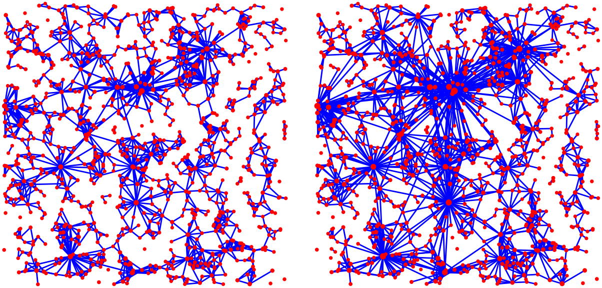
Spreading processes on spatial contact networks

Júlia Komjáthy

joint w: Bas Lodewijks; John Lapinskas, Johannes Lengler, Ulysse Shaller

November 2, 2020



Spreading processes on spatial networks

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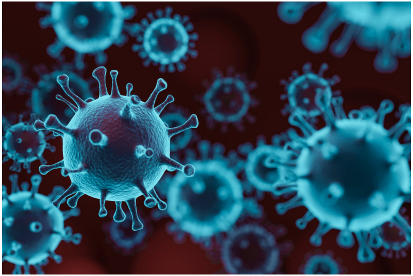

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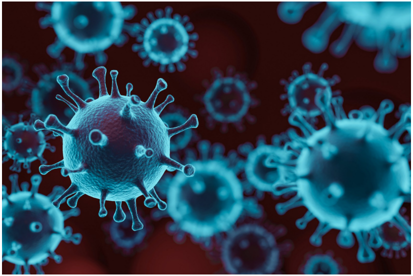

Properties of real spatial contact networks

Most large real networks are *statistically very similar.* If you understand one, you understand others.

[Adapted from *Network Science* (2015) by Albert László Barabási]

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- **Hierarchical**: most nodes are connected to at least one node with more connections than they have themselves

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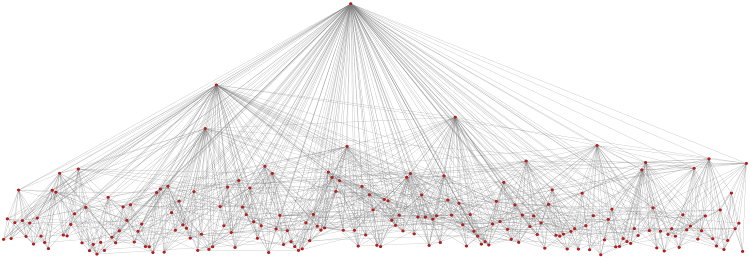
History of spreading models

- 1750's – now: Differential equations: versatile but only complex models can include geometry / individuals



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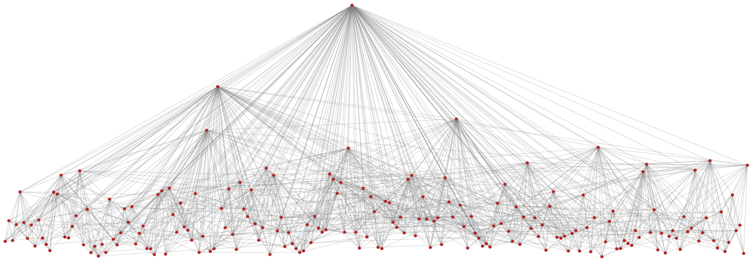

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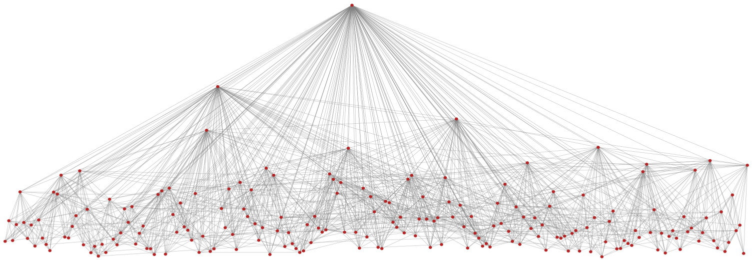

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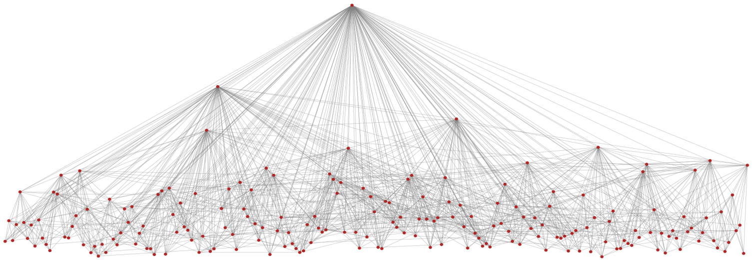

