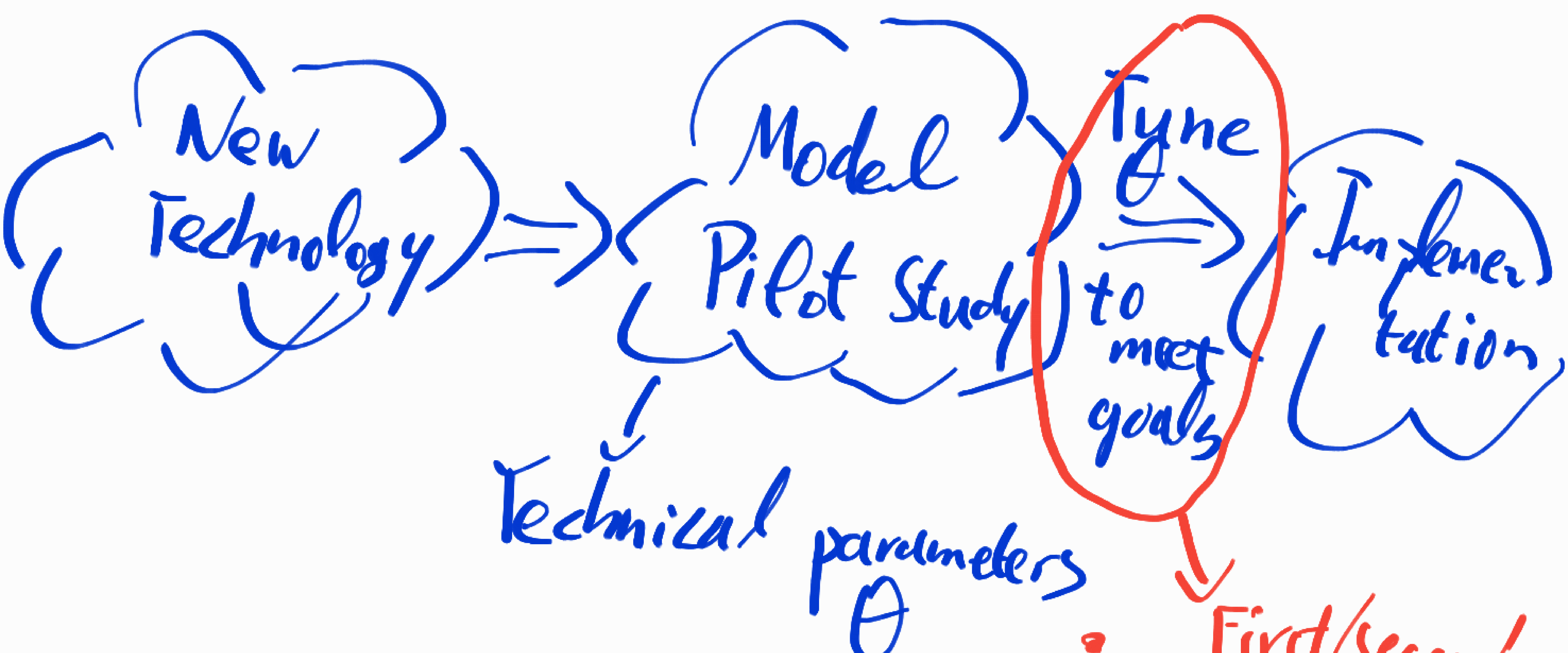
Lecture 1: Large Deviations

I. Background

How to build next-gen. wireless network?

Goal: Design network such that key performance indicators (connectivity; QoS; capacity.) meet targets



First/second-order approx

⇝ Network works well on average

$\boxed{\text{Challenge}}$: Design networks which work with a very high probability

- Can we identify root causes for network failures?

Large Deviation Theory

Information Theory

Chemical Reaction Networks

Statistical Physics

Monte Carlo simulation

...

II. Large deviations 101: Coin tosses

$\boxed{Q}$ Throw coin 10^6 times. What is the probability see at most 250,000 heads?

Let $\{X_i\}_{i\geq 1}$ be iid $\mathrm{Ber}(p)$ and set

$$S_n := X_1 + \dots + X_n$$

Take $s \in (0,1)$

$$\boxed{\mathbb{P}(S_n = ns)} = \binom{n}{ns} p^{ns} (1-p)^{n-ns}$$

$$\frac{n!}{(ns)!\,(n-ns)!} \;\underset{\text{Stirling}}{\sim}\; \frac{\cancel{n^n}}{(\cancel{n}s)^{ns}\,(\cancel{n}-\cancel{n}s)^{n-ns}}\sqrt{\dots}$$

Stirling $k! \sim \left(\frac{k}{e}\right)^k \sqrt{2\pi k}$

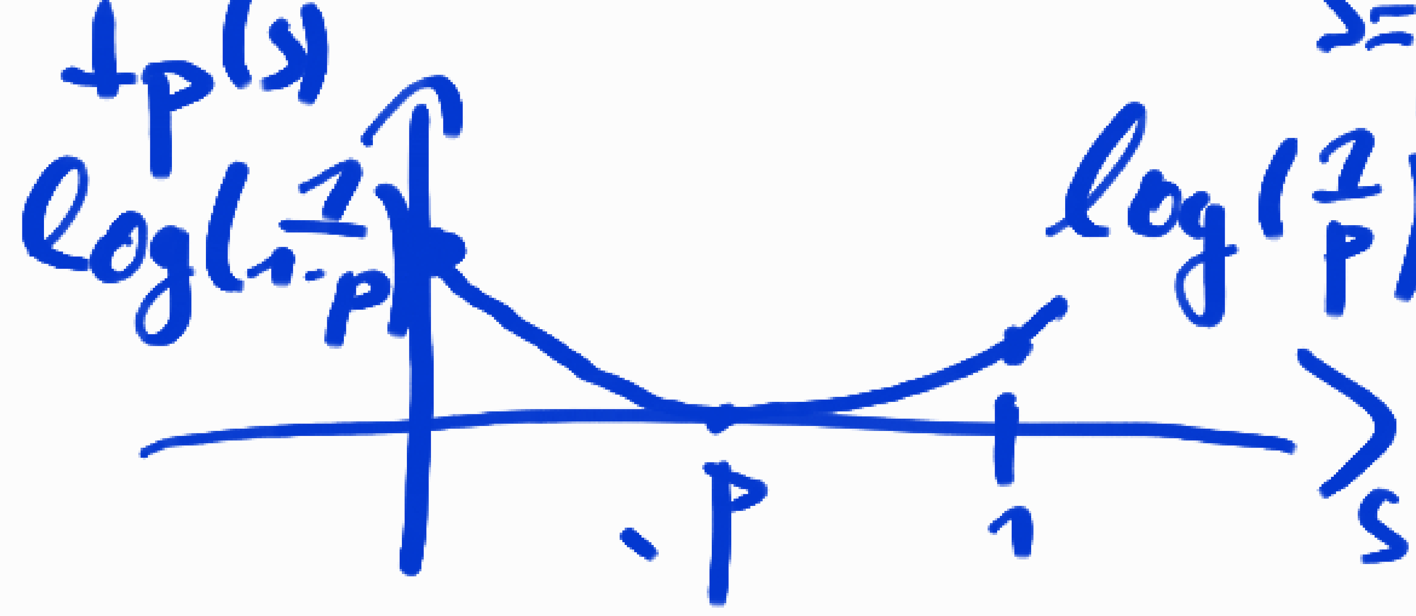
$$\underbrace{\frac{n^n}{(ns)^{ns}(n-ns)^{n-ns}}}_{s^{-ns}(1-s)^{n(1-s)}}$$

$$\sim \underbrace{\left(\frac{p}{s}\right)^{ns}\left(\frac{1-p}{1-s}\right)^{n(1-s)}}_{e^{-n I_p(s)}} \sqrt{\dots}$$

~~where~~ with rate function

$$\boxed{I_p(s) := s\log\left(\frac{s}{p}\right) + (1-s)\log\left(\frac{1-s}{1-p}\right)}$$

$s=p \rightsquigarrow I_p(s)=0$



$$\Rightarrow \boxed{\frac{1}{n}\log \mathbb{P}(S_n = ns) \longrightarrow -I_p(s)}$$

Why $I_p(s)$?

