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On hyperuniformity and rigidity of point processes

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joint work with
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1. Point processes

Setting

$(\Omega, \mathcal{A}, \mathbb{P})$ is a fixed probability space.

Definition

A point process on $\mathbb{R}^d$ is a random element Φ in the space **N** of all locally finite subsets of $\mathbb{R}^d$ equipped with a suitable σ-field. Write $\Phi(B)$ for the (random) number of points of Φ in a Borel set $B \subset \mathbb{R}^d$.

Definition

A point process Φ on $\mathbb{R}^d$ is stationary if $\Phi + x \stackrel{d}{=} \Phi$ for all $x \in \mathbb{R}^d$ and ergodic if $\mathbb{P}(\Phi \in A) \in \{0, 1\}$ for all translation invariant measurable $A \subset \mathbf{N}$. The intensity of a stationary point process is defined by $\gamma_\Phi := \mathbb{E}[\Phi([0, 1]^d)]$.

Remark

If Φ is a stationary point process then

$$\mathbb{E}[\Phi(B)] = \gamma_\Phi \lambda_d(B), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where λ_d denotes Lebesgue measure on $\mathbb{R}^d$.

Definition

Let Φ be a point process on $\mathbb{R}^d$. The intensity function ρ_1 of Φ is measurable function $\rho_1 : \mathbb{R}^d \to [0, \infty)$ satisfying

$$\mathbb{E}[\Phi(dx)] = \rho_1(x)dx.$$

The correlation function of a point process Φ is a measurable function $\rho_2 : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ satisfying the heuristic equation

$$\rho_2(x, y)dxdy = \mathbb{E}[\Phi(dx)\Phi(dy)], \quad x \neq y.$$

If Φ is stationary, then ρ_1 is the intensity of Φ and $\rho_2(x, y) \equiv \rho_2(y - x)$. The function

$$g(x) := \gamma_\Phi^{-2}\rho_2(x), \quad x \in \mathbb{R}^d,$$

is the pair correlation function of Φ.

Remark

Assume that Φ is a locally square-integrable point process with pair correlation function g. Let $W \subset \mathbb{R}^d$ be bounded and measurable. Then

$$\mathbb{V}\mathrm{ar}[\Phi(W)] = \gamma_\Phi^2 \int \lambda_d(W \cap (W + x))(g(x) - 1)\, dx + \gamma \lambda_d(W).$$

Definition

The structure function of a stationary point process Φ with pair correlation function g is defined by

$$S(k) := 1 + \gamma_\Phi \int (g(y) - 1)e^{-ikx}\, dx, \quad k \in \mathbb{R}^d.$$

Theorem

Assume that Φ is a stationary point process with a pair correlation function g such that $g - 1$ is integrable. Then

$$\gamma_\Phi S(k) = \lim_{r\to\infty} \lambda_d(rW)^{-1} \left(\mathbb{E}\Big| \sum_{x\in\Phi\cap rW} e^{-ikx} \Big|^2 - \Big| \mathbb{E} \sum_{x\in\Phi\cap rW} e^{-ikx} \Big|^2 \right).$$

2. Hyperuniformity

Definition

A locally square-integrable point process Φ is said to be hyperuniform if

$$\lim_{r\to\infty} \frac{\mathbb{V}\mathrm{ar}[\Phi(rW)]}{\lambda_d(rW)} = 0,$$

for each convex and compact set with $\lambda_d(W) > 0$.

Remark

The local behavior of a hyperuniform point process can very much resemble that of a weakly correlated point process. Only on a global scale a regular geometric pattern might become visible.

Examples

- Perturbed lattices (Gacs and Szaz '75).
- Ginibre ensembles (Ghosh and Lebovitz '17)
- Coulomb gas in $d = 1, 2$ and in $d \geq 3$ (Chatterjee '17).
- Hyperuniformity in condensed matter physics and materials science: Torquato '18 (survey).

Theorem

Assume that Φ is a stationary point process with a pair correlation function g such that $g - 1$ is integrable. Then Φ is hyperuniform iff

$$1 + \gamma_\Phi \int (g(x) - 1)\, dx = 0.$$

3. Number rigidity

Definition

A point process Φ on $\mathbb{R}^d$ is called number rigid if for each bounded Borel set $B \subset \mathbb{R}^d$ the random number $\Phi(B)$ is almost surely determined by $\Phi \cap B^c$.

Examples

- Gaussian perturbed lattices with sufficiently small variance (Peres and Sly '14).
- Ginibre ensembles (Ghosh and Peres '16).
- Zeros of random analytic functions (Ghosh and Peres '16)
- Hyperuniform point processes for $d = 1, 2$ with $g(r) \leq cr^{-2d-\varepsilon}$ for $r \geq 1$ (Ghosh and Lebovitz '17).
- Sine-Beta Gibbs processes (Dereudre et al. '18).

