Random field induced order in two dimensions

(joint work with N. Crawford and prelude with A. van Enter)

Wioletta M. Ruszel

Mathematical Institute - Utrecht University - The Netherlands

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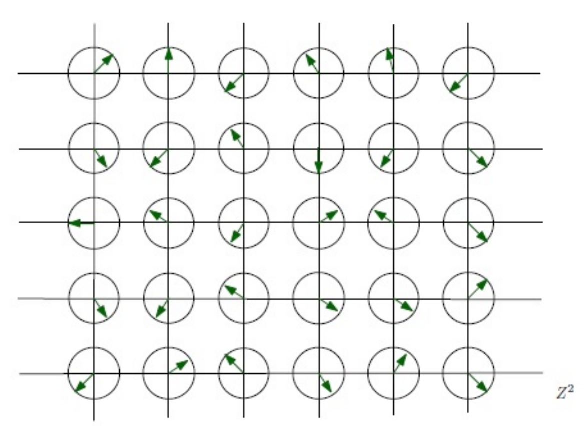
Table of contents

Introduction

Classical XY model



Introduction: classical XY model

- classical XY model (or $O(2)$ vector model) one of the simplest models with continuous symmetry
- various applications in studies related to superfluid helium, thin-films, superconductivity, liquid crystals, dielectric plasma
- Mermin-Wagner effect: no continuous symmetry breaking in $d = 2$
- XY model related to integer valued GFF

Introduction: Berezinskii-Kosterlitz-Thouless transitions

- BKT transition: between the phases with exponentially and power-law decaying correlation functions
- phases linked to delocalization of their dual integer (Fröhlich-Spencer (1981), Kharash-Peled (2017), van Engelenburg-Lis (2021), Aizenman-Harel-Peled-Shapiro (2021))
- statistical reconstruction of DGFF from F(DGFF) undergoes BKT transition (Garban-Sepulveda (2020))

Introduction: adding frozen-in disorder

- frozen-in disorder can have the potential to drastically alter some physically interesting properties of homogeneous media
- random field Ising model in $d \geq 3$: first-order phase transition persists for weak disorder (Bricmont-Kupiainen (1987), Imry-Ma (1975))
- rounding effect of disorder for $d = 2$ for the RFIM (Aizenman-Wehr (1990)), unique Gibbs state for almost all realizations of disorder
- no infinite-volume gradient measure under weak disorder (van Enter-Külske (2006), Bovier-Külske (1995), Külske (1998))

Introduction: XY with random field

- since '80ties discussion about ordering or not
- Dotsenko-Feigelman ('81+'82) gave approximative arguments for no order in $d = 1, 2$
- mean-field arguments and simulations suggesting order (Wehr-Niederberger-Sanchez-Palencia-Lewenstein (2006))
- Gibbs-non Gibbs transitions (van Enter-R. (2009))
- order for
  - Kac potentials (Crawford (2011) [2])
  - order for nearest-neighbour potential in $d = 3$ (Crawford (2014) [1])
- question: what happens for $d = 2$?

The model

The model

- spin configuration $\sigma = (\sigma_x)_{x \in \mathbb{Z}^2}$, $\sigma_x \in \mathbb{S}^1$
- random field $\alpha = (\alpha_x(\omega))_{x \in \mathbb{Z}^2}$, i.i.d. $N(0,1)$ random variables, $\omega \in \Omega$
- (random) Hamiltonian for $\Lambda \subset \mathbb{Z}^2$:

$$-\mathcal{H}_\Lambda^\omega(\sigma|\sigma^0) = -\frac{1}{2} \sum_{x \sim y; x,y \in \Lambda} (\sigma_x - \sigma_y)^2 + \epsilon \sum_{x \in \Lambda} \alpha_x(\omega) e_2 \cdot \sigma_x$$

  where σ^0 is the boundary condition, $\epsilon > 0$ and $e_1 = (1,0), e_2 = (0,1)$
- Gibbs measure

$$\mu_\Lambda^{\omega,\sigma^0}(A) = Z_\Lambda^{-1} \int_A \prod_{x \in \Lambda} \nu(d\sigma_x) \exp\left(-\beta \mathcal{H}_\Lambda^\omega(\sigma|\sigma^0)\right)$$