Consider change of measure:
Under $\mathbb{Q}$ ~~where~~ we have $\{X_i\}_{i\geq 1}$ iid Ber(s)

$$\mathbb{P}(S_n = ns) = \mathbb{E}[\mathbb{1}\{S_n = ns\}]$$

$$= \mathbb{E}_{\mathbb{Q}}\Big[\underbrace{\frac{d\mathbb{P}}{d\mathbb{Q}}(X_1,\dots,X_n)}_{\left(\frac{p}{s}\right)^{X_1}\left(\frac{1-p}{1-s}\right)^{1-X_1}\cdots\left(\frac{p}{s}\right)^{X_n}\left(\frac{1-p}{1-s}\right)^{1-X_n}} \mathbb{1}\{S_n = ns\}\Big]$$

$\sum X_i = S_n = ns$

$$= \mathbb{E}_{\mathbb{Q}}\Big[\underbrace{\left(\frac{p}{s}\right)^{ns}\left(\frac{1-p}{1-s}\right)^{n-ns}} \mathbb{1}\{S_n = ns\}\Big]$$

$$= e^{-n I_p(s)}\, \mathbb{Q}(S_n = ns)$$

$\hookrightarrow \mathbb{E}_{\mathbb{Q}}[S_n] = ns$

rel. entropy

$$\rightsquigarrow \boxed{I_p(s) = H(\mathrm{Ber}(s) \,|\, \mathrm{Ber}(p))}$$

Take-Away: $\mathrm{Ber}(p) \rightsquigarrow \mathrm{Ber}(s)$ leastly costly change of measure in which rare event becomes typical

Reduce rare-event analysis to deterministic optimization problem

III. Interference at a base station

(Ganesh / Torrisi, 2008)

Let $\{x_i\}$ be a collection of transmitters in $\mathbb{R}^d$ ($d=2$), then the interference at o is given

$$V := \sum_{i\geq 1} \ell(|x_i|)$$

path-loss function

$\ell(r) = 1 \wedge r^{-\alpha}$ $\quad \alpha > 2$

$\alpha = 2$

$\mathbb{P}(V=\infty) = 1$

Q: Assume Transmitters scattered at random in plane. What is the prob. to see a $\boxed{\text{large interference}}$?

Poisson point process with int. λ

A: tail behavior of V

$$\boxed{\frac{1}{n\log(n)} \log P(V \geq n x) \xrightarrow{n\to\infty} -x}$$

$$V^0 := \sum_{|x_i|\le 1} \underbrace{\ell(|x_i|)}_{\substack{1\!\!1\,|x_i|^{-\alpha} \\ 1}} = \underbrace{\#\{i : |x_i| \le 1\}}_{\text{Poisson RV}}$$

$$\mathbb{P}(V^0 \ge nx) = \mathbb{P}(\mathrm{Poi}(\lambda|B_1|) \ge nx)$$

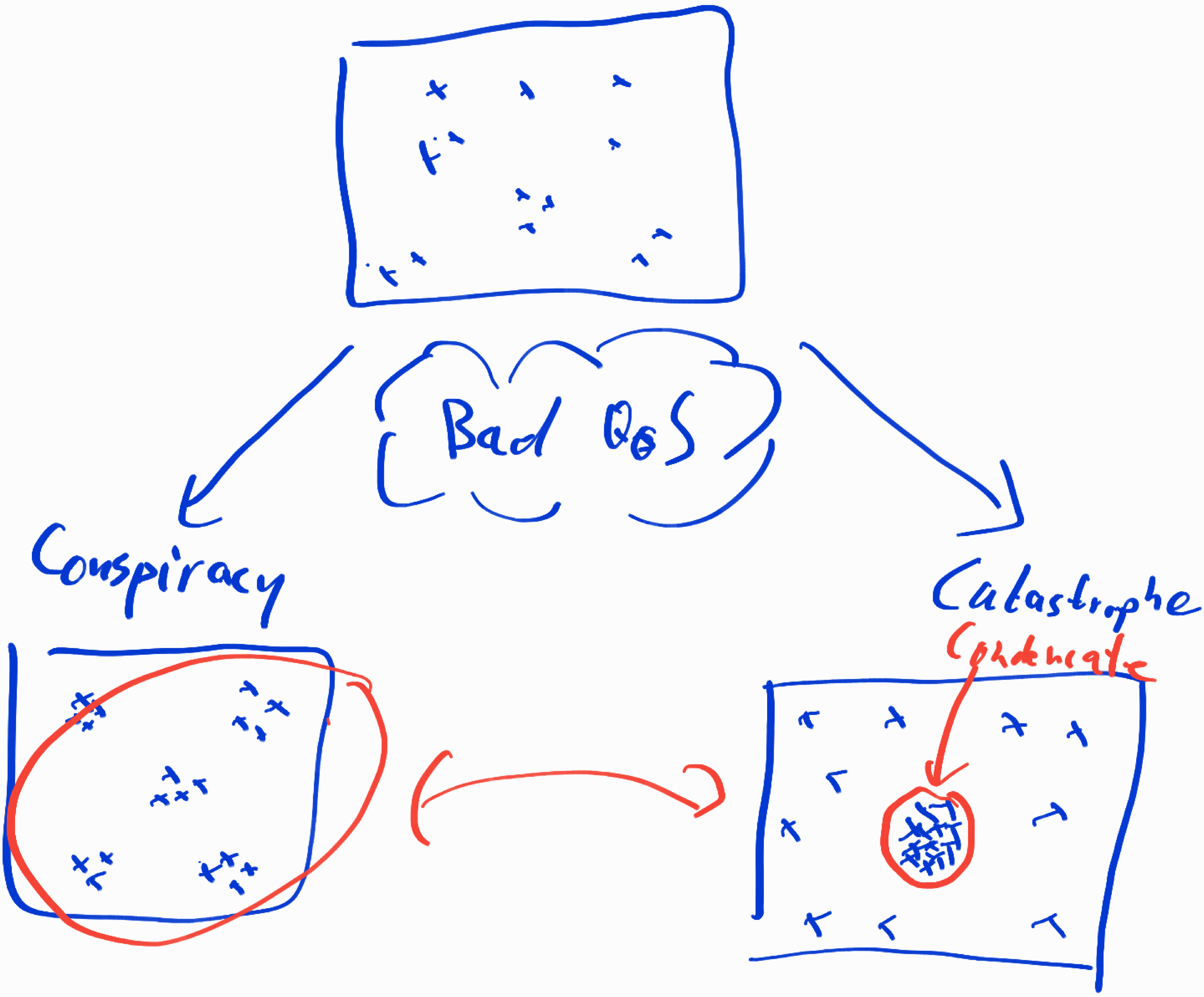
$$\le \exp\left(-\lambda|B_1| \cdot h\left(\frac{nx}{\lambda|B_1|}\right)\right)$$

where $\boxed{h(y) = y\log(y) - y + 1}$

IV Conspiracy & Catastrophe

So far: Large Deviation of interf. at o.

Problem: Characteristics of many users in large domain



Example [Gilbert graph]

Graph connects x_i, x_j if $|x_i - x_j| \le 1$

Consider PPP X in box

$$Q_n := \left[-\frac{n}{2}, \frac{n}{2}\right]^d$$

[Q] Prob. Gilbert graph in Q_n has exceptionally many/few edges?

$\boxed{\text{Upper Tails}}$

Set $M_n := \frac{1}{n^d} \# \{ \text{Edges in } Q_n \}$

$$\mathbb{P}(M_n \geq \mu + \delta) \xrightarrow[n \to \infty]{} ?$$

$\mu = \mathbb{E}[M_n]$

Idea (Chatterjee & Harel, 2020)

Place $k := \lceil 2 \delta n^d \rceil$ Poisson points in ball of diameter 1

$\rightsquigarrow \binom{k}{2} \sim \delta n^d$ extra edges

$$\mathbb{P}(\# (X \cap B_{\frac{1}{2}}) \geq k)$$

$$\sim \exp \Big(- \overbrace{\lambda |B_{\frac{1}{2}}|}^{\lambda' :=} \, h \Big(\frac{k}{\lambda |B_{\frac{1}{2}}|} \Big) \Big)$$

$$\sim \exp\left(-k\log\left(\frac{k}{\lambda'}\right) + k - \lambda'\right)$$

$$\sim \exp\left(-k\log(k) + o(k\log(k))\right) \geq \exp(-c_n^d)$$

$\Rightarrow$ ~~condensation~~ clumping

<u>lower tail</u>

$$\mathbb{P}(\underline{H_n} \leq \mu - \delta)$$

<u>key tool</u>

Consider empirical process R_n

$$\boxed{R_n = \frac{1}{n^d}\sum_{X_i \in Q_n} \delta_{\left(X - X_i,\ \theta_{X_i}(X)\right)}}$$

R_n contains a lot of information

Consider $F(\varphi) = \boxed{\#\left(\varphi \cap B_1(0) \setminus \{0\}\right)}$

"vertex at o"

<u>not bounded</u>

$$\boxed{\int F(\varphi)\, R_n(d\varphi)} = \frac{1}{n^d} \sum_{x_i \in Q_n} F(X - x_i)$$

$$= \langle M_n \rangle$$

Blueprint

$\boxed{\text{LDP for } R_n}$ $\xrightarrow{\int F(\cdot)}$ LDP for M_n

"contraction principle"

Georgii & Zessin '92 τ-Topology smallest top. evf: M1

T: continuous transformation, bounded

$n^{\frac{d}{2}}$

$$\frac{1}{n^d} \log \mathbb{E}\left[\exp\left(T(R_n)\, n^d\right)\right] \xrightarrow{n \to \infty} \sup_{\substack{Q:\text{ stationary} \\ \text{pt. processes}}} \left\{ T(Q^\circ) - \mathrm{Ent}(Q \mid \mathrm{Pois}) \right\}$$

$\to 0$

where

stationary

$$\mathrm{Ent}(Q \mid \mathrm{Pois}) = \lim_{n \to \infty} \frac{1}{n^d} \mathbb{E}_Q\left[\log\left(\frac{dQ|_{[0,n]^d}}{d\mathrm{Pois}|_{[0,n]^d}}\right)\right]$$

Computationally tractable ?

smallest top:

$ev_f : \mu \mapsto \int f(q)\, \mu(dq)$

are cont

for all f : local

& bounded

⇒ solve lower tail

by 1) Truncation

2) Sprinkling

Lecture 2: Percolation

I. Background

Classical wireless networks rely on a simple user $\longleftrightarrow$ base station

Issue: Waste of spectrum

- Device-to-device communication (D2D)

$\boxed{Q}$
- Can D2D be implemented on large scale?
- Can substantial proportion of users send messages over long distances through relaying?