Theorem (Gosh, Lebowitz '16)

Assume that Φ is a stationary (or $\mathbb{Z}^d$-stationary) hyperuniform point process on $\mathbb{R}$ or $\mathbb{R}^2$. Assume also that Φ has a pair-correlation function g such that there exist $c, \varepsilon > 0$ with

$$|g(x,y) - 1| \le c(\|x-y\| + 1)^{-1}$$

for $d = 1$ or

$$|g(x,y) - 1| \le c(\|x-y\| + 1)^{-4+\varepsilon}$$

for $d = 2$. Then Φ is number rigid.

Question

Are hyperuniform point processes always number rigid? No!

Theorem (Peres, Sly '16)

Consider an i.i.d. perturbed lattice Φ in $\mathbb{R}^d$, where the perturbations follow a centred normal distribution with variance σ^2. Then there exists $\sigma_r > 0$ such that Φ is number rigid for $\sigma < \sigma_r$ and not number rigid for $\sigma > \sigma_r$.

4. Maximal rigidity

Definition

A point process Φ is said to be maximally rigid if for each bounded Borel set $B \subset \mathbb{R}^d$, the point process Φ_B is almost surely a function of Φ_{B^c}.

Definition

A point process Φ is called generalized stealthy if its structure function vanishes in a non-empty open set.

Theorem (Ghosh and Lebowitz '18)

A generalized stealthy point process is maximally rigid.

5. Stable matchings

Setting

φ, ψ are locally finite subsets of $\mathbb{R}^d$.

Definition

A (partial) matching of (φ, ψ) is a mapping $\tau\colon \varphi \cup \{\infty\} \to \psi \cup \{\infty\}$ such that τ is injective on $\varphi \cap \{\tau < \infty\}$. (If $\tau(x) = \infty$ then x has no matching partner.) The (suitably defined) inverse mapping from ψ to $\varphi \cup \{\infty\}$ is also denoted by τ.

Definition

Let τ be a matching of (φ, ψ). A pair $(p, x) \in \varphi \times \psi$ is called unstable if

$$|p - x| < \min\{|p - \tau(p)|, |x - \tau(x)|\}.$$

A matching is called stable if there is no unstable pair.

Remark

Stable matchings were introduced by Holroyd, Pemantle, Peres and Schramm (2009). More general version were studied by Gale and Shapley (1962).

Definition

We call (φ, ψ) non-equidistant if there do not exist $p, q \in \varphi$ and $x, y \in \psi$ with $\{p, x\} \neq \{q, y\}$ and $\|x - p\| = \|y - q\|$.

Definition

A sequence (z_n) of points in $\mathbb{R}^d$ is called an infinite descending chain in (φ, ψ) if $z_i \in \varphi$ for odd $i \in \mathbb{N}$, $z_i \in \psi$ for even $i \in \mathbb{N}$ and $\|z_{i+1} - z_i\| < \|z_i - z_{i-1}\|$ for each $i \in \mathbb{N}$ with $i \geq 2$.

Theorem (Holroyd, Pemantle, Peres, Schramm '09)

Assume that (φ, ψ) is non-equidistant and that there is no infinite descending chain in (φ, ψ). Then there is a unique stable matching τ of (φ, ψ). Moreover, we either have $\{p \in \varphi : \tau(p) = \infty\} = \emptyset$ or $\{x \in \psi : \tau(x) = \infty\} = \emptyset$.

Remark

The stable matching can be constructed recursively, using the mutual nearest neighbour matching.

6. Stable matchings between point processes

Setting

Φ and Ψ are point processes on $\mathbb{R}^d$ such that (Φ, Ψ) is almost surely non-equidistant and has no infinite descending chain. Let τ denote the stable matching of (Φ, Ψ) and define

$$\Phi^\tau := \{p \in \Phi : \tau(p) \neq \infty\},$$
$$\Psi^\tau := \{x \in \Psi : \tau(x) \neq \infty\}.$$

These are the processes of matched points from Φ and Ψ, respectively.

Theorem

Assume that Φ *and* Ψ *are jointly stationary and ergodic with intensities 1 and* $\alpha \geq 1$, *respectively. Then* $\mathbb{P}(\Phi^\tau = \Phi) = 1$. *If* $\alpha = 1$, *then also* $\mathbb{P}(\Psi^\tau = \Psi) = 1$.

Theorem

Assume that Ψ *is a stationary point process with intensity* $\alpha \geq 1$, *ergodic under translations from* $\mathbb{Z}^d$. *Let* $\Phi := \mathbb{Z}^d + U$, *where* U *is a* $[0, 1)^d$*-valued random variable, independent of* Ψ. *Then the assertions of the preceding theorem hold.*

Theorem (Klatt, L. and Yogeshwaran ’20)

Assume that $\Phi = \mathbb{Z}^d$ and that Ψ is a stationary Poisson process with intensity $\alpha > 1$. Then Ψ^τ is hyperuniform.

