The weight-depending random connection model

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Based on joint work with Peter Gracar (Cologne), Remco van der Hofstad (Eindhoven), Günter Last (Karlsruhe), Kilian Matzke (Munich), Christian Mönch (Mainz), Peter Mörters (Cologne)

What is a complex network?

Aim: Study realistic models for real-life networks.

Many real-world networks, such as WWW, social, financial, neural, or biological networks, exhibit general pattern ("stylized facts"):

- the length of a smallest path between two vertices is small w.r.t. the system size (small world),
- the degrees of vertices exhibit a power law (a scale-free network),
- vertices that are "geographically" close are likely to be connected (geometric clustering),
- vertices with high degree are likely to be connected even if far away from each other (hierarchies).

The **weight-dependent** random connection model

Poisson process of unit intensity on $\mathbb{R}^d \times [0,1]$. Interpret Poisson point $\mathbf{x} = (x, s)$ as a vertex at position x with weight s^{-1}.

Two vertices $\mathbf{x} = (x, s)$ and $\mathbf{y} = (y, t)$ are connected by an edge independently with probability $\phi(\mathbf{x}, \mathbf{y})$ for a connectivity function

$$\phi\colon (\mathbb{R}^d \times [0,1]) \times (\mathbb{R}^d \times [0,1]) \to [0,1],$$

We assume throughout that ϕ has the form

$$\phi(\mathbf{x}, \mathbf{y}) = \phi\big((x, s), (y, t)\big) = \rho\big(h(s, t, |x - y|)\big)$$

for a non-increasing, integrable profile function $\rho\colon \mathbb{R}_+ \to [0,1]$ and a suitable kernel function $h\colon [0,1] \times [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$.

W.l.o.g. assume $\int_{\mathbb{R}^d} \rho(|x|)\, dx = 1$.
Different kernel function yields different network properties...

Various connection kernels

- *Plain kernel* as

$$h^{\mathrm{plain}}(s,t,v) = \frac{1}{\beta} v^d.$$

  Special case: $\rho(r) = 1_{[0,a]}$ for suitable a (*Gilbert disc model*).
  Yields *random connection model*.

- *Sum kernel*

$$h^{\mathrm{sum}}(s,t,v) = \frac{1}{\beta}\left(s^{-\gamma} + t^{-\gamma}\right)^{-1} v^d.$$

  Special case: $\rho(r) = 1_{[0,a]}$ for suitable a (*Boolean model*).
  Further variant: *min-kernel* defined as

$$h^{\mathrm{min}}(s,t,v) = \frac{1}{\beta}(s \wedge t)^{\gamma} v^d,$$

  but $h^{\mathrm{sum}} \le h^{\mathrm{min}} \le 2h^{\mathrm{sum}}$.

- *Product kernel*

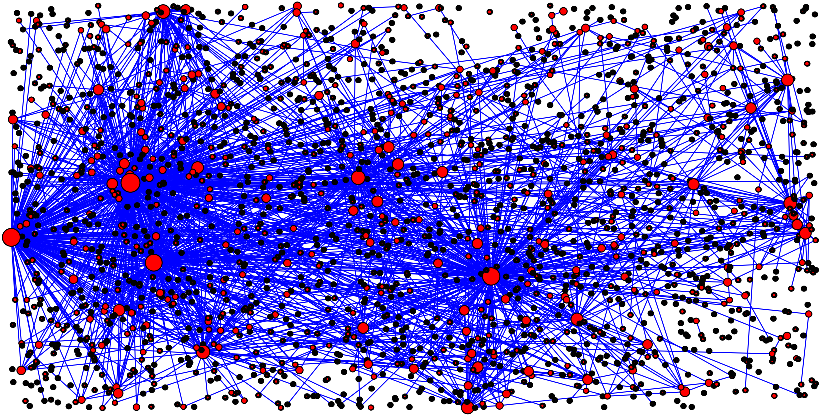
$$h^{\mathrm{prod}}(s,t,v) = \frac{1}{\beta} s^{\gamma} t^{\gamma} v^d,$$

  Continuum version of *scale-free percolation*

- *Preferential attachment kernel*

$$h^{\mathrm{pa}}(s,t,v) = \frac{1}{\beta}(s \vee t)^{1-\gamma}(s \wedge t)^{\gamma} v^d,$$

Example: Product kernel



Random graph with product kernel ($\gamma = 0.6$, $\delta = 2$).