→ Continuum percolation

Base stations

Streets

Cox percolation

Interference

→ afternoon

Bounded hops

mobility

...

II. Poisson case

Let $X = \{X_i\}_{i \geq 1}$ be PPP of devices with intensity $\lambda > 0$. ~~Say~~ Write $X_i \leftrightarrow X_j$ if $|X_i - X_j| < r$ [$\rightsquigarrow$ Gilbert Graph $G(X,r)$

Percolation theory $\rightsquigarrow$ large components of $G(X,r)$

$G(X,r)$ <u>percolate</u> if it has unbounded connected componen

$$\boxed{\Theta = \Theta(\lambda, r) = \mathbb{P}_\lambda\big(o \leftrightarrow \infty \text{ in } G(X \cup \{o\})\big)}$$

Percolation probability

$$\boxed{\lambda_c(r) = \inf\{\lambda > 0 : \Theta(\lambda) > 0\}}$$

Theorem [Phase transition]: $\boxed{0 < \lambda_c(r) < \infty}$

II. Cox percolation (H., Jahnel, Cali, 2019)

Idea: Devices are $\boxed{\text{not}}$ scattered homogeneously at random in space but ~~dri~~ the location is driven by random environment

Cox process: Poisson process in random environment

More precisely: Let Λ be a stationary random measure on $\mathbb{R}^d$. A $\boxed{\text{Cox process}}$ X is a point process, that when conditional on Λ is a Poisson process with IM Λ

intensity measure

Examples [Modulated Process]

$$\boxed{\Lambda(dx) = f_x\, dx}$$ for stationary random field $\{f_x\}_{x \in \mathbb{R}^d}$

- Street system

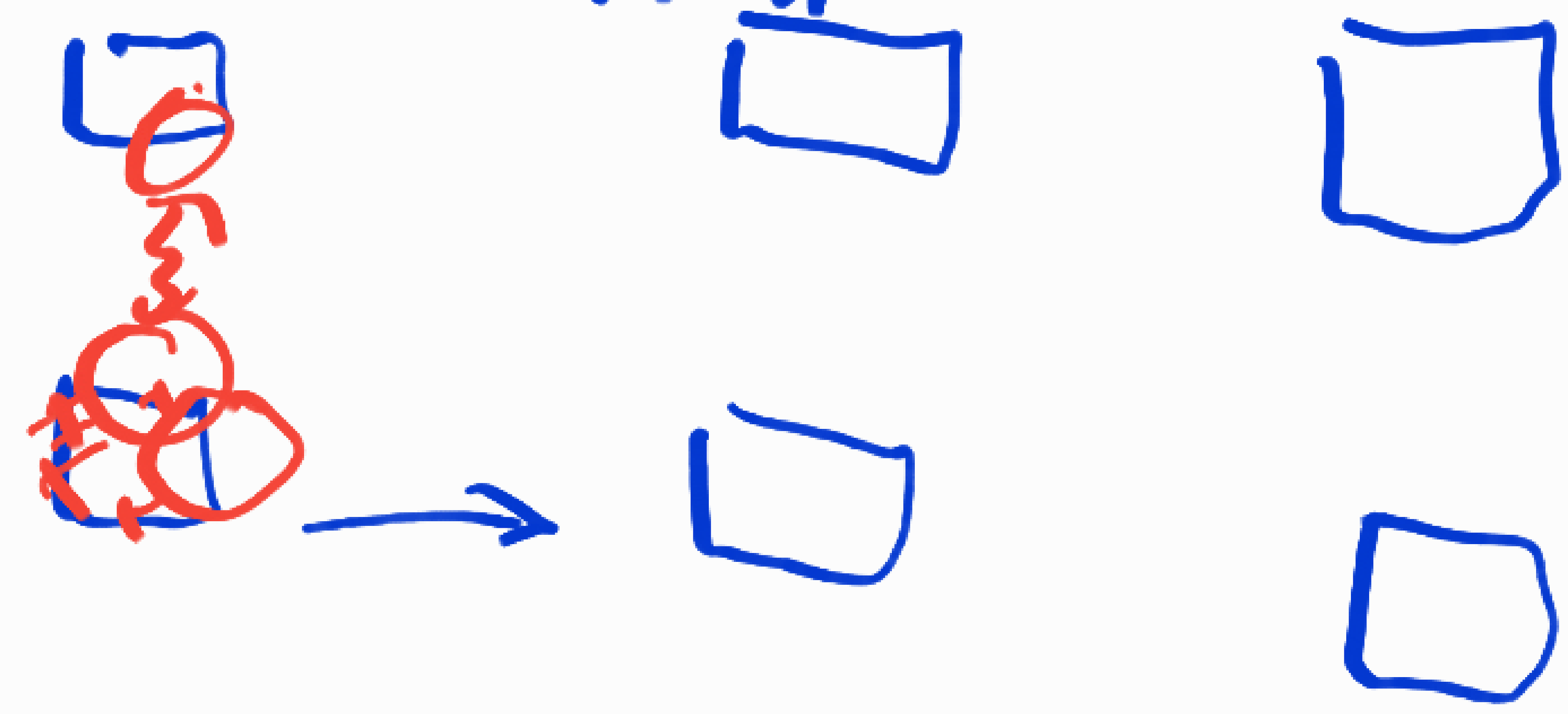
$$\boxed{\Lambda(dx) = \nu_1(S \cap dx)}$$ with S stationary segment system

Assume normalization $\mathbb{E}[\Lambda([0,1]^d)] = 1$ and X^{λ} Cox process with IM $\lambda \cdot \Lambda$

Q] Phase transition?

↪ for $G(X, r)$

↪ fixed



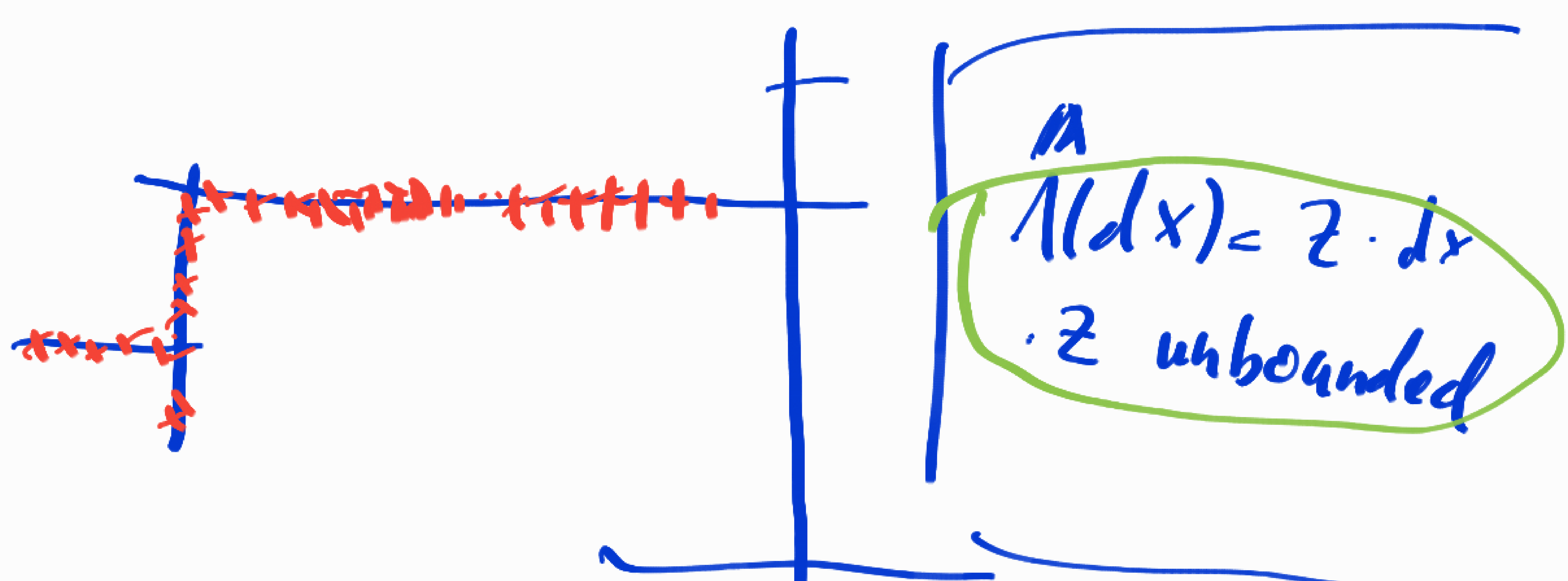
Counterexample 1

$\lambda_c(r) = \infty$

Counterexample 2

$\boxed{\lambda_c(r) = 0}$ (⇝ Bartek / Yogesh)