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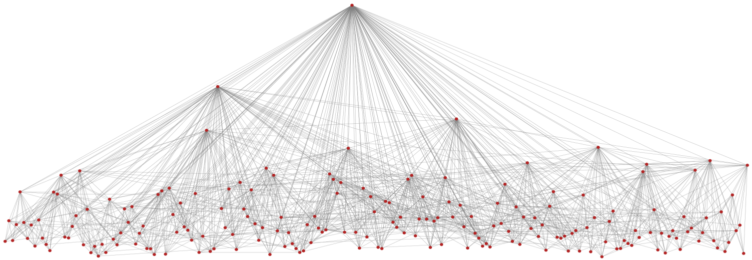
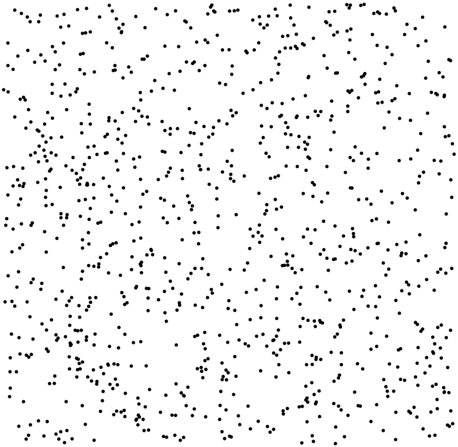
- 1960's – now: Random graphs: simple models, but no geometry, clustering
- 2010's – now: Hyperbolic random graphs and spatial scale-free network models: simple models, all desired properties



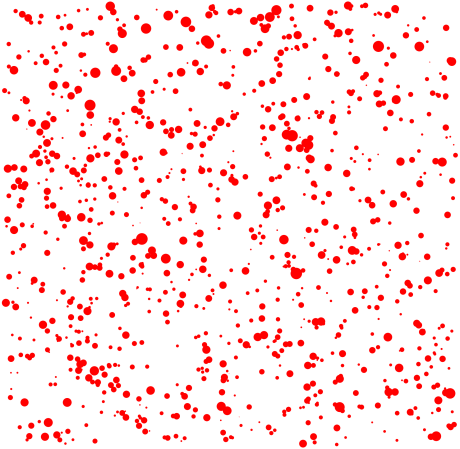
Spatial Scale-free Network Models

Ingredient 1: point process for the location of nodes



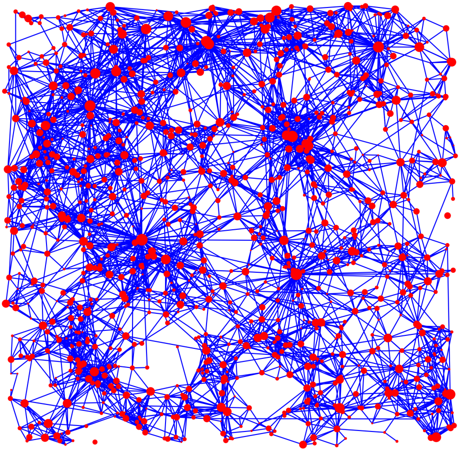
Spatial Scale-free Network Models

Ingredient 2: i.i.d. fitnesses for nodes, e.g. fat tailed, $\mathbb{P}(W > x) \asymp x^{1-\tau}$



Spatial Scale-free Network Models

Ingredient 3: random connections between nodes
probability increasing with fitness and decaying with distance.



Spatial, clustered, long-range models

Hyperbolic geometric graphs (by Krioukov *et al.* '10)

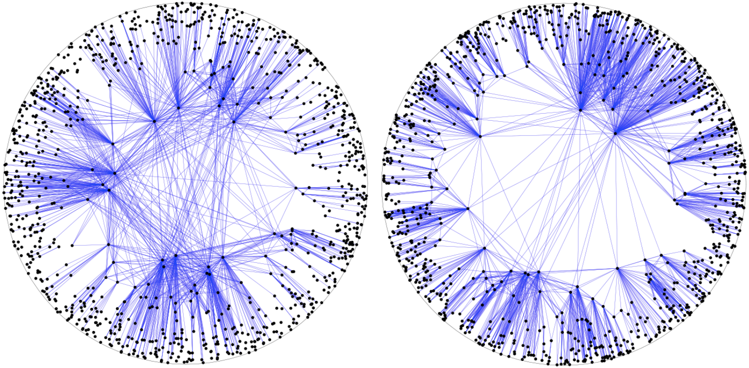


Figure: Hyperbolic random graph simulations by Tobias Müller

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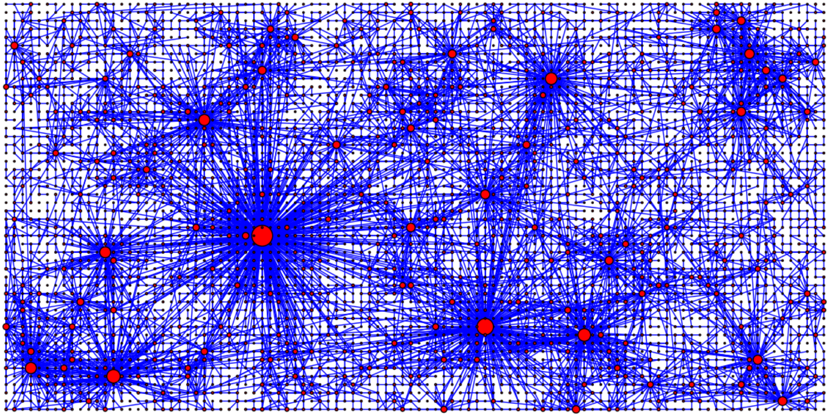