Remark

The theorem remains true for a determinantal point process Ψ with a sufficiently fast decaying kernel.

Remark

We conjecture the theorem to be true if Φ is a randomized lattice or even a more general hyperuniform process.

Strategy of the proof

- There exists $c_1 > 0$ such that

$$\mathbb{P}(\|\tau(\mathbb{Z}^d, \Psi, 0)\| > r) \le c_1 e^{-c_1^{-1} r^d}, \quad r \ge 0.$$

  This part of the proof was inspired by Hoffman, Holroyd and Peres (2009).
- Construction of a matching flower $F(\Psi, q)$, which is a stopping set that determines the matching partner of $q \in \mathbb{Z}^d$.
- Control the size of the matching flower.
- Write the variance of $\Psi^\tau(rW)$ as a double series over $p, q \in \mathbb{Z}^d$ and apply dominated convergence in a suitable way.

Theorem (Klatt, L. and Yogeshwaran ’20)

Assume that $\Phi = \mathbb{Z}^d$ and that Ψ is a stationary Poisson process with intensity $\alpha > 1$. Then Ψ^τ is number rigid.

7. Poisson hyperplane intersection processes

Question

Is it true (as it has been common belief) that for $d \geq 2$ a rigid (and ergodic) point process must be hyperuniform?

Setting

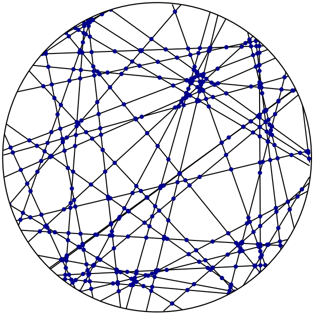
η is a stationary Poisson process on the space $\mathbb{H}^{d-1}$ of all hyperplanes in $\mathbb{R}^d$ with intensity measure

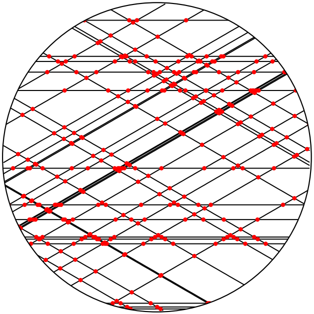
$$\lambda = \gamma \iint \mathbf{1}\{H_{u,s} \in \cdot\}\, ds\, \mathbb{Q}(du),$$

where $\gamma > 0$ is an intensity parameter and the directional distribution $\mathbb{Q}$ is an even probability measure on the unit sphere. Here,

$$H_{u,s} := \{y \in \mathbb{R}^d : \langle y, u \rangle = s\}.$$

We assume that $\mathbb{Q}$ is not concentrated on a great subsphere.





Definition

The intersection process (associated with η) is the point process Φ of all points $x \in \mathbb{R}^d$ satisfying

$$\{x\} = H_1 \cap \cdots \cap H_d$$

for some $H_1, \ldots, H_d \in \Phi$.

Theorem (Heinrich, Schmidt and Schmidt '06)

There exists a finite $c > 0$ such that

$$\lim_{r\to\infty} r^{-(2d-1)}\,\mathbb{V}\mathrm{ar}[\Phi(rW)] = c.$$

Remark

For $d \geq 2$ the intersection process Φ is hyperfluctuating. Moreover, this process is mixing and has (in the isotropic case) a polynomially decaying pair-correlation function.

Theorem

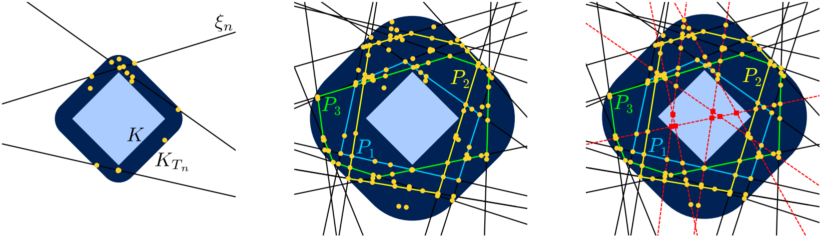
Let $K \subset \mathbb{R}^d$ be convex and compact. Then there exists a Φ-stopping set Z with $Z \subset K^c$ and such that the hyperplanes intersecting K (and in particular Φ_K) are almost surely determined by $\Phi \cap Z$. Moreover, there exist constants $c_1, c_2 > 0$ such that

$$\mathbb{P}(R(Z) > s) \leq c_1 e^{-c_2 s}, \quad s \geq 1,$$

where $R(Z)$ is the radius of the smallest ball centred at the origin and containing Z.

Idea of the proof

To determine a hyperplane from η we need $2d - 1$ intersection points. (In general this number cannot be reduced.)



$n = 16$ $\quad$ $n = 112$ (i) $\quad$ $n = 112$ (ii)

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