System of long rods



Homework:

- rods of length s occur with prob $\sim p s^{-3}$
- 1D intensity on rod of lengths s as $(\log s)^2$

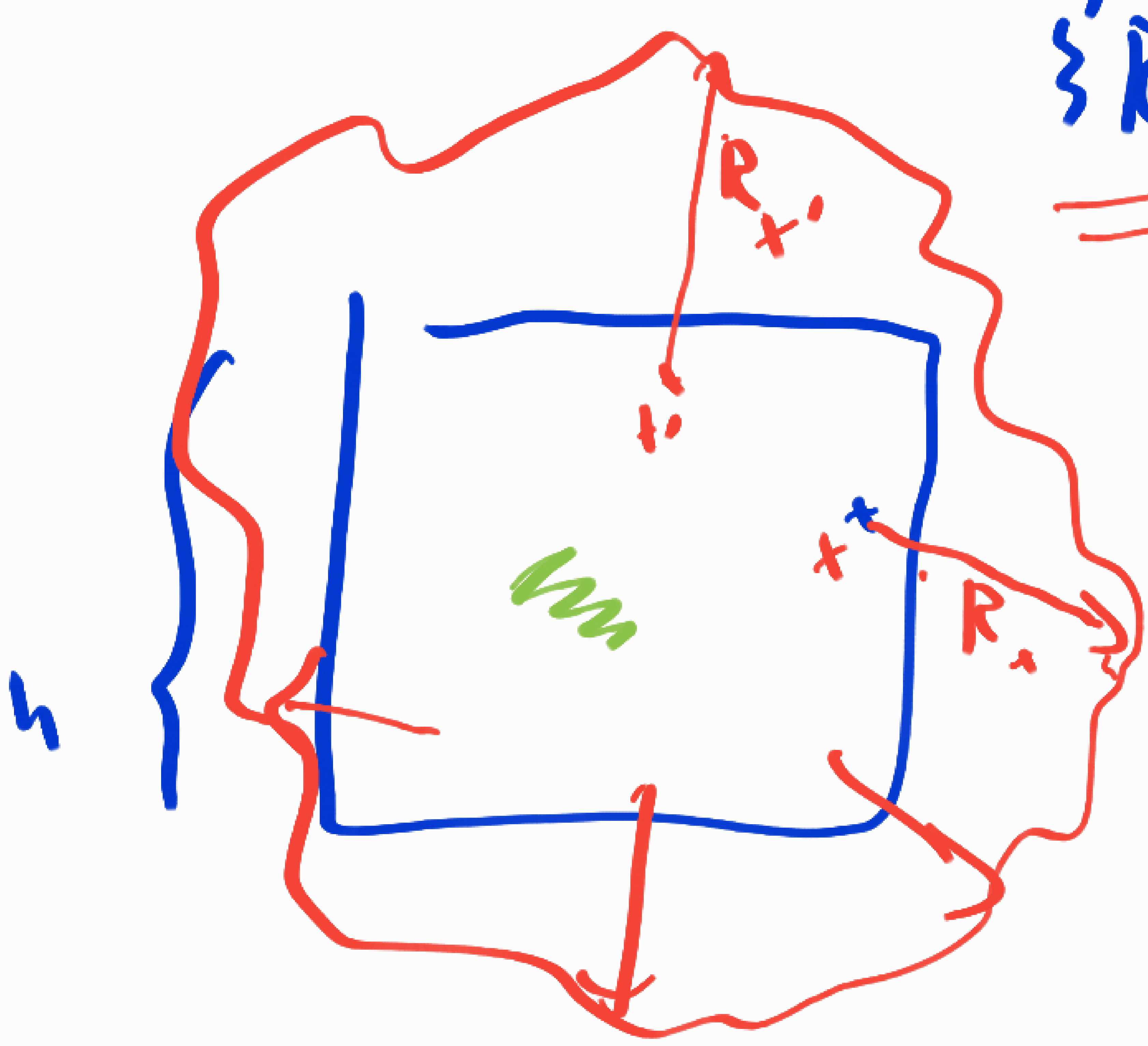
Theorem [Phase transition]

Assume Λ is: 1. stabilizing [~ Penrose & Yukich 03]

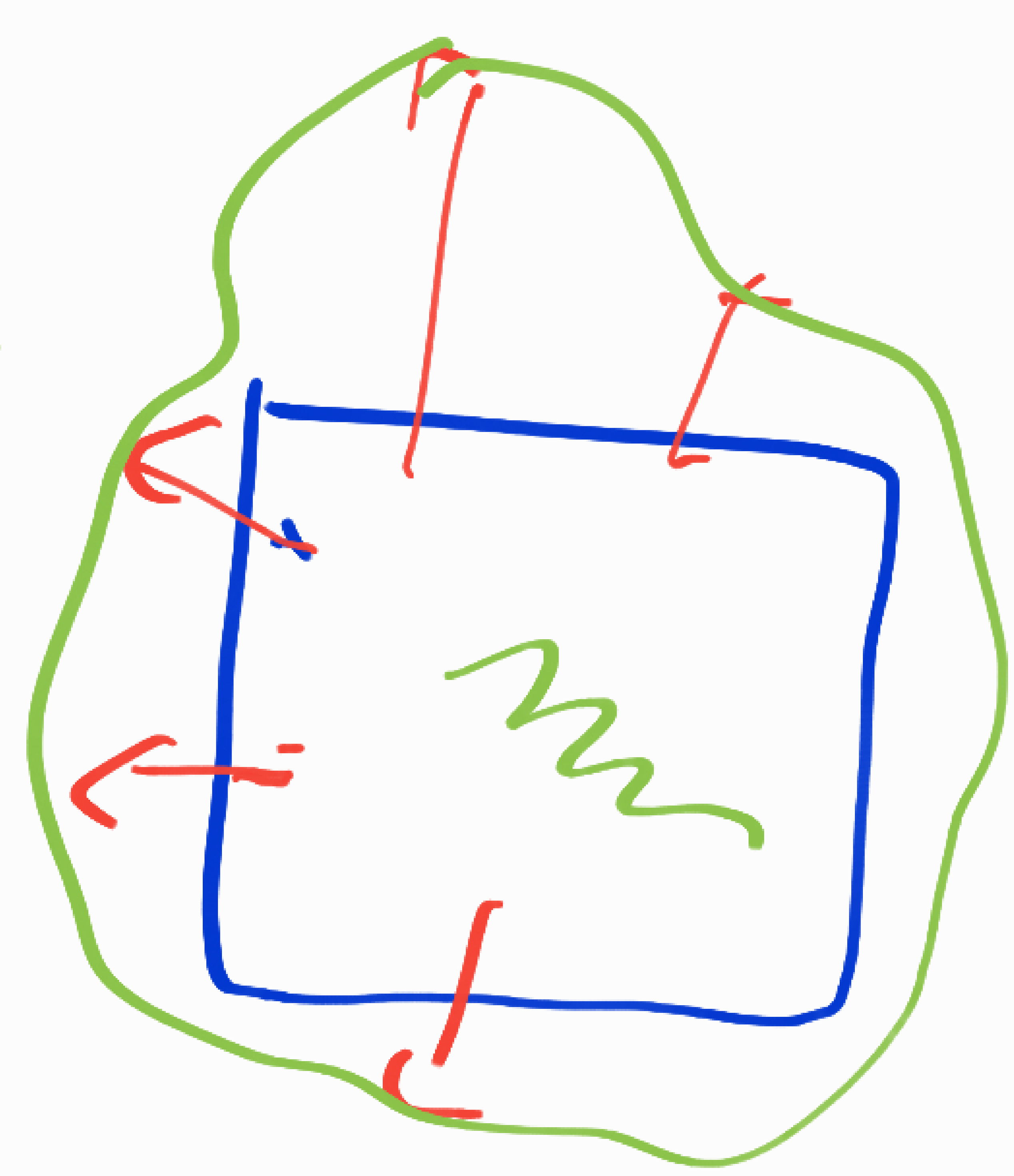
2. asymptotically essentially connected [~ Aldous '09]

$$\implies \boxed{0 < \lambda_c(r) < \infty} \quad \lambda_c = ?$$

Stabilizing: Correlation lengths bounded by some stabilization radii $\{R_x\}_{x \in \mathbb{R}^d}$



$r = \emptyset$



$$\left\{ f(\Lambda\Gamma_{Q_n(x)})\, \mathbb{1}\{\sup_{y\in Q_n(x)} R_y < n\} \right\}_{x\in A}, \quad A=\{x_1,\dots,x_k\}$$