Figure: Scale-free percolation, by Joost Jorritsma

Spatial, clustered, long-range models

Geometric inhomogeneous random graphs

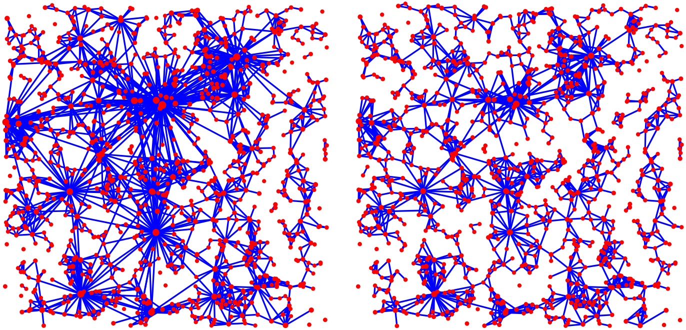

Figure: GIRG simulation by Joost Jorritsma

Infinite Geometric Inhomogeneous Random Graphs: IGIRG

- d = dimension
- Vertices: a homogeneous Poisson Point Process $\mathcal{V}$ on $\mathbb{R}^d$
- Vertex-fitnesses: *iid fitness* W_v to each vertex $v \in \mathcal{V}$
- Edges: Connect $u, v \in \mathcal{V}$ *conditionally independently* w/p

$$\mathbb{P}(u \leftrightarrow v | W_u, W_v) := h(u, v, W_u, W_v),$$

where $h : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to [0, 1]$ measurable.

Infinite Geometric Inhomogeneous random graphs 2.

Choice of parameters:

- Fitnesses: W_v power law with exponent $\tau > 1$:

$$\mathbb{P}(W \geq x) \asymp 1/x^{\tau-1}$$

(slowly varying correction term is allowed)

- Edges: Connection probability satisfies

$$h\big(u, v, W_u, W_v\big) = \Theta\Big(\min \Big\{ 1, \Big(\tfrac{W_u W_v}{\|u-v\|^d} \Big)^{\alpha} \Big\} \Big),$$

- Threshold GIRG: Connection probability satisfies

$$h\big(u, v, W_u, W_v\big) = \mathbb{1}\{ \|u - v\|^d \leq \Theta(W_u W_v) \}.$$

History of the models:
vertex set $\mathbb{Z}^d$: Scale-free percolation; Deijfen, v/d Hofstad, Hooghiemstra '13;
vertex set *PPP* on $\mathbb{R}^d$: Deprez, Hazra, Wüttrich, '15
threshold h: Hyperbolic random graphs, Krioukov, et al
n vertices in $[0, 1]^d$: Bringmann, Keusch, Lengler '15
general connection prob: Lodewijks & K '19+

Some properties of GIRGs/ IGIRGs

Theorem (BKL‘17, BKL‘16)

Let $\alpha > 1$. Fitness distribution W power law with $\tau > 2 \Rightarrow$ degree distribution power law with $\tau > 2$.

Theorem (DHH’13)

If $\alpha \leq 1$ or $\tau < 2$, each vertex has infinite degree.

Theorem (Bhattacharjee, Schulte ‘19)

The Hill’s estimator is consistent for these models.

Some properties of GIRGs/ IGIRGs

Theorem (DHH’13, BKL‘17, KLL‘19+)

Let $\alpha > 1, \tau \in (2, 3)$. Then there is a unique infinite component.

For $\tau > 3$, there is a unique infinite component above a threshold edge-density.

For finite versions, there is a unique linear sized giant-component.

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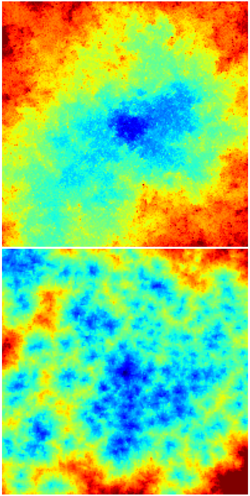
For finite versions, there is a unique linear sized giant-component.
Average distance within a Volume N box:

$$\overline{\mathrm{Dist}}(N) = \frac{1}{\binom{N}{2}} \sum_{u,v} d_G(u,v) = \begin{cases} \Theta(\log\log N) & \textit{when } \tau \in (2,3), \alpha > 1 \\ \Theta\big((\log N)^{\zeta}\big) & \textit{when } \tau > 3, \alpha \in (1,2) \\ \Theta(\sqrt{N}) & \textit{when } \tau > 3, \alpha > 2, \end{cases}$$

Spreading processes on networks

Susceptible-Infected model:

- At time $t = 0$ the source node is infected, all other nodes susceptible.
- if, on an edge $\{u, v\}$, u is infected and v is not, then v becomes infected after a random transmission delay $L_{(u,v)}$.