Terminology of the models in the literature

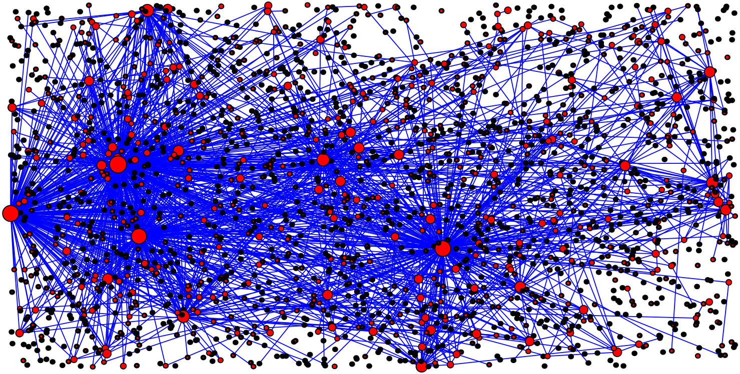
Profile	Kernel	Model	Reference
indicator	plain	Random geometric graph, Gilbert disc model	Penrose '93
general	plain	Random connection model	Meester-Penrose-Sarkar '97
		Soft random geometric graph	Penrose '16
indicator	sum	Boolean model	Hall '85, Meester '94
indicator	min	Scale-free Gilbert graph	Hirsch '17
polynomial	prod	Inhomogeneous long-range percolation	Deprez-Hazra-Wüthrich '15
		Continuum scale-free percolation	Deprez-Wüthrich '18
general	prod	Geometric inhomogeneous random graphs	Bringmann-Keusch-Lengler '19
general	pa	Age-dependent random connection model	Gracar et al. '19

Remarks:

- All models except *plain kernel* are scale-free with power-law exponent $\tau = 1 + \frac{1}{\gamma}$.
- Henceforth assume power-law profile function: $\rho(v) \approx v^{-\delta}$, $v \to \infty$.
- If $\delta > 1$ and $\gamma < 1$, then resultig graph is locally finite for all $\beta > 0$.

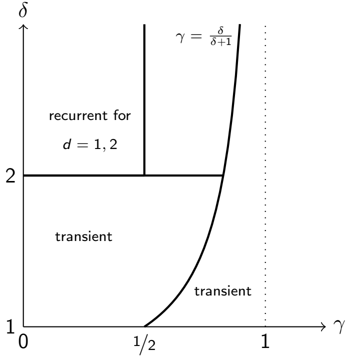
What are the structural properties of the (a.s. unique) infinite component?

Random walk on graphs: recurrent or transient?

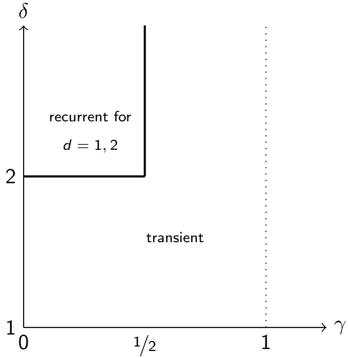


Random walk is recurrent if a.s. returns to starting point.
Random walk is transient if never returns with positive probability.

Theorem. (Joint work with P. Gracar, C. Mönch, P. Mörters)
Consider the weight-dependent random connection model with profile function $\rho(v) \approx v^{-\delta}$. Then the the infinite connected component (if it exists) is recurrent resp. transient according to the following graph:



Preferential attachment kernel, Sum kernel, Min kernel.



Product kernel.

Rem.: Product kernel in discrete version in H., Hulshof, Jorritsma (2017).

Transience for $\gamma > 1/2$ in *product kernel*

Theorem. For product kernel, the supercritical cluster is **transient** whenever $\gamma > 1/2$.

Use a renormalization argument via **multiscale ansatz**:

- Group vertices in boxes, call boxes 'good' or 'bad' depending on vertices and edges *inside* these boxes.
- Iterate this process by considering larger and larger boxes. Also these larger boxes are 'good' or 'bad' depending on number of 'good' sub-boxes and edges between the sub-boxes.
- This implies transience if a fixed point is *on all scales* in a 'good' box (possible if β large enough).