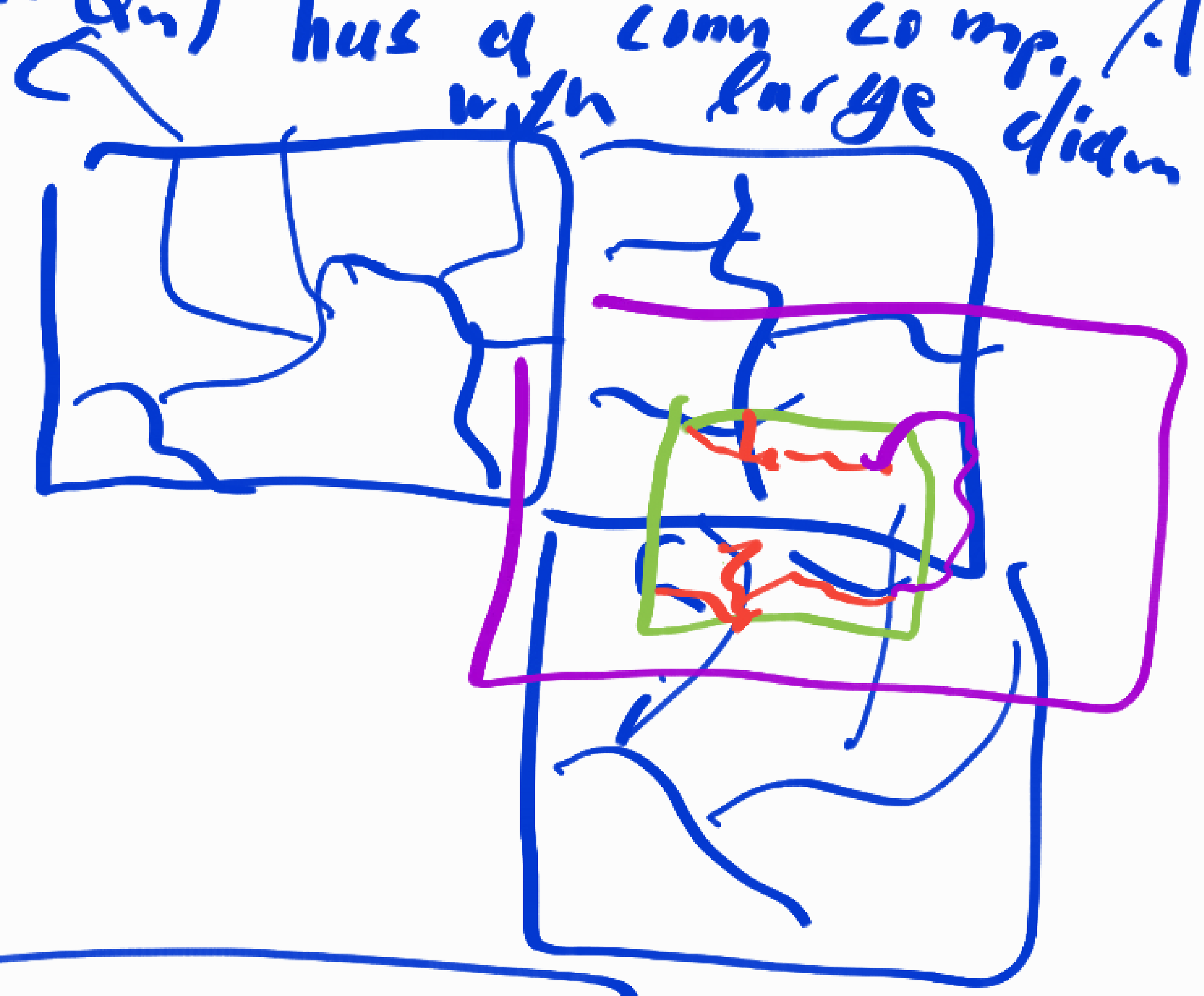
$|x_i - x_j| \geq 3n$

independent

$\mathbb{P}(R_x > n) \leq \exp(-c\cdot n)$

supp$(\Lambda\Gamma_{Q_n})$ has a conn comp. with large diam

Asymptotic essential connectedness



$\boxed{\text{Limits in } \Lambda, \Gamma}$

~~th~~ Poisson case:

If X is PPP with int. λ and $a>0$ Parameter

$\rightsquigarrow$ aX is PPP with intensity $\boxed{a^{-d}\lambda}$

$$\Rightarrow \boxed{\theta(a^{-d}\lambda, ar) = \theta(\lambda, r)}$$

$\rightsquigarrow$ $\boxed{Q}$ Analogs for Cox percolation

"Zoom-out": $\boxed{r\to\infty,\ \lambda\to 0 \\ \lambda r^d = \rho}$ with

Thm. 2.9 $\Rightarrow$ erg & stab.

$$\boxed{\boxed{\theta(\lambda, r) \xrightarrow{\lambda\to 0} \theta_{Pois}(\rho, 1)}}$$

$\Rightarrow$ Universal behavior

Franceschetti & Meester "Communication networks"

Zoom-in : $r \to 0$
$\lambda \to \infty$
$\lambda r^d = \varrho$ const.

1. $\Lambda(dx) = f_x \, dx$

2 obstructions

I: fail to percolate out of microscopic neighborhood

II: fail to be in giant comp. of the random environment

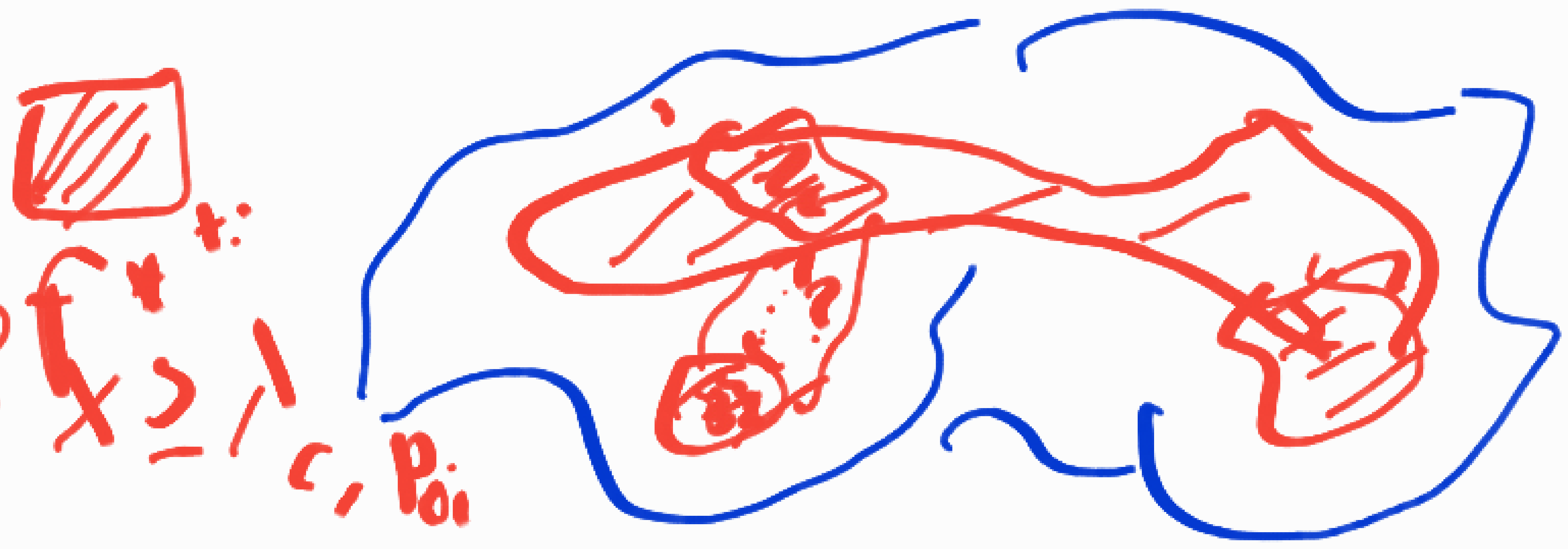
Thm. 2.10 $\rightsquigarrow$

$$\Theta(\lambda, r) \longrightarrow \mathbb{E}\Big[\overset{\mathrm{I}}{\mathbb{1}\{0 \overset{L_{\geq 0}}{\longleftrightarrow} \infty\}} \; \overset{\mathrm{II}}{\theta_{Pois}(\varrho f_0^*)}\Big]$$

f^*: size-biased version of f:

$$\mathbb{E}[g(f^*)] = \mathbb{E}[f_0 \, g(f)]$$

- $L_{\geq 0} := \{x \in \mathbb{R}^d : \varrho f_x^* \geq \lambda_{c,Pois}(1)\}$



