) \to [0,\infty)$: continuous, monotone decreasing, describes propagation of signal strength over distance.
E.g.: $\ell(r) = (1+r)^{-\alpha}$, $\alpha > 0$.
For two different points X_i, X_j of the point process $X^\lambda = (X_k)_{k\in I}$, define

$$\mathrm{SINR}(X_i,\ X_j,\ \mathbf{X}^\lambda) = \frac{P_i \ell(|X_i - X_j|)}{N_0 + \gamma \sum_{k\neq i,j} P_k \ell(|X_k - X_j|)}.$$

Parameters: $N_0 \geq 0$ is an external noise. The sum in the denominator is the interference, $\gamma \geq 0$ is the interference cancellation factor.

Fix an SINR threshold $\tau > 0$.

SINR graph $g_\gamma(\mathbf{X}^\lambda)$ based on vertex set X^λ: connect X_i, X_j $(i \neq j)$ by an edge whenever $\mathrm{SINR}(X_i, X_j, \mathbf{X}^\lambda) > \tau$ and $\mathrm{SINR}(X_j, X_i, \mathbf{X}^\lambda) > \tau$.

Main questions

$$\mathrm{SINR}(X_i,\ X_j,\ \mathbf{X}^\lambda) = \frac{P_i\ell(|X_i - X_j|)}{N_0 + \gamma\sum_{k\neq i,j} P_k\ell(|X_k - X_j|)} > \tau.$$

1. Notice: given all other parameters, the SINR graph is (stochastically) monotone decreasing in γ.
   Question: given that there is percolation for $\gamma = 0$ if the intensity λ is large enough, can one also guarantee percolation for $\gamma > 0$ sufficiently small (for some, possibly very large, λ)?
   This talk: particular interest in the case of Cox point processes.
2. SINR graphs have bounded degrees (see below).
   How large should the degree bound be such that there is no percolation independent of λ?
3. A sharpening of the first question:
   Is it true that if λ is above the critical threshold for percolation for $\gamma = 0$, then there is always also percolation for some $\gamma > 0$?

Some basic properties of SINR graphs

$$\mathrm{SINR}(X_i,\ X_j,\ X^\lambda) = \frac{P_i\ell(|X_i - X_j|)}{N_0 + \gamma\sum_{k\neq i,j} P_k\ell(|X_k - X_j|)} > \tau.$$

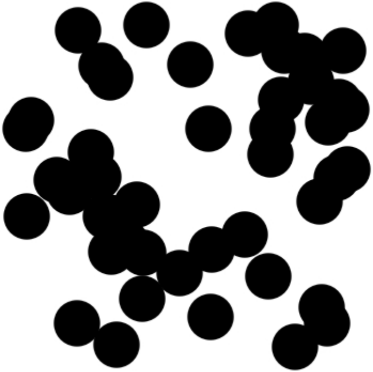
- If $\gamma = 0$: no interference,

  $$X_i \to X_j \text{ is a successful transmission } \Leftrightarrow |X_i - X_j| < \ell^{-1}(\tau N_0/P_i)$$

  $\Rightarrow$ Gilbert graph/random geometric graph with random radii.
  E.g. if $P_i \equiv P > 0$: two points are connected whenever their distance is less than $r_{\mathrm{B}} := \ell^{-1}(\tau N_0/P)$.
- First main question: given that percolation can be guaranteed for $\gamma = 0$, can one verify percolation for $\gamma > 0$ sufficiently small?
  Main technical difficulty: infinite-range dependencies of SINR graphs.