The epidemic curve

The function that counts the total number of infected nodes before time t:

$$I(t) = \#\{\, \textit{infected nodes before time } t \,\}$$

The shape of the epidemic curve

Question

What does the epidemic curve look like for spreading on **real** networks?

Is it typically. . .

The shape of the epidemic curve

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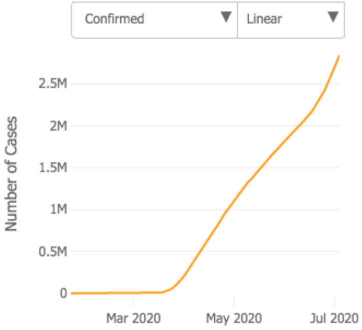
Answer

Yes.

Qualitatively. Quantifying these statistically is very difficult.

Epidemic curves in real life



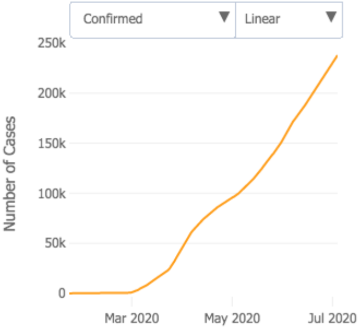


Figure: Covid-19 epidemic curves: US (left), Iran (right). Source: Johns Hopkins University Corona Dashboard

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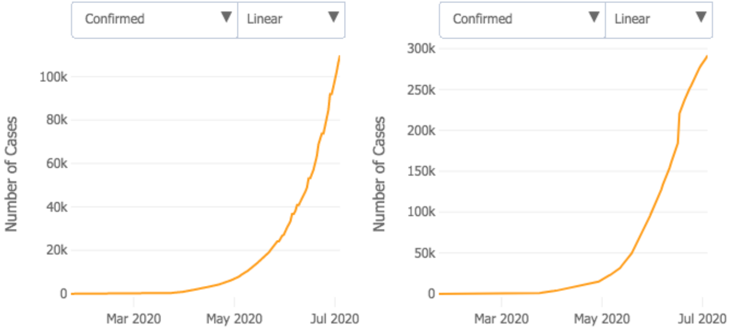
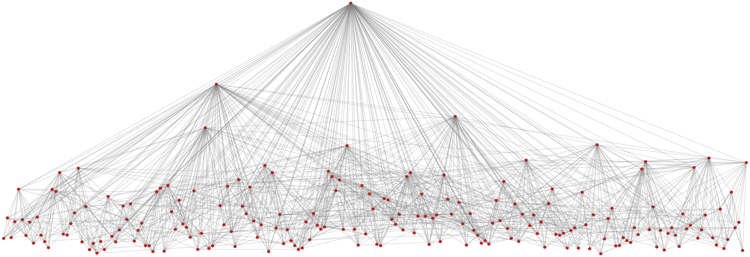


Figure: Covid-19 epidemic curves: Colombia (left), Chile (right). Source: Johns Hopkins University Corona Dashboard

The epidemic curve on SSNMs

Distance \ Fitnesses	**fat-tailed**	**light-tailed**
weak decay		
strong decay		



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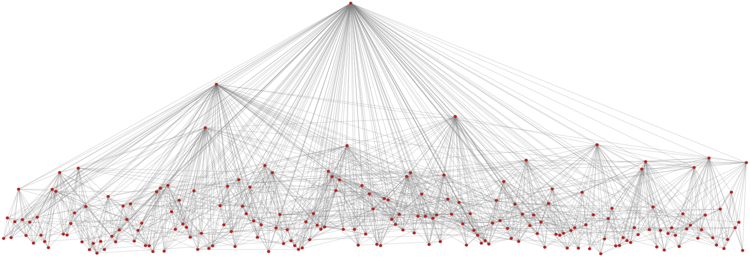

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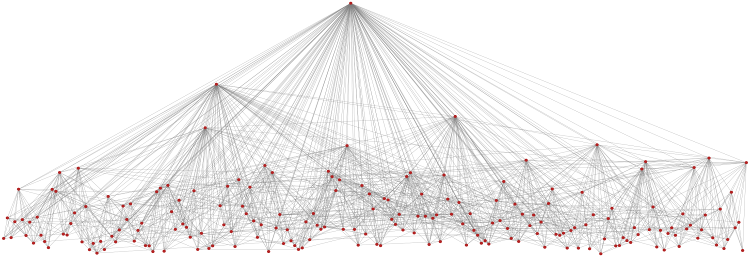