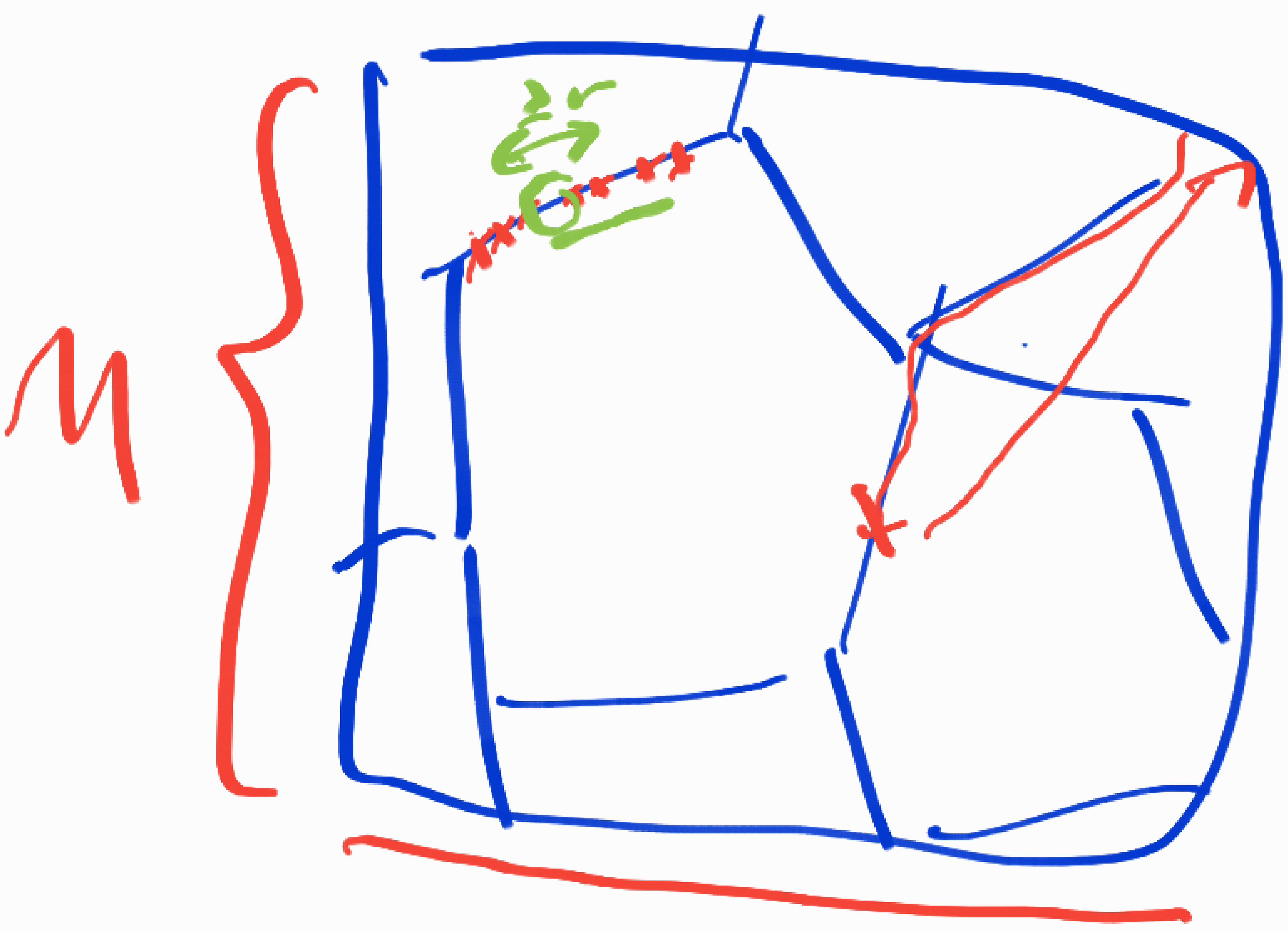
$gf^{*}_{x} \geq \lambda_{c,Poi}$

b2. $\Lambda(d_x) = \nu_1(S \cap d_x)$

$\lambda \to \infty$

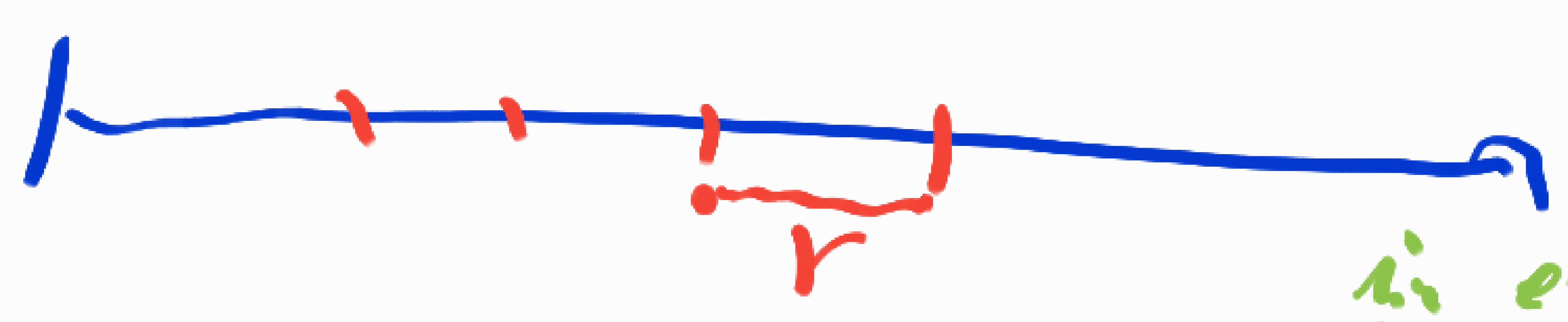
$r \to 0$

$\lambda e^{-\lambda r} = c$



Gaps of length $\geq r$ make entire edge unusable

Gaps occur at "intensit" $\lambda e^{-\lambda r} \cdot c$



in edge of length ℓ: $\exp(-c\ell)$

Poisson $\Longrightarrow$ Limit prob. of no gaps $\approx \exp(-c)$

$\Longrightarrow \theta^{n}(\lambda, r) \longrightarrow \theta^{n}_{Ber\text{-}inhom}(\exp(-c))$

where in $\theta_{Ber\text{-}inh}(a)$ *: inhom. bond percolation where bonds of length ℓ are retained $w\ell^{a}$

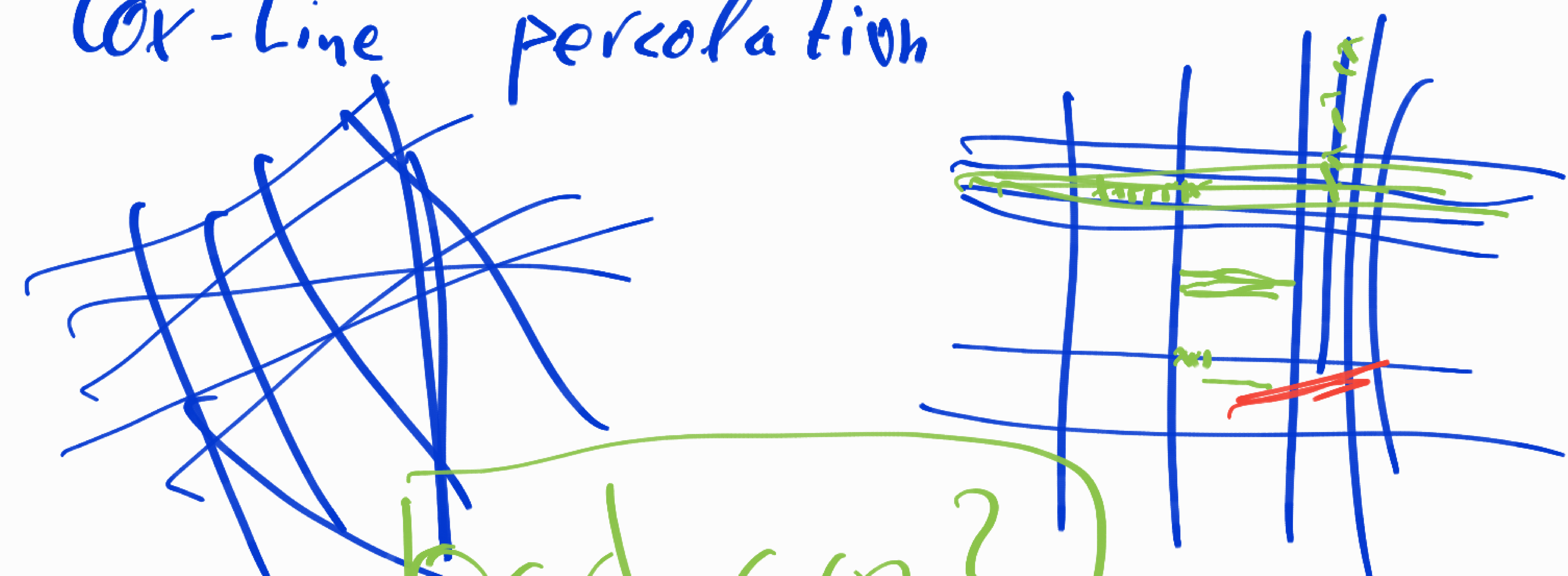
Idea proof:

1. Establish percolation outside finite boxes ($\rightsquigarrow$ weak converges for point processes)
2. General upper-semicontinuity $\rightsquigarrow$ „$\le$"
3. "Tightness"-type result:

$$\boxed{\lim_{k\to\infty} \lim_{r\to 0/\infty} \mathbb{P}\left(o \not\leftrightarrow \infty;\ o \leftrightarrow \partial Q_{k \cdot s(r)}\right) = 0}$$