The model: some notation

- $\Lambda_N \subset \mathbb{Z}^2$, $|\Lambda_N| = N^2$, $L = 2^k$ for some $k \in \mathbb{N}$ and define box (block) $Q_L(z)$, $z \in L\mathbb{Z}^2$ by

$$Q_L(z) = z + \{0, 1, \cdots, L-1\}^2$$

- block average magnetization

$$M_z = \frac{1}{|Q_L|} \sum_{x \in Q_L(z)} \sigma_x$$

- for $\omega \in \Omega$ define
$\mathbb{D}_\omega = \{$union of regions for which there are too many small or large boxes with "large randomness"$\}$

Main result

Main result

Theorem (Crawford, R. 2021 [3])
Let $\xi \in (0,1)$ be sufficiently small. $\exists\ \epsilon_0(\xi) > 0$ so that for $\epsilon < \epsilon_0$ and for almost all $\omega\ \exists\ \mathbb{D}_\omega$, $N_0(\omega)$ and $\delta > 0$ such that

$$|\mathbb{D}_\omega \cap \Lambda_N| \leq Ce^{-c|\log(\epsilon)|^\delta} |\Lambda_N| \text{ for all } N \geq N_0(\omega).$$

Moreover $\exists \beta_0(\epsilon) > 0$ so that if $\beta > \beta_0$ then for each $z \in \Lambda_N$ with $Q_L(z) \cap \mathbb{D}_\omega = \emptyset$, we have

$$\left\| \mathbb{E}_{\mu^{\omega, e_1}_{\Lambda_N}}(M_z) - e_1 \right\|_2 \leq \xi.$$

Prelude: spin-flop transitions

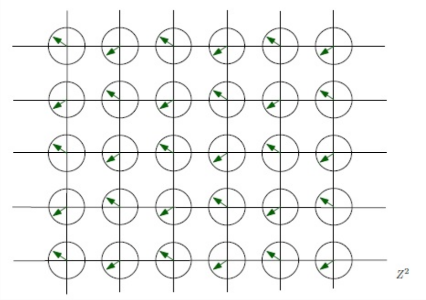
Prelude: spin-flop transitions

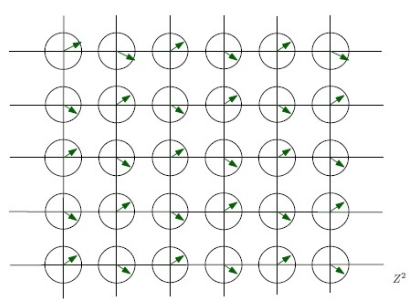
- van Enter and R. (2008) [4] studied Gibbs-non-Gibbs phase transitions for XY model on $\mathbb{Z}^2$
- (formal) Hamiltonian of the double-layer system w.r.t specific configuration:

$$-\beta \mathcal{H}^{\tau^{spec}}(\theta) = \beta J \sum_{x \sim y} \cos(\theta_x - \theta_y) + \sum_x h_t (-1)^{|x|} \sin(\theta_x)$$

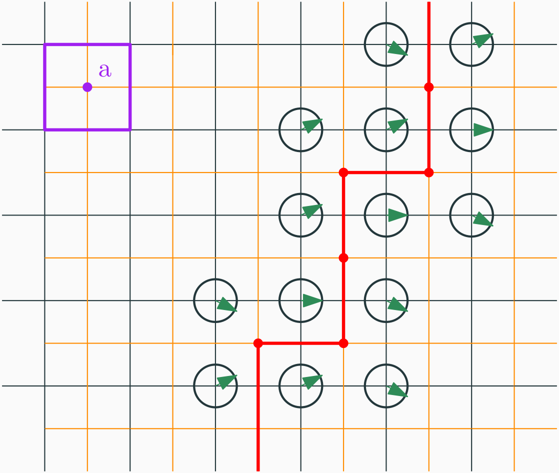
- competition: interaction term wants angles close, external field angles close to $\pm e_2$
- induced discrete symmetry and ordering perpendicular to field around $\pm e_1$ ($\rightarrow$ spin-flop transitions)

Prelude: spin-flop transitions





Prelude: low-energy clusters percolate



Prelude: low-energy clusters percolate

Ideas of the proof:

- identify each point a in the dual lattice $(\mathbb{Z}^2)^*$ with surrounding cube in $\mathbb{Z}^2$
- write: $\beta \mathcal{H}^{\tau^{spec}}(\theta) = \sum_{a \in (\mathbb{Z}^2)^*} \Phi_\beta(a, \theta)$
- for fixed $\delta > 0$ and configuration θ, look at the graph with vertex set

$$V_\delta = \{a \in (\mathbb{Z}^2)^* : \Phi_\beta(a, \theta) \leq (\inf \Phi_\beta) + \delta\}$$

- let C_δ be the "low-energy percolation cluster" on the sites of G_δ
- with probability going to 1, $\exists ! \{|C_\delta| = \infty\}$ for $\beta > \beta_c$ for at least one t.i. Gibbs measure
- discrete symmetry breaking via splitting $\{\exists ! |C_\delta| = \infty\}$ into disjoint events

Ideas of the proof

Ideas of the proof: What happens in small boxes?

Let us first look at small boxes Q_ℓ:

- Hamiltonian with free boundary condition:

$$
\begin{aligned}
-\mathcal{H}_{Q_\ell}(\sigma) &= -\frac{1}{2}\sum_{x\sim y; x,y\in Q_\ell} (\sigma_x - \sigma_y)^2 + \epsilon \sum_{x\in Q_\ell} \hat{\alpha}_x(\omega) e_2 \cdot \sigma_x \\
&= \sum_{x\sim y; x,y\in Q_\ell} (\cos(\theta_x - \theta_y) - 1) + \epsilon \sum_{x\in Q_\ell} \hat{\alpha}_x(\omega) \sin(\theta_x)
\end{aligned}
$$

  with $\hat{\alpha}_x = \alpha_x - \frac{1}{|Q_\ell|}\sum_{z\in Q_\ell} \alpha_z$

- optimize for $\theta_x = \Psi + \hat{\theta}_x$:

$$
-\mathcal{H}_{Q_\ell}(\sigma) \approx -\frac{1}{2}\sum_{x\sim y; x,y\in Q_\ell} (\hat{\theta}_x - \hat{\theta}_y)^2 + \epsilon \cos(\Psi) \sum_{x\in Q_\ell} \hat{\alpha}_x(\omega)\hat{\theta}_x + \mathcal{O}(\epsilon|\sum_{x\in Q_\ell} \alpha_x|)
$$

Ideas of the proof: ground states

- the optimal deviation is

$$\hat{\theta}_x = \cos(\Psi)\epsilon(-\Delta^{-1})\hat{\alpha}_x = \cos(\Psi)g_x^N$$

- this leads to

$$\sup_{(\theta_x)_{x\in Q_\ell}\approx(\Psi)} -\mathcal{H}_{Q_\ell}(\theta) = \frac{\epsilon^2}{2}\cos^2(\Psi)\sum_{x\in Q_\ell}\hat{\alpha}_x(-\Delta^{-1})\hat{\alpha}_x + \mathcal{O}\left(\epsilon|\sum_{x\in Q_\ell}\alpha_x|\right)$$

and $\Psi \in \{0, \pi\}$

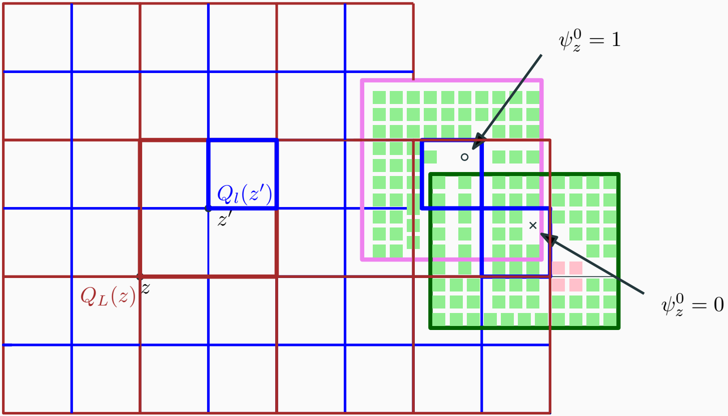
- the first term is of order $\epsilon^2\ell^2\log(\ell)$ and the error term typically $\epsilon\ell$
- energetic costly if

$$\mathcal{E}_{Q_\ell}(\sigma) = \frac{1}{2}\sum_{x\sim y;x,y\in Q_\ell}(\sigma_x - \sigma_y)^2 \geq C\epsilon^2\log(\ell)\ell^2$$