Poisson case: scale invariance, example for $d = 2$

Poisson case: the Gilbert graph ($\gamma = 0$) is scale invariant: the graph with connection radius r_{B} and intensity λ has the same percolation properties as the one with radius r'_{B} and intensity λ' if $\lambda r_{\mathrm{B}}^d = \lambda' r_{\mathrm{B}}'^d$.
Underlying Boolean model associated to the Gilbert graph:



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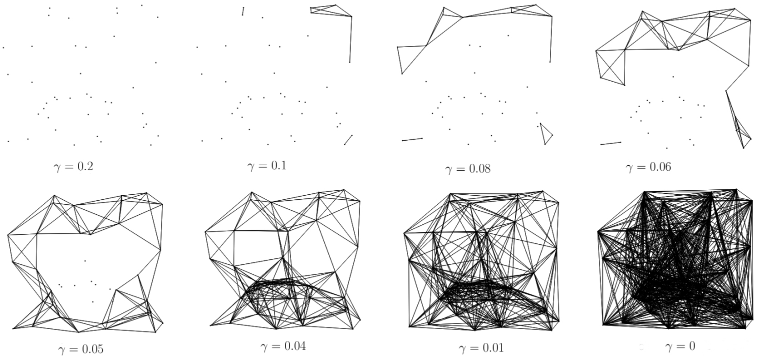
For $d \geq 2$, if λr_{B}^d is sufficiently large, there is percolation a.s. for $\gamma = 0$ (see Gilbert (1961)).
For sufficiently small $\gamma > 0$, most of the connectivity of the graph is preserved.



Positive results about percolation in SINR graphs

Usual setting: fix all model parameters but λ, γ (for $P_i \equiv P$, this in particular means fixing the Boolean radius r_{B}).
Show: if λ is sufficiently large such that there is percolation and strong enough connectivity for $\gamma = 0$, then

$$\gamma^*(\lambda) = \inf\{\gamma > 0 \colon \mathbb{P}(g_\gamma(\mathbf{X}^\lambda) \text{ percolates}) > 0\}$$

is positive. No monotonicity of $\lambda \to \gamma^*(\lambda)$! Often, $\lim_{\lambda \to \infty} \gamma^*(\lambda) = 0$.

Positive percolation results about SINR graphs, e.g.:

- PPPs, $d = 2$, constant radii: Dousse, Baccelli, and Thiran (2003); Dousse, Franceschetti, Macris, Meester, and Thiran (2005). The latter paper is the main source of proof techniques in the field.
- Sub-Poisson point processes, $d = 2$, constant radii: Błaszczyszyn and Yogeshwaran (2013).
- Cox point processes, $d \geq 2$, constant radii: T. (2018),
- PPPs, $d \geq 2$, random radii: Löffler (2019),
- Cox point processes, $d \geq 2$, random radii: Jahnel and T. (2019).

Here, years correspond to the first preprint versions of the works.

Schematic approach for verifying percolation for $\gamma > 0$

Proof scheme originating from Dousse et al. (2005): fix all parameters but λ, γ.

1. Define a suitable auxiliary discrete site/edge percolation model. Call a site/edge open if the following two conditions are satisfied:
   - Connectivity: In a certain neighbourhood of the site/edge, the underlying Gilbert graph ($\gamma = 0$) satisfies a certain strong connectivity property. For this, need to make λ sufficiently large.
   - Interference control: In this neighbourhood of the site/edge, all interferences are bounded by a certain constant M.
2. Show that for large enough λ, given a suitable choice of auxiliary parameters, the discrete model percolates with probability 1 $\to$ Peierls argument.
   - Connectivity property $\to$ need: spatial decorrelation and strong local connectivity of X^λ.
   - Interference control property $\to$ infinite range dependent if the path-loss function ℓ has unbounded support. Need: $\int_0^\infty r^{d-1}\ell(r)\mathrm{d}r < \infty$. Often helpful: exponential moment assumptions on the power variables/number of points of X^λ in $[0,1]^d$.
3. Conclude from this that also the SINR graph percolates for such λ and small enough (but positive!) $\gamma > 0$.

The case of Cox point processes

Take a stationary random measure Λ on $\mathbb{R}^d$ such that $\mathbb{E}[\Lambda([0,1]^d)] = 1$.
For $\lambda > 0$, the Cox point process X^λ with directing measure $\lambda\Lambda$ is characterized by the property that conditional on Λ, X^λ is a PPP with intensity $\lambda\Lambda$.