VI. Nice problems

- Cox-Line percolation



$\boxed{0 < \lambda_c < \infty \,?}$

- Sharp-threshold

  → $\boxed{\text{No FKG inequality !}}$

  "Percolation without FKG" Beffara & Gayet '17

  " see also Muirhead; Belidev Rivera:

  " ⇝ Sharp-thresholds without FKG

Lecture 3: Optimal markings

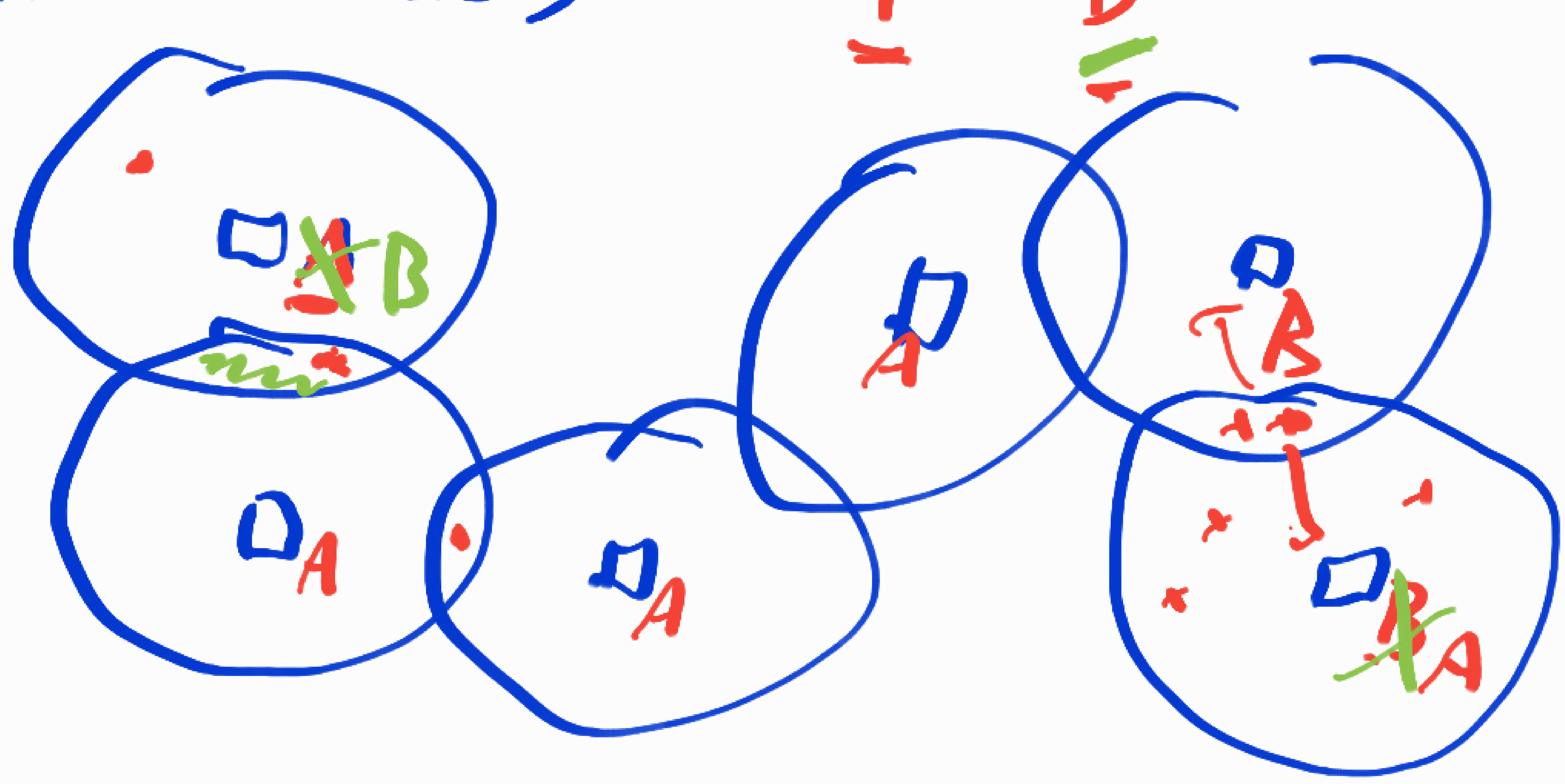
(joint with G. Last
with B. Błaszczyszyn)

I. Background

Caching:

- Data-intensive services (video-streaming)
- Caching: Distribute most popular content in a spatial network ~~ba~~ of lightweight base station as infra-structure nodes

A > B



? - How to distribute content **optimally** among nodes?

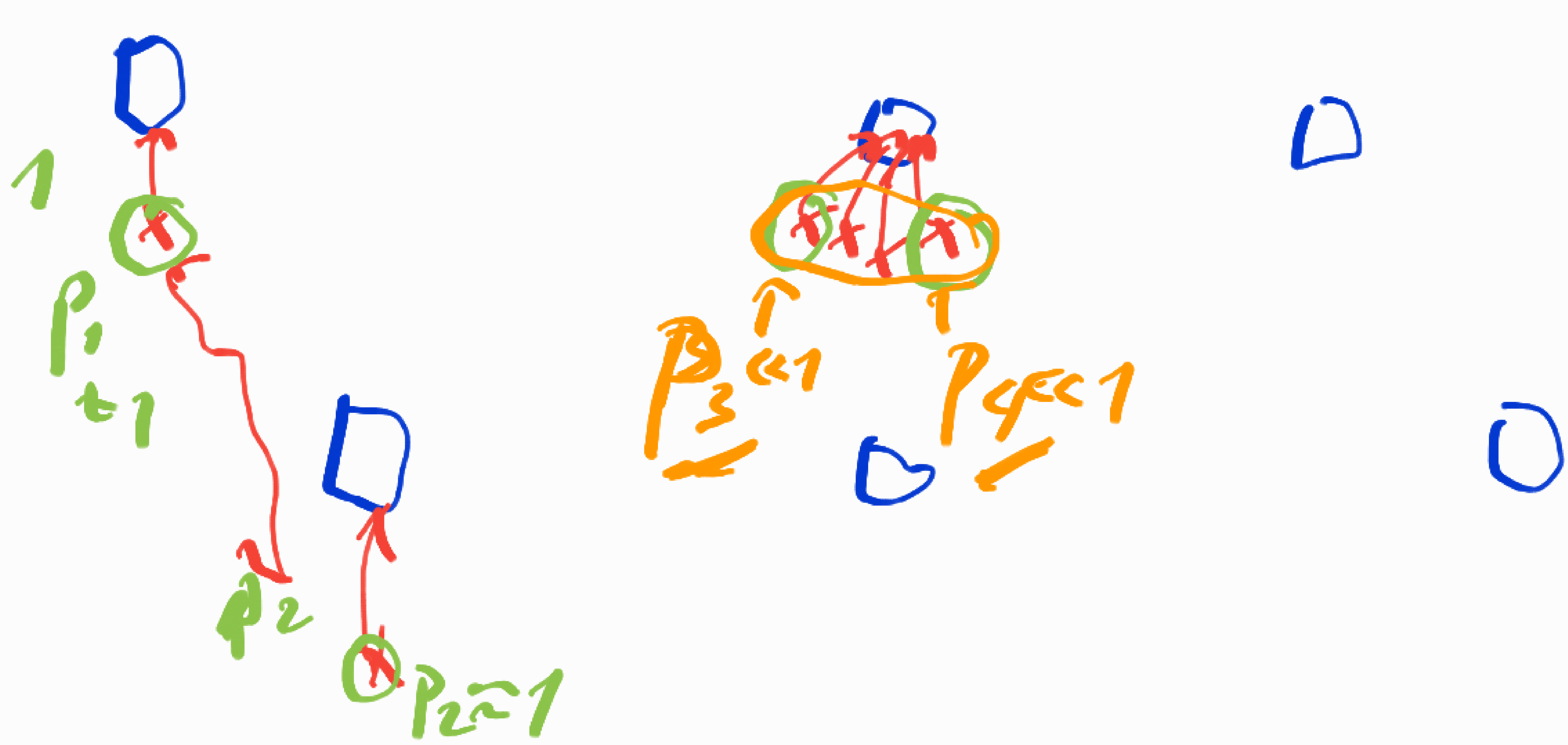
- How to proceed for unboundedly large networks?

Medium-Access-Control (MAC)

⇝ Baccelli, Blaszczyszyn, Singh, '14

→ Wireless networks suffer from interference

MAC: Regulate which devices may access medium at certain points in time



ALOHA-protocol:

- partition transmission into time slots
- simplest: with ~~p~~ some probability $p \in [0,1]$ a device independently sends a message to a base station in a given slot

How to choose ~~a~~ probabilities $\{p_i\}_i$ adapted to the ~~su~~ neighborhood?

$\Rightarrow$ Rigorous definition of optimal marking for stationary processes

$\Rightarrow$ Existence?
Uniqueness?