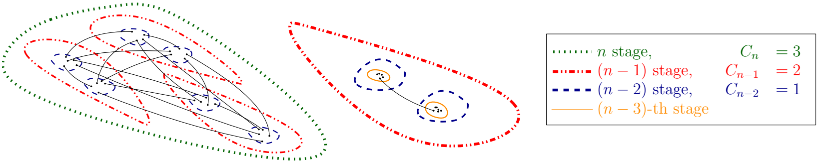
Transience proof for $\gamma > 1/2$ and β sufficiently large

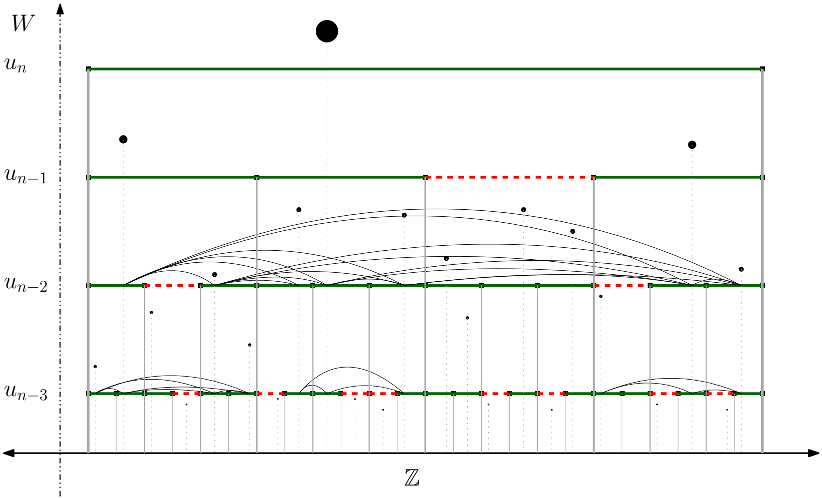
Multiscale ansatz: Fix sequences $(C_n)_n$, $(D_n)_n$, $(u_n)_n$.

1-stage boxes have side length D_1. Good if $\geq C_1$ vertices in box, and largest vertex ("1-dominant vertex") has weight $\geq u_1$.

For $n \geq 2$: n-stage boxes are formed of D_n^d $(n-1)$-stage boxes.
Good if

- $\geq C_n$ good $(n-1)$-stage boxes,
- in one of the good $(n-1)$-stage boxes there is a vertex with weight $\geq u_n$,
- in any good $(n-1)$-stage box, the $(n-2)$-dominant vertices form a clique.





Sketch of the renormalization scheme in $d = 1$ for
$D_n = 4, D_{n-1} = 3, D_{n-2} = 2, \quad C_n = 3, C_{n-1} = 2, C_{n-2} = 1.$
'Good' boxes are marked with a solid line, 'bad' boxes have a dashed line.

Transcience proof for $\gamma > 1/2$ and β sufficiently large

A choice that works:

$$D_n := 2(n+1)^2, \qquad C_n := (n+1)^{2d},$$

$$u_n := d^{\alpha/2}(n+2)^{d(2-\gamma)/2} 2^{(n+2)\alpha/2}((n+3)!)^{\alpha}.$$

$L_n := \mathbb{P}(n\text{-box containing 0 is good})$.

Can show: $\mathbb{P}\left(\bigcap_{n=1}^{\infty} L_n\right) > 0$ provided λ large enough and ε small enough.

Lemma [Berger 2002]: Graph generated by edges in good boxes is **transient** if $\sum_{n=1}^{\infty} 1/C_n < \infty$.

- Extra argument ("coarse-graining") for "small" $\beta > \beta_c$.
- Arguments for other kernels are similar.
  Preferential attachment kernel is most complicated.

Weight-dependent random connection model

Summary:

- Weight-dependent random connection model has high universality.
- Multiscale-Ansatz is strong technique in order to prove structural properties of random graphs in great generality.
- Gives excellent control in **supercritical** regime (= if infinite components exist).

Open:

- What happens if infinite components are about to appear (so-called **critical** behaviour)?

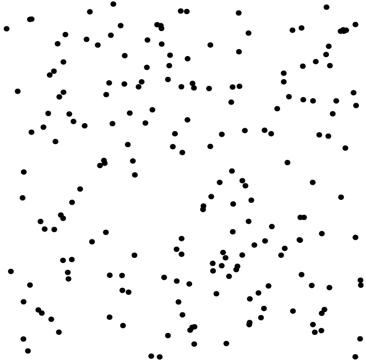
How are large components arising?

From now on:

- only ‘plain kernel’, i.e. connection probability (only) dependent on spatial distance of points;
- fix transition probabilities and vary the intensity of the Poisson process.