Ideas of the proof: construct contours

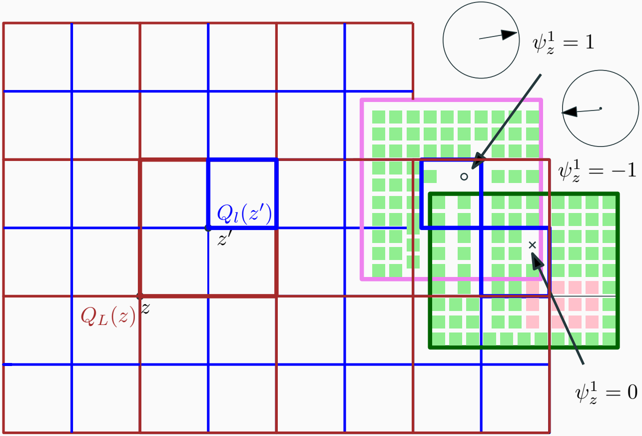
- artificial microscopic surface tension by considering two scales ℓ and L, such that $\ell \ll$ "interaction length" $\ll L$, (Presutti, Zahradnik)
- the larger L, the worse GPS bounds
- fundamental length scale: $\epsilon^{-1}|\log(\epsilon)|^{-1/2}$ (width of $0 \leftrightarrow \pi$)
- choose $\ell \sim \epsilon^{-1}|\log(\epsilon)|^{-1/2-1/64}$ and $L \sim \epsilon^{-1}|\log(\epsilon)|^{-1/2+1/64}$
- contours will be defined relative to L
- roughly a box Q_ℓ will behave bad if either the Dirichlet energy is too large or the average configuration is far from $\pm e_1$
- a contour Γ is a maximally connected union of boxes Q_L so that within $2L$ $\exists$ Q_ℓ which is bad

Ideas of the proof: constructing contours



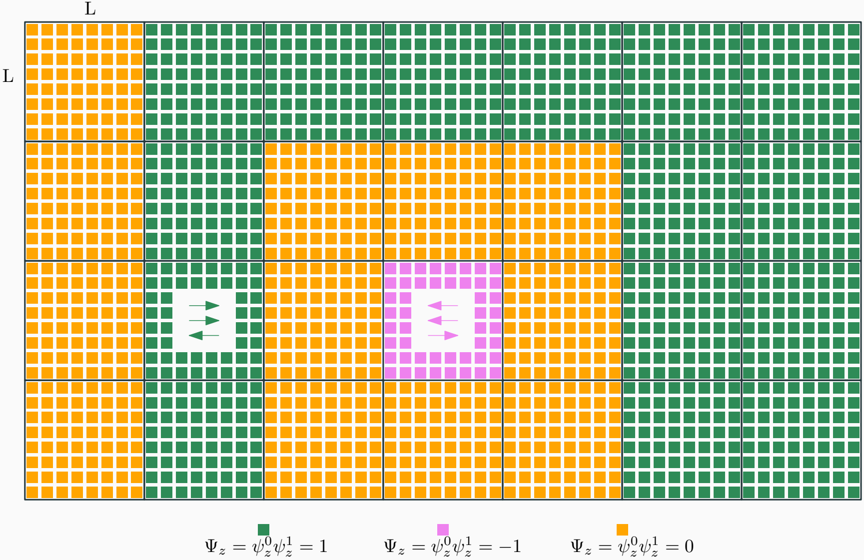
$\psi_z^0 = 1$ if in an enlarged neighbourhood the Dirichlet energy is small
$\mathcal{E}_{Q_\ell}(\sigma) \leq C\epsilon^2 |\log(\epsilon)|^{1+\chi} |Q_\ell|$

Ideas of the proof: constructing contours



$\psi_z^1 = \pm 1$ if in an enlarged neighbourhood the average orientation is close to $\pm e_1$