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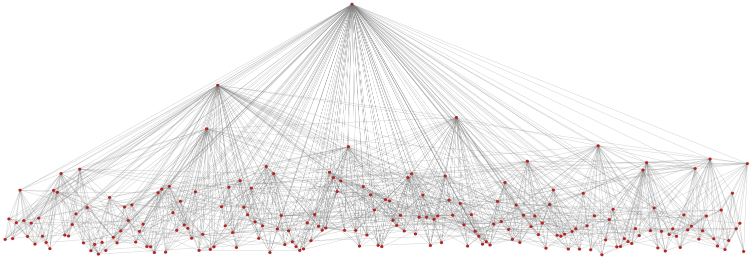

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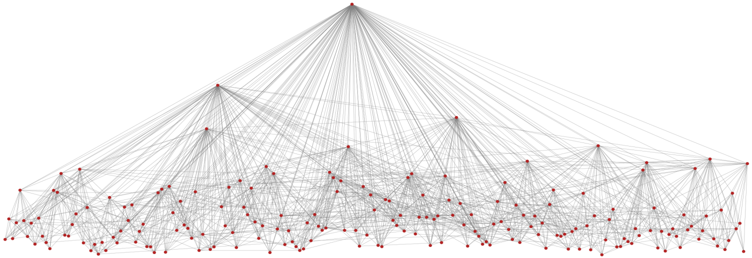
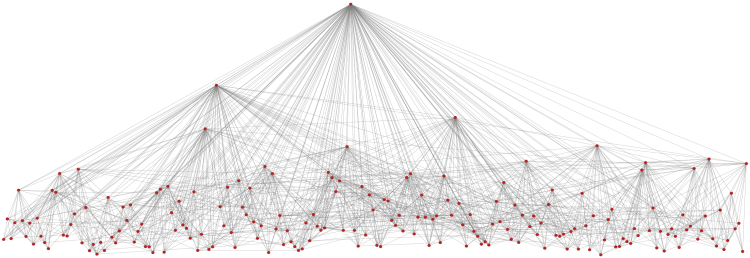


The epidemic curve on SSNMs

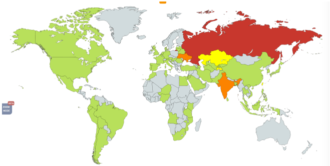
Distance \ Fitnesses	**fat-tailed** $\tau \in (2,3)$	**light-tailed** $\tau > 3$
weak decay $\alpha \in [1,2)$	doubly-exponential or explosive	(stretched) exponential
strong decay $\alpha \in [2,\infty]$	doubly-exponential or explosive	linear/polynomial



What is explosion?

On infinite networks

A spreading process is explosive on an infinite network if $I(t) = \infty$ for some $t < \infty$.



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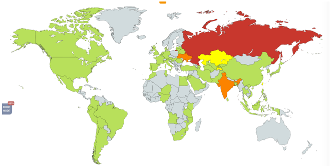

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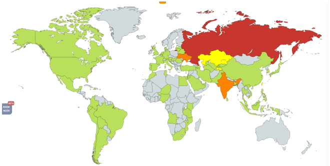

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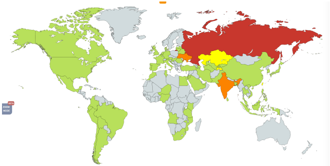

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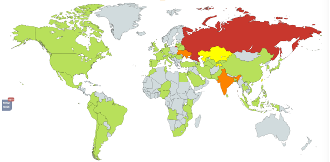

Explosion on scale-free spatial networks

Recall: infection uses i.i.d. transmission times $L_{u,v}$ on edges.

Theorem (Komjáthy, Lodewijks, *SPA,* 2020)

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Finite model: $\frac{I_N(t)}{N} \xrightarrow{d} f(t - E_1)$, *where* $f(t) = \mathbb{P}(E_2 < t)$, *and* E_1 *is a random shift.*

Explosion on scale-free spatial networks

Recall: infection uses i.i.d. transmission times $L_{u,v}$ on edges.

Theorem (Komjáthy, Lodewijks, *SPA*, 2020)

Explosion happens for SI on spatial scale-free network models when:

- $\tau \in (1, 2]$*: (but trivially and not realistic), or*
- $\tau \in (2, 3)$, *if and only if*

$$\sum_{k=1}^{\infty} F_L^{(-1)}\left(1/e^{e^k}\right) < \infty,$$

where F_L is the cumulative distribution function of delays L.

- $\tau > 3$ *explosion never happens.*

Moreover, if we have explosion, we can characterise the epidemic curve.

Finite model: $\frac{I_N(t)}{N} \xrightarrow{d} f(t - E_1)$, where $f(t) = \mathbb{P}(E_2 < t)$, and E_1 is a random shift.