An example: The Boolean model (Gilbert disk model)

low-density Poisson process



high-density Poisson process

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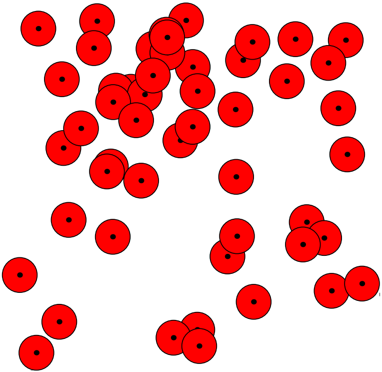

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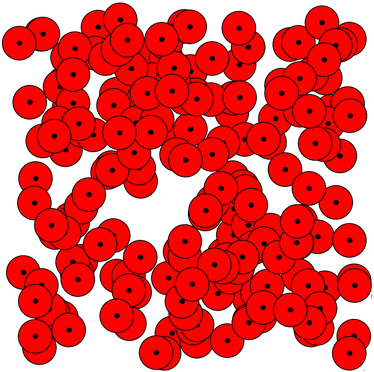

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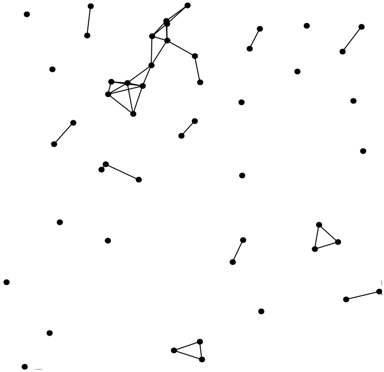

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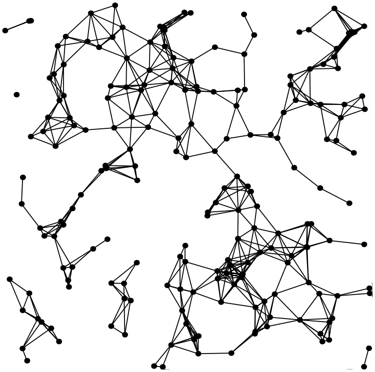

high-density Poisson process

Yields random graph embedded in $\mathbb{R}^d$. Connectivity properties are determined by **density** of underlying Poisson process.
First studied by Gilbert 1961.

Generalizing: The **Random Connection Model**

Construction:

- η is Poisson point process with intensity $\lambda > 0$.
- Add *edge* between $x, y \in \eta$ with probability $\varphi(x,y) = \varphi(|x-y|)$.

Example: Boolean percolation with fixed radius R: $\varphi(x,y) = \mathbb{1}_{|x-y|\le 2R}$.

Aim: Study this model for *critical* percolation

$$\boxed{\lambda_c := \inf\{\lambda \mid \text{the random graph has an infinite component}\}}$$

A little history:

M. Penrose '91: $0 < \lambda_c < \infty$ whenever $\int \varphi < \infty$.

R. Meester '95: $\forall \lambda < \lambda_c \; \exists c > 0 : \; \mathbb{P}_\lambda(|\mathcal{C}| > k) \le e^{-ck}$
("sharp phase transition").

R. Meester, M. Penrose, A. Sarkar '97: $\lambda_c \int \varphi \to 1$ as $d \to \infty$.

A little more formal:

$$\mathbb{R}^{[2d]} = \text{ 2-element subsets of } \mathbb{R}^d$$

Enumerate Poisson points $\eta = \{X_i \colon i \in \mathbb{N}\}$.
Interpret random connection model as point process on $\mathbb{R}^{[2d]} \times [0,1]$:

$$\xi := \big\{(\{X_i, X_j\}, U_{i,j}) \colon X_i < X_j, i,j \in \mathbb{N}\big\}.$$

Add edge between X_i and X_j whenever $U_{i,j} \le \varphi(|X_i - X_j|)$.

ξ^x is ξ with an extra point added at $x \in \mathbb{R}^d$
(together with random connections from x to all other points)

$\xi^{x,y}$ is ξ with two extra points added at $x, y \in \mathbb{R}^d$
(together with random connections to all other points)

Cluster $\mathcal{C}(x) := \{y \in \eta^x \colon x \leftrightarrow y \text{ in } \xi^x\}$

2-pt-fct $\tau(x,y) := \mathbb{P}_\lambda(x \longleftrightarrow y \text{ in } \xi^{x,y})$

Now can define:
$\theta(\lambda) = P_\lambda(|\mathcal{C}(0)| = \infty)$ $\qquad$ $\lambda_c = \inf\{\lambda \mid \theta(\lambda) > 0\}$.