II. Intensity & local optimality

- $\Phi = \{(X_i, M_i)\}$ stationary $\mathbb{M}$-marked process in $\mathbb{R}^d$ for some Polish space $\mathbb{M}$

- Consider stationary markings of Φ

[optimize]

$\Psi = \{(X_i, M_i, M_i')\}_{i \geq 1}$ stationary $\mathbb{M} \times \mathbb{M}'$-marked pt process such that projection to $\mathbb{R}^d \times \mathbb{M}$ has the same law as Φ

.

Measure quality of marking ψ through a ~~so~~ score function

$$\xi : \mathcal{N}^0_{\mathbb{R}^d \times M \times M'} \to [0,\infty) \cup \{-\infty\}$$
$$\to [-\infty, 0]$$

space of locally finite conf. in $\mathbb{R}^d \times M \times M'$ with an atom at 0

$\xi(\psi) \rightsquigarrow$ "score of a typical node at the origin"

For arbitrary (x_i, m_i, m_i') score attached to it is $\xi(\underbrace{\psi - x_i}_{\Theta_{x_i}(\psi)})$

<u>Goal</u> Define markings maximize total aggregate ξ-scores

Infinite System?

<u>Intensity-optimality</u> $\overset{?}{\longleftrightarrow}$ <u>Local-optimality</u>

<u>Intensity-optimality</u>

Define <u>ξ-intensity</u> as the expected score in a unit box:

$$\boxed{\gamma_\xi(\overset{\mathcal{L}(\Psi)}{\ }) = \mathbb{E}_\Psi\Big[\sum_{x_i \in \Psi \cap [0,1]^d} \xi(\Psi - x_i)\Big] = \lambda \cdot \mathbb{E}^0[\xi(\Psi)]}$$

optimal ξ intensity: $\gamma_{\xi,opt} = \sup_{\Psi:\text{ stat. mark.}} \gamma_\xi(\Psi)$

$\mathcal{L}(\Psi)$ is <u>intensity-optimal</u> if $\boxed{\gamma_\xi(\Psi) = \gamma_{\xi,opt}}$

Local optimality: "No improvement after changing finite number of ~~bot~~ marks"

Set

$$\varphi' \Delta_{\xi,x} \varphi = \xi(\varphi'-x) - \xi(\varphi-x)$$

Would to optimize

$$\varphi' \Delta_{\xi} \varphi := \sum_{x \in \Phi} \varphi' \Delta_{\xi,x} \varphi = \underline{\sum_{\lambda} \xi(\varphi'-x) - \xi(\varphi-x)}$$

⚠ However, may not be well defined

Therefore:

φ is locally optimal $\boxed{\varphi' \Delta_\xi \varphi \le 0}$
whenever

$\boxed{\#(\varphi' \setminus \varphi) < \infty}$ and

$$\sum_{x \in \Phi} (\varphi' \Delta_{\xi,x} \varphi)_- < \infty$$

Examples

- Hard-core thinning:

  $\mathbb{M}$: compact convex bodies

  $\mathbb{M}' = \{0,1\}$ retention indicator

$$\xi(\varphi) = \begin{cases} -\infty & \text{if hard-core condition is violated at } \mu_0 \\ |\mu_0| \cdot \mu_0' & \text{otherwise} \end{cases}$$

typical grain at o $\in \{0,1\}$ survival indicator

MAC bipolar model

Each transmitter has its own receiver located at distance 1 from it

$\mathbb{M} = \partial B_1 \rightarrow$ receiver location

$\mathbb{M}' = (0,1] \rightarrow$ medium-access probability

Maximize expected log-throughput:
$\{(X_i, Y_i, P_i)\}_i$

$$\xi(\varphi) = \log(P_0) + \sum_{i \neq 0} \underbrace{\log\left(1 - \frac{P_i}{1 + |X_i - Y_0|^3}\right)}_{\boxed{\mathbb{P}(\text{SINR} \geq \tau \mid \{X_i, Y_i\})}}$$

$$\sum_j F_{j,i}\, \ell(|X_j - X_i|)$$

III. Existence & Equivalence

Existence?

<u>Idea</u>: Tightness/compactness argument

1. Consider markings φ_n: $\gamma_\xi(\varphi_n) \xrightarrow{n \to \infty} \gamma_{\xi_{opt}}(\varphi)$

2. $\{\psi_n\}$ are markings of a common process Φ

$\Rightarrow$ $\boxed{\{\psi_n\}_n \text{ tight}}$ as pt process

$\rightsquigarrow$ Extract subsequential limit ψ^* maybe $\psi^* = \emptyset$

3. Show that $\boxed{\gamma_\xi(\psi^*) = \gamma_{\xi, opt}}$

$\Rightarrow$ Need condition on: 1) continuity ξ

2) Tightness of mark space

Intensity-optimal $\Rightarrow$ local-optimality

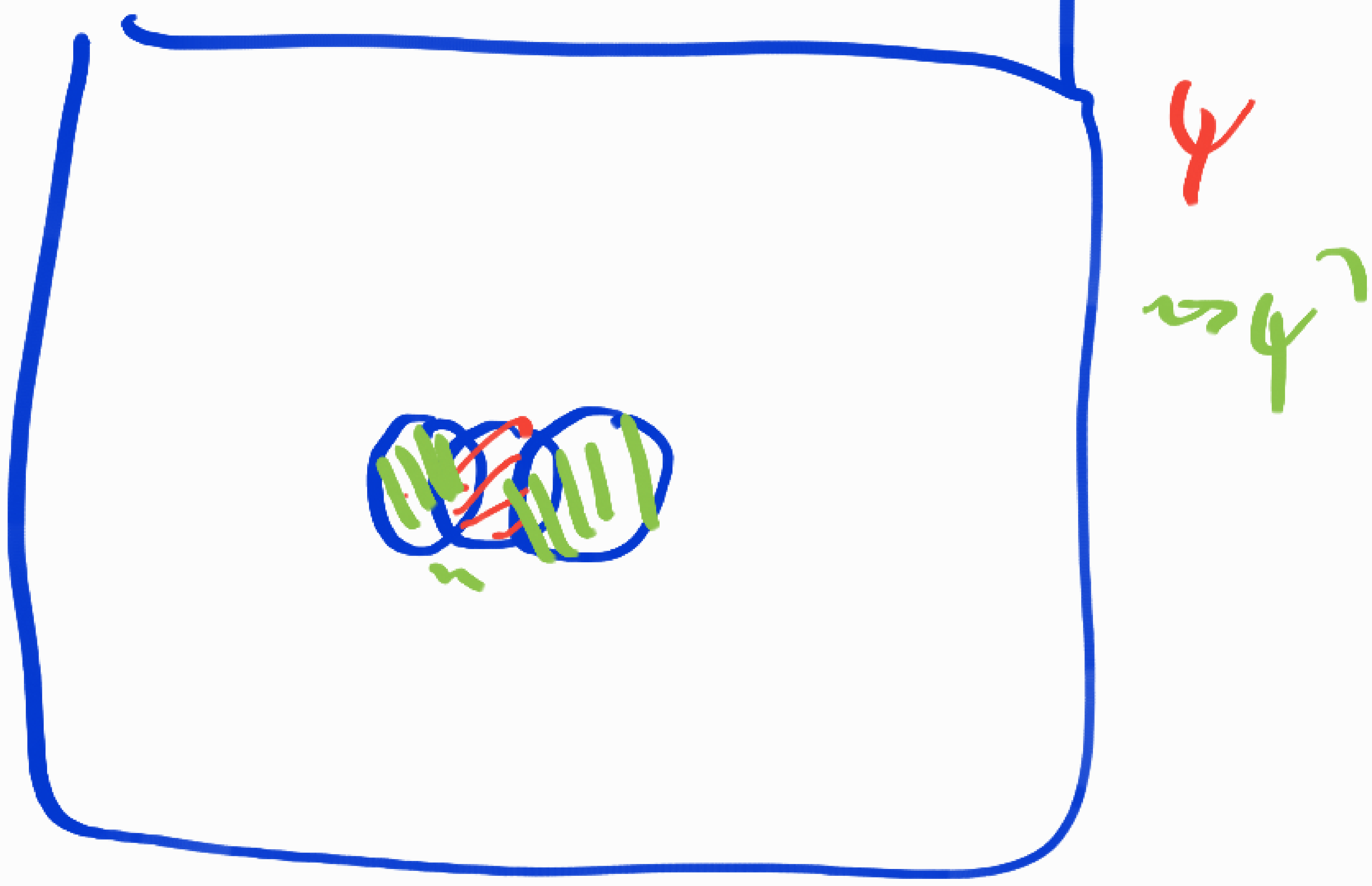
Proof by contraposition

Say if marking ψ is not

locally optimal.

$\Rightarrow \exists$ finite modif ψ' : $\psi' \Delta_\xi \psi > 0$

"valid swap"



- valid swaps form stationary process
- implement swaps one after another $\rightsquigarrow$ better solution wrt γ_ξ !

IV. Uniqueness : $\boxed{\text{Open Problem}}$

V. Challenges

- Computation of optimal marking