Special case: $\Lambda \equiv \mathrm{Leb} \Rightarrow X^\lambda$ PPP with intensity λ.
A more interesting example: PPPs on random street systems. Here: Λ given by a Poisson–Voronoi tessellation.

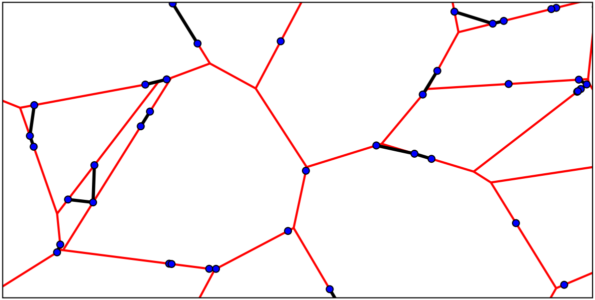


Figure: Part of an SINR graph of a Cox point process with Λ being a Poisson–Voronoi tessellation.
All figures with red Voronoi tessellations originate from Alexander Hinsen.

Important properties of directing measures

We will often refer to the following properties of intensity measures Λ of Cox point processes (the precise definitions can be seen on extra slides):

- Stabilization: a strong spatial decorrelation property of Λ.
  - Special case: *b*-dependence, i.e., independence of restrictions of Λ to areas whose distance is larger than b.
- Asymptotic essential connectedness: stabilization + strong local connectivity of the support of the intensity measure.
  - Satisfied by Poisson–Voronoi and Poisson–Delanuay tessellations (which are stabilizing but not b-dependent).

Hirsch, Jahnel, and Cali (2017) showed the following in case of a Gilbert graph ($\gamma = 0$) with fixed connection radius r_{B}.

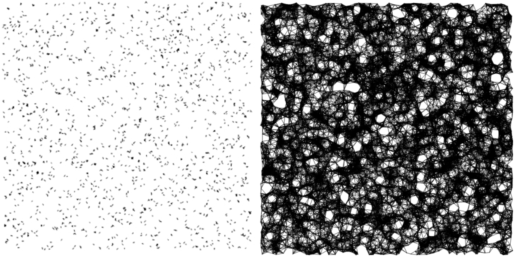
- If Λ is stabilizing, then the Gilbert graph does not percolate if $\lambda > 0$ is sufficiently small.
- If Λ is asymptotically essentially connected, then the Gilbert graph percolates for $\lambda > 0$ sufficiently large with positive probability (and thus with probability 1, since stabilization implies ergodicity).

The case of only stabilizing intensity

Let us assume that Λ is merely stabilizing (i.e., it has nice decorrelation properties but it is possibly very disconnected).

- Small connection radius r_{B}: it can happen that the Gilbert graph does not percolate for any $\lambda > 0$.
- Large r_{B}: there is percolation for all large enough $\lambda > 0$.
  This observation comes from T. (2018) and is in turn an easy corollary of some results of Hirsch, Jahnel, and Cali (2017).

Figure: a Cox point process with large λ and a very disconnected stabilizing (and even b-dependent) Λ. The Gilbert graph only percolates for large r_{B}.



Percolation in the Cox–SINR graph with random radii

Theorem (B. Jahnel and T. (2019))

Let $d \geq 2$, $N_0, \tau > 0$, let the distribution of powers P_i have unbounded support, and let Λ be stabilizing. Further, let ℓ satisfy the following assumptions:

(i) *ℓ is continuous and strictly decreasing as long as it does not vanish*, and*

(ii) $\int_0^\infty r^{d-1}\ell(r)\mathrm{d}r < \infty$ 　　* *The strict decrease can slightly be relaxed.*

Then $\exists \lambda > 0$ such that $\gamma^*(\lambda) > 0$ if

(a) *ℓ has unbounded support, Λ is b-dependent, and $\mathbb{E}[\exp(\alpha\Lambda([0,1]^d))] < \infty$ as well as $\mathbb{E}[\exp(\alpha P_i)] < \infty$ holds for some $\alpha > 0$, or if*

(b) *ℓ has bounded support, $\mathbb{E}[P_i] < \infty$, and Λ is asymptotically essentially connected or $\sup \mathrm{supp}(\ell)$ is sufficiently large depending on Λ.*

Explanation of the case (b) of boundedly supported ℓ: need asymptotic ess. connectedness of Λ or large connection radii.