Main result: An infrared bound

Theorem. (Joint work with R. van der Hofstad, G. Last, K. Matzke)
If (A) $d > 6$ sufficiently large (and φ "well-behaved")
or (B) $\alpha > 0$, $d > 3(\alpha \wedge 2)$ and $\varphi(r) = \frac{1}{(r/L)^{d+\alpha}} \vee 1$ (L suff. large)
then exists $A > 0$ such that for all $\lambda < \lambda_c$:

$$|\hat{\tau}_\lambda(k)| \leq \frac{A}{\hat{\varphi}(0) - \hat{\varphi}(k)}, \quad k \in \mathbb{R}^d \quad \text{(infrared bound)}.$$

N.B. $\hat{f}$ denotes Fourier transform of $\mathbb{R}^d$-valued function f.

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Consequences:

1. $\Delta_{\lambda_c} := \tau_{\lambda_c}^{*3}(0) = \int \hat{\tau}_{\lambda_c}(k)^3 \, dk < \infty$, the famous triangle condition.
2. No percolation at criticality: $\theta(\lambda_c) = 0$
3. Critical exponents have mean-field values,e.g. $\mathbb{E}_\lambda |\mathcal{C}(0)| \asymp (\lambda - \lambda_c)^{-1}$
4. Bounds on λ_c: $1 \leq \lambda_c \int \varphi \leq \begin{cases} 1 + O(d^{-1/4}) \text{ in case (A)} \\ 1 + O(L^{-d}) \text{ in case (B)} \end{cases}$

Proving the infrared bound using lace expansion I

2 important ingredients:

- Mecke-formula for all $f\colon (\mathbb{R}^{[2d]} \times [0,1]) \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$:

$$\mathbb{E}_\lambda\Big(\sum_{x\in\eta} f(\xi,x)\Big) = \lambda \int_{\mathbb{R}^d} \mathbb{E}_\lambda\big(f(\xi^x,x)\big)\,dx.$$

  Example: Expected cluster size

$$\mathbb{E}_\lambda\big(|\mathcal{C}(0)| \text{ in } \xi^0\big) = 1 + \mathbb{E}_\lambda\Big(\sum_{x\in\eta} \mathbb{1}_{\{0\leftrightarrow x \text{ in } \xi^0\}}\Big)$$

$$= 1 + \lambda \int_{\mathbb{R}^d} \tau_\lambda(x)\,dx = 1 + \lambda\hat{\tau}_\lambda(0).$$

- van den Berg - Kesten ("BK") inequality: Let $A \circ B$ denote *spatial disjoint occurrence* of increasing events A and B.
  Then $\mathbb{P}_\lambda(A\circ B) \leq P_\lambda(A)\,P_\lambda(B)$.

Proving the infrared bound using lace expansion II

Proof strategy: adapt lace expansion to point processes.
(Lace expansion for percolation on $\mathbb{Z}^d$ by T. Hara & G. Slade '90.)

Random connection model: $\tau(x,y) := \mathbb{P}(x \longleftrightarrow y \text{ in } \xi^{x,y})$

$$\tau(x,y) = \varphi(x,y) + \Pi(x,y) + \lambda \int ((\varphi(x,z) + \Pi(x,z))\, \tau(z,y)\, dz$$

Expansion of $\tau(x,y)$ identifies (complicated) function $\Pi(x,y)$ in above equation. This resembles completely different model:

Random walk on $\eta^{x,y}$: $G(x,y) := \sum_{n \geq 1} \mathbb{P}(x \xrightarrow{n} y \text{ in } \eta^{x,y})$ (Green's fct)

$$G(x,y) = \varphi(x,y) + \lambda \int \varphi(x,z)\ G(z,y)\, dz$$

Strategy: Show that $|\Pi(x,y)|$ sufficiently small in order to deduce that $\tau(x,y) \approx G(x,y)$. Works best in Fourier space.

Summary

Completed:

- Have derived lace expansion for random connection model.
- Analysis of lace expansion obtains triangle condition, understand critical behaviour.
- Proof works directly in continuum (no discretization).

Challenges:

- *Weight-dependent* random connection model
- Random connection model with more general point processes (beyond Poisson)