Comment: Explosion insensitive to τ, as long as $\tau \in (2, 3)$. All polynomial F_L's are explosive.

Degree-dependent Susceptible-Infected models

Observation

Disease spreading, real-world communication: Large-degree nodes have a limited “time-budget” to meet and infect.

Miritello *et. al.* ‘13, Feldman Janssen ‘17, Giuraniuc *et al.* ‘16, Karsai *et. al.* ‘11

Model: Degree-penalised transmission delays

- Transmission delay through an edge:

$$T_{(u,v)} = L_{(u,v)} \cdot f(\deg(u), \deg(v), \|u - v\|)$$

- Random component: i.i.d. random variables $L_{(u,v)} \geq 0$
- Budget factor: $f(\deg(u), \deg(v), \|u - v\|)$ depends on the degrees and spatial distance

Result: Explosion with degree-penalties

Is explosion still possible with penalty factors?

Theorem (Komjáthy, Lapinskas, Lengler (2020+))

$$F_L(t) \geq t^{\beta} \quad on \quad [0, t_0].$$

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- *Yes, when* $\tau < 3$*, and*

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- *Yes, when* $\tau < 3$*, and*
- $f = poly(\mathrm{deg}(u), \mathrm{deg}(v))$ *is a polynomial: Explosive* *if and only if* *for some* $\beta < \beta_c = (3-\tau)/\,\mathrm{deg(poly)}$,

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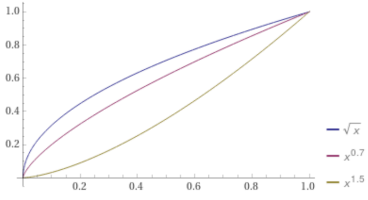
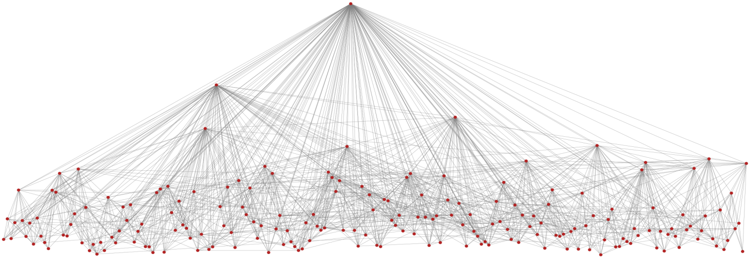
Otherwise, the model is not explosive.

Explosion with penalties requires a steep polynomial increase of L at 0.
Compare: without penalty factor, much easier, many sub-polynomials are explosive.



The epidemic curve on SSNMs

Distance \ Fitnesses	**fat-tailed**	**light-tailed**
weak decay		
strong decay		



The epidemic curve on SSNMs

Distance \ Fitnesses	**fat-tailed**	**light-tailed**
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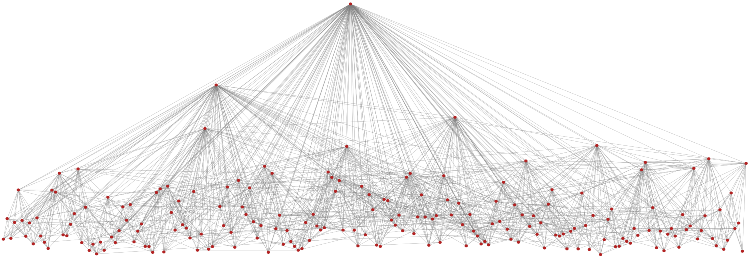
strong decay		linear/polynomial (grid-like)



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weak decay		(stretched) exponential
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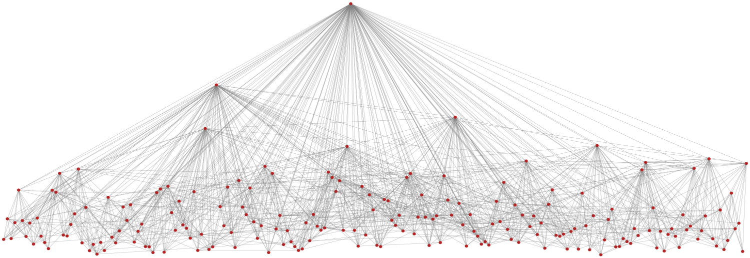

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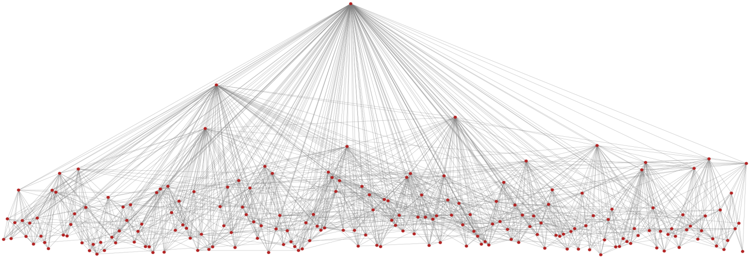