Similarly, for bounded P_i with values in $[0, P_{\max})$, with $\mathbb{P}(P_i > 0) > 0$:
If also $\ell(0) > \frac{\tau N_0}{P_{\max}}$, then (a), (b) still hold for Λ asymptot. essentially connected.
For Λ only stabilizing, need again large connection radii, i.e., $P_{\max}$ and $\sup \mathrm{supp}(\ell)$ both have to be large enough.

Negative results about percolation: degree bounds

However, there exists $\gamma > 0$ such that for any $\lambda > 0$, $\mathbb{P}(g_\gamma(\mathbf{X}^\lambda)$ percolates$) = 0$.

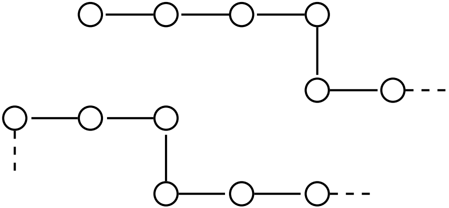
This is due to the degree bounds of SINR graphs with $\gamma > 0$:

Theorem (Dousse, Baccelli, Thiran, 2003)

Let $\gamma > 0$. Then any vertex in $g_\gamma(\mathbf{X}^\lambda)$ has degree less than $1 + \frac{1}{\tau\gamma}$.

Consequences:

1. $\gamma \geq \frac{1}{\tau}$: degree bound is at most 1, no percolation!
2. $\frac{1}{\tau} > \gamma \geq \frac{1}{2\tau}$: degree bound is at most 2, conjecture: also no percolation. If there is an infinite cluster, it must be a path (no cycles, no additional branches), infinite in one or two directions.



Cox point processes: the two-degree case

Example revisited: Λ given by a Poisson–Voronoi tessellation.

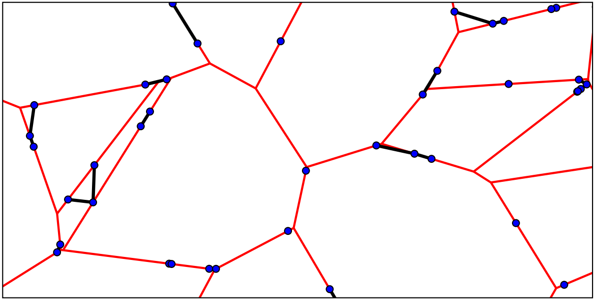


Figure *by Alexander Hinsen*: Part of the SINR graph of a Cox point process with Λ being a Poisson–Voronoi tessellation, degree bound $= 2$.
Many users, but highly disconnected, no cycles, no macroscopic clusters.

No percolation for degree bound = 2

Theorem (B. Jahnel, T., 2019)

For $d \geq 1$, for any stationary and nonequidistant Cox point process, for any distribution of the power variable $P_i \geq 0$, and for $\gamma \geq \frac{1}{2\tau}$ (in which case the degree bound is at most 2),

$$\mathbb{P}\big(g_\gamma(\mathbf{X}^\lambda) \text{ percolates}\big) = 0, \qquad \forall \lambda > 0.$$

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For $d \geq 1$, for any stationary and nonequidistant Cox point process, for any distribution of the power variable $P_i \geq 0$, and for $\gamma \geq \frac{1}{2\tau}$ (in which case the degree bound is at most 2),

$$\mathbb{P}\big(g_\gamma(\mathbf{X}^\lambda) \text{ percolates}\big) = 0, \qquad \forall \lambda > 0.$$

A few words about the proof:

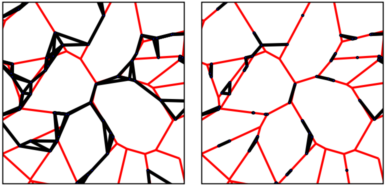
- Exclude infinite paths having an endpoint ($\sim$mass-transport principle).
- Use: in the 2-degree case, if a point $X_i \in X^\lambda$ is connected to two points $X_j, X_k \in X^\lambda$, then $P_j \ell(|X_j - X_i|)$ and $P_k \ell(|X_k - X_i|)$ are the two largest ones among the values $P_l \ell(|X_l - X_i|)$, $l \neq i$.
  $\rightarrow$ A "weighted" 2-nearest neighbour relation!
- Edge-preserving property: if we remove points of X^λ, all edges between remaining points in the SINR graph are preserved.
- Use these together with basic properties of Cox point processes to derive a contradiction assuming that there is an infinite cluster.

Relation to the bidirectional k-nearest neighbour graphs

For constant powers P, if two points X_i, X_j are connected by an edge in the SINR graph, then they are mutually among the k nearest neighbours of each other, where k is the degree bound.

Hence, the SINR graph is a subgraph of the bidirectional k-nearest neighbour (**B**-kNN) graph introduced by Balister and Bollobás (2008).

- High-confidence result by Balister and Bollobás: if X^λ is a stationary PPP in $\mathbb{R}^2$, then the **B**-kNN graph does not percolate for $k \leq 4$.
- From **B**-kNN graphs to SINR graphs: this would imply no percolation in the corresponding SINR graph for $\gamma \geq \frac{1}{4\tau}$.
- Figure: for $k = 5$, the **B**-kNN graph already percolates, but the SINR graph still seems to be very disconnected.



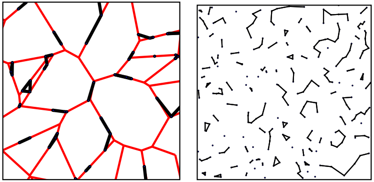
Extension of the non-percolation result to the B-2NN graph

- From SINR graphs to bidirectional 2-nearest neighbour graphs: in turn, almost the exact same proof as for the SINR graphs in the 2-degree case yields non-percolation in the **B**-2NN graph.

Theorem (B. Jahnel, T. (2020))

The **B***-2NN graph of a stationary and nonequidistant Cox point process does not percolate in any dimension.*

- Case of a 2-dim. PPP $\Rightarrow$ partial verification of the high-confidence result.
- Extensions to other kinds of point processes (e.g., Gibbs p.p.) and to generalizations of the **B**-2NN graph are possible.
- The degree bound of 2 and stationarity are essential.
- Figure: **B**-2NN graph of a Cox p.p. based on a Poisson–Voronoi tessellation and the one of a PPP.



Equality of critical densities

Positive results about percolation in SINR graphs are often of the form 'if λ is large enough, then $g_\gamma(\mathbf{X}^\lambda)$ percolates for some $\gamma > 0$'.
Question: does this happen for all λ being supercritical for percolation for $\gamma = 0$?
Let us consider constant powers $P_i \equiv P > 0$, recalling that $r_{\mathrm{B}} = \ell^{-1}(\tau N_0/P)$.
The critical density of the Gilbert graph is defined as follows:

$$\lambda_{\mathrm{c}}(r_{\mathrm{B}}) = \inf \left\{\lambda > 0 \colon \mathbb{P}(g_0(\mathbf{X}^\lambda) \text{ percolates}) > 0\right\}.$$

We call $\lambda < \lambda_{\mathrm{c}}(r_{\mathrm{B}})$ subcritical, $\lambda = \lambda_{\mathrm{c}}(r_{\mathrm{B}})$ critical and $\lambda > \lambda_{\mathrm{c}}(r_{\mathrm{B}})$ supercritical.
Let us further define

$$\lambda_{\tau,N_0,P} = \inf\{\lambda > 0 \colon \exists \gamma > 0 \colon \mathbb{P}(g_\gamma(\mathbf{X}^\lambda) \text{ percolates}) > 0\}.$$

Always true: $\lambda_{\tau,N_0,P} \geq \lambda_{\mathrm{c}}(r_{\mathrm{B}})$ since $\gamma \mapsto g_\gamma(\mathbf{X}^\lambda)$ is decreasing.