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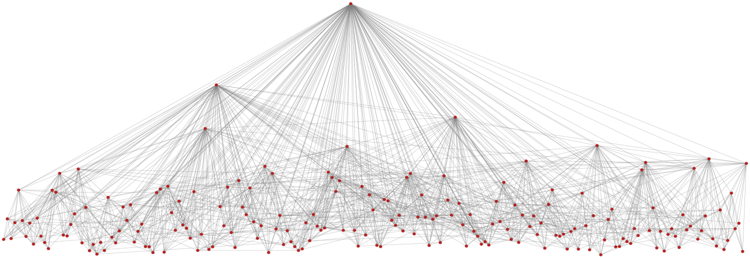
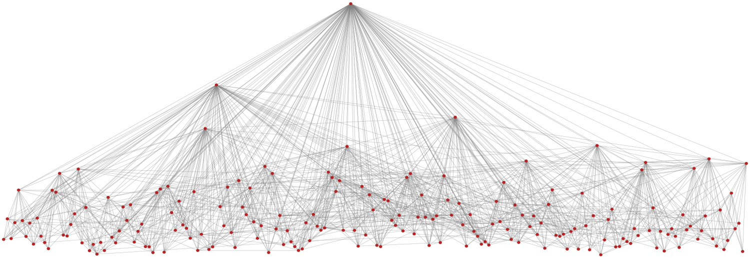


The epidemic curve on SSNMs

Distance \ Fitnesses	**fat-tailed** $\tau \in (2, 3)$	**light-tailed** $\tau > 3$
weak decay $\alpha \in [1, 2)$	doubly-exponential or explosive	(stretched) exponential
strong decay $\alpha \in [2, \infty]$	doubly-exponential or explosive	linear/polynomial (grid-like)



Current and future work

The epidemic curve with degree penalties

Penalty & Decay \ Fitnesses	**fat-tailed** $\tau \in (2,3)$
small	doubly exponential or explosive
medium	
high	
very high	

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Current and future work

The epidemic curve with degree penalties

Fitnesses / Penalty &Decay	**fat-tailed** $\tau \in (2,3)$
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Current and future work

The epidemic curve with degree penalties

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Current and future work

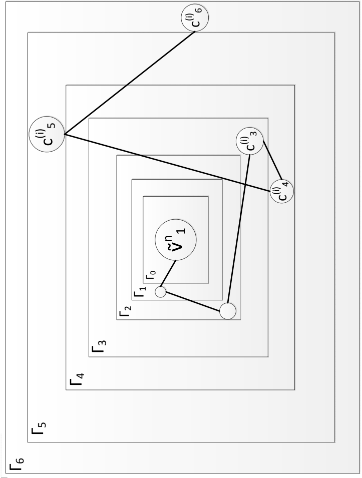
The epidemic curve with degree penalties

Penalty &Decay \ Fitnesses	**fat-tailed** $\tau \in (2, 3)$
small $\deg(f) < (3-\tau)/\beta$	doubly exponential or explosive
medium $\deg(f) < 2(3-\tau)/\beta$ or $\alpha \in (1, 2)$	stretched exponential
high $\deg(f) < \frac{2}{d} + 2(3-\tau)/\beta \vee 2\frac{\alpha-\tau+1}{d(\alpha-2)}$ and $\alpha > 2$	polynomial (faster than grid-like)
very high $\deg(f) > \frac{2}{d} + 2(3-\tau)/\beta \vee 2\frac{\alpha-\tau+1}{d(\alpha-2)}$ and $\alpha > 2$	linear (grid-like)

Proof ideas

Proof of explosion when $\deg f < (3 - \tau)/\beta$

Construction of a greedy path with finite total length



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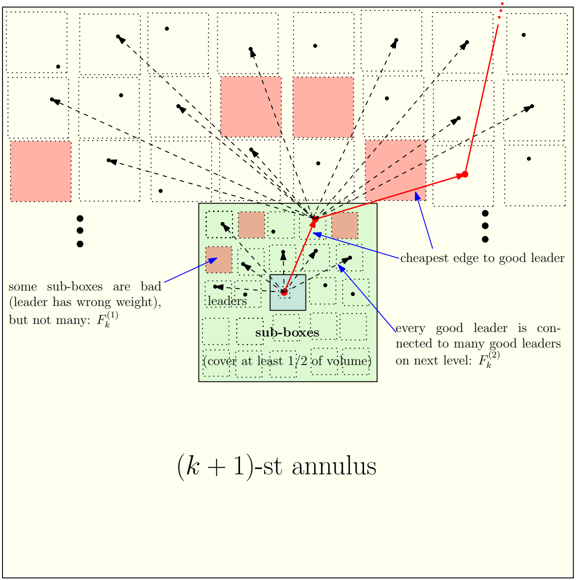

Construction of a greedy path with finite total length

- Let $M, A, B > 1$, $\text{Annulus}(\text{k})_{k \geq 1}$ be consecutive annuli of volume

$$\text{Vol}_k := M^{AB^k}$$

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- $\#\{\text{leader neighbors in Annulus}(k+1) \text{ of a leader}(k)\}$

$$\text{LeaderDeg}(k) = cM^{(A-1)B^{k+1}(1-\varepsilon)}$$

with summable error probability as long as $\frac{1-\delta}{\tau-1}(1+B) \geq AB$.