Theorem (Dousse, Franceschetti, Macris, Meester, Thiran (2005))

For the homogeneous Poisson process X^λ with intensity λ in $\mathbb{R}^2$, we have $\lambda_{\tau,N_0,P} = \lambda_{\mathrm{c}}(r_{\mathrm{B}})$.

Equality of critical densities, cont'd

Theorem (Dousse, Franceschetti, Macris, Meester, Thiran (2005))

For the homogeneous Poisson process X^λ with intensity λ in $\mathbb{R}^2$, we have $\lambda_{\tau,N_0,P} = \lambda_c(r_B)$.

This relies on Russo–Seymour–Welsh type arguments for the underlying Poisson–Boolean model: any supercritical Boolean model crosses $n \times 3n$ rectangles in the hard direction with high probability as $n \to \infty$.
Question: what can one do in higher dimensions?

Idea (Penrose and Pisztora (1996)): even for $d \geq 3$, any supercritical Poisson–Boolean model satisfies a certain asymptotic essential connectedness property.
This together with scale invariance of the Boolean model yields a suitable connectivity argument for percolation in the SINR graph for all $\lambda > \lambda_c(r_B)$.

Theorem (B. Jahnel, T. (2019))

For the homogeneous Poisson process X^λ with intensity λ in $\mathbb{R}^d$, $d \geq 2$, we have $\lambda_{\tau,N_0,P} = \lambda_c(r_B)$.

Thank you for your attention!

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Stabilization and asymptotic essential connectedness I.

For $n \geq 1$ and $x \in \mathbb{R}^d$, let us write $Q_n(x) = x + [-n/2, n/2]^d$ and $Q_n = Q_n(o)$.

Definition (Hirsch–Jahnel–Cali (2017))

Λ is *stabilizing* if $\exists$ a random field $R = (R_x)_{x \in \mathbb{R}^d}$ of *stabilization radii* such that

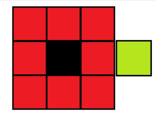
(1) (Λ, R) are jointly stationary,

(2) $\lim_{n \to \infty} \mathbb{P}(\sup_{x \in Q_n \cap \mathbb{Q}^d} R_x < n) = 1$,

(3) for all $n \geq 1$, for any $f \colon \mathbb{M} \to [0, \infty)$ measurable and for any $\varphi \subseteq \mathbb{R}^d$ finite with $\mathrm{dist}(x, \varphi \setminus \{x\}) > 3n$, the random variables

$$\Big\{ f(\Lambda|_{Q_n(x)}) \mathbb{1}\Big\{ \sup_{y \in Q_n(x) \cap \mathbb{Q}^d} R_y < n \Big\} \Big\}_{x \in \varphi}$$

are independent. Here $\mathbb{M}$ is the set of all Borel measures on $\mathbb{R}^d$.



Stabilization and asymptotic essential connectedness II.

For a (possibly singular) Borel measure ν on $\mathbb{R}^d$, we define its support as

$$\mathrm{supp}(\nu) = \{x \in \mathbb{R}^d : \nu(Q_\varepsilon(x)) > 0, \forall \varepsilon > 0\}.$$

Definition (Asymptotic essential connectedness (HJC17))

The stabilizing random measure Λ with stabilization radii R is *asymptotically essentially connected* if for all $n \geq 1$, whenever $\sup_{x \in Q_n \cap \mathbb{Q}^d} R_x < n/2$, we have that

1. $\mathrm{supp}(\Lambda|_{Q_n})$ contains a connected component of diameter at least $n/3$,
2. any two connected components of $\mathrm{supp}(\Lambda|_{Q_n})$ of diameter at least $n/9$ are contained in the same connected component of $\mathrm{supp}(\Lambda|_{Q_{2n}})$.