Construction of a greedy path with finite total length

Greedy path

- Assume $0 \in \mathcal{C}_\infty$ of IGIRG
- From 0, follow a path to leader(0) (its length is some finite random variable $X(\mu, L)$)
- Take the edge *with minimal L* between leader(0) and its leader(1) neighbors.
- continue with this rule

Cost of the greedy path

$$\text{Cost of } \pi_{\text{greedy}} \leq \text{Cost to go to leader of Annulus(0)}$$

$$+ \sum_{k=0}^{\infty} W^{\mu}_{\text{leader}(k)} W^{\mu}_{\text{leader}(k+1)} \cdot \min_{j \leq LeaderDeg(k)} L_{kj}$$

$$W_{\text{leader}(k)} = cM^{B^k \frac{1 \pm \delta}{\tau - 1}}$$

$$\text{LeaderDeg}(k) = cM^{(A-1)B^{k+1}(1-\varepsilon)},$$

$$\min_{j \leq LeaderDeg(k)} L_{kj} \leq F_L^{(-1)}(\xi(k)/LeaderDeg(k))$$

$$F_L^{(-1)}(Z) \leq Z^{1/\beta}$$

Plug everything in, we need that the sum is finite:

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Plug everything in, we need that the sum is finite:

$$\sum_{k=0}^{\infty} M^{B^k \left(\mu(1+B)\frac{1+\delta}{\tau-1} - (A-1)B(1-\varepsilon)/\beta\right)} < \infty$$

Cost of the greedy path

Greedy path has to exist and have finite cost when:

$$\sum_{k=0}^{\infty} M^{B^k\left(\mu(1+B)\frac{1+\delta}{\tau-1}-(A-1)B(1-\varepsilon)/\beta\right)} < \infty$$

Path is present:

$$\frac{1-\delta}{\tau-1}(1+B) \geq AB$$

Finite-cost:

$$\mu(1+B)\frac{1+\delta}{\tau-1} - (A-1)B(1-\varepsilon)/\beta < 0$$

This system of inequalities have a solution for $A, B > 1$ and $\varepsilon, \delta > 0$ if $\tau \in (1, 3)$ and

$$2\mu\beta < 3 - \tau.$$

Greedy path has finite cost. □

Non-explosive regimes

Weighted model point-of-view

- In SI-model, an edge is used precisely once.
- Pre-sample all transmission delays $(C_e)_{e\in\mathcal{E}}$ before the spread starts.
- Infection time $d_C(u,v)$ becomes: weighted distance wrt to the metric:

Infection time = weighted distance

$$d_C(v,u) = \min_{\pi:\ \text{path } v\to u} \Big(\sum_{e\in\pi} C_e \Big)$$

Non-explosive regimes

Non-explosion when $\deg f > (3-\tau)/2\beta$

Tricky (truncated) path counting methods.

Stretched exponential and polynomial growth

See jamboard.

- Upper bounds: Constructing bridges (ala Kleinberg or ala Biskup)
- Lower bounds: Robust renormalisation techniques (ala Berger)

Thank you for the attention!

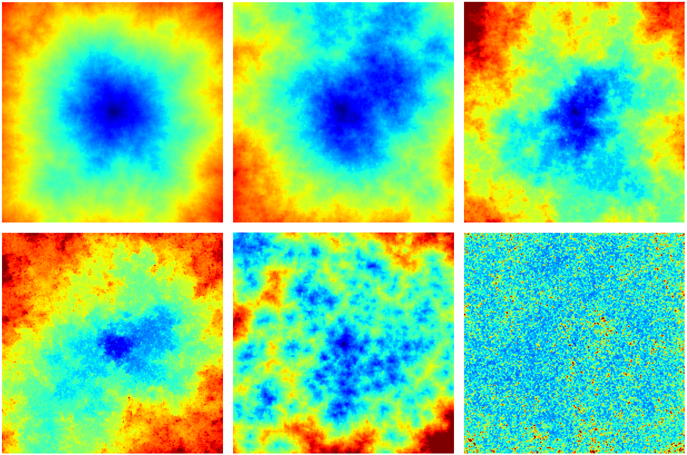

Figure: Six instances of an infection spreading on a two-dimensional SSNM with different parameters τ and α.