Ideas of the proof: constructing contours



contour Γ, union of squares on scale L separates $\Psi = 1$ from $\Psi = -1$

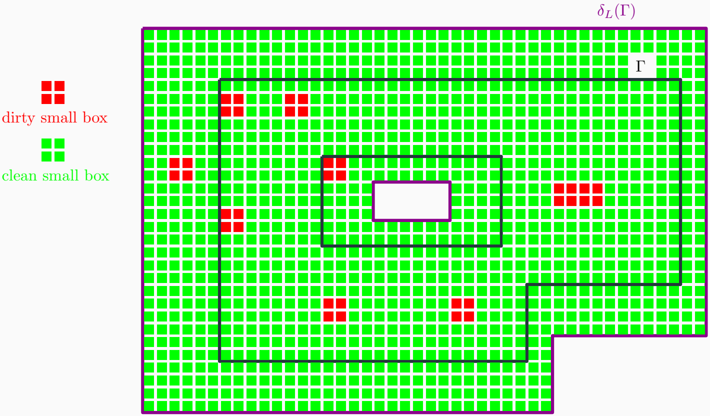
Ideas of the proof: construct new configuration $\overline{\sigma}$ from σ

- regular regions: if the thickened region $\delta(R)$ has not too many boxes with randomness which behaves atypical (dirty boxes Q_{L_0})
- show that

$$\begin{aligned}&\mathbb{P}(\Gamma \text{ is not regular})\\ &\leq C \exp\left(-|\log(\epsilon)|^{\delta} \sharp|\text{boxes on scale } L_0 \text{ which cover } \Gamma|\right)\end{aligned}$$

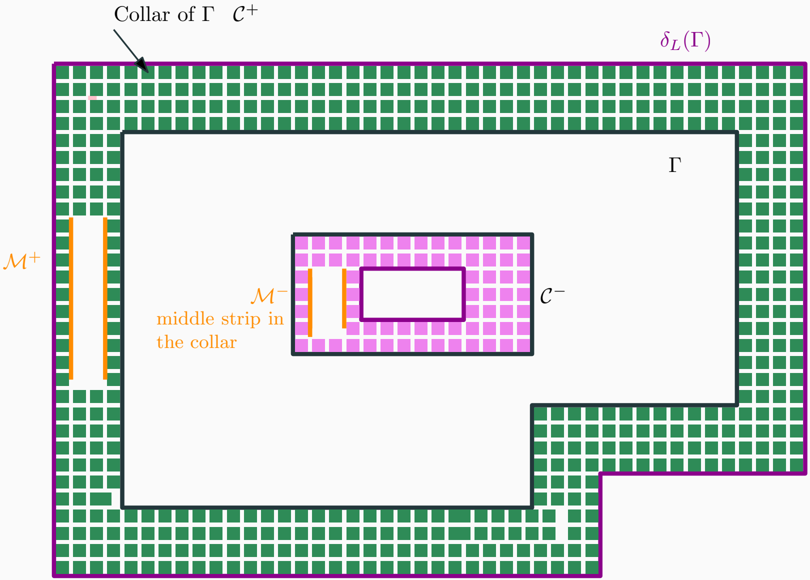
- on regular contours Γ, construct $\overline{\sigma}$ from σ:
  - on $\Lambda_N \setminus \delta(\Gamma)$ set $\overline{\sigma} = \sigma$ or the reflection of σ on e_2
  - on $\delta(\Gamma)$ do surgery and interpolate smoothly to σ
- regular contours can contain dirty boxes but energy contribution small
- Peierls argument: large contours are unlikely and surgery provides energy gain

Ideas of the proof: bulk of a contour



- on $\delta_L(\Gamma) \setminus \Gamma$ the cubes Q_ℓ are good and Ψ of constant sign
- for dirty boxes set $\overline{\sigma}_x = e_1$ and for clean boxes $\overline{\sigma}_x = (\cos(g^{\lambda,D}_{x,Q_\ell}), \sin(g^{\lambda,D}_{x,Q_\ell}))$ (replace g^N by $g^{\lambda,D}$)
- energy gain: $-\mathcal{H}_{\delta_L(\Gamma)}(\overline{\sigma}) + \mathcal{H}_{\delta_L(\Gamma)}(\sigma) \geq C\epsilon^2 |\log(\epsilon)|^{1-\delta'} |\Gamma|$

Ideas of the proof: surgery on the boundary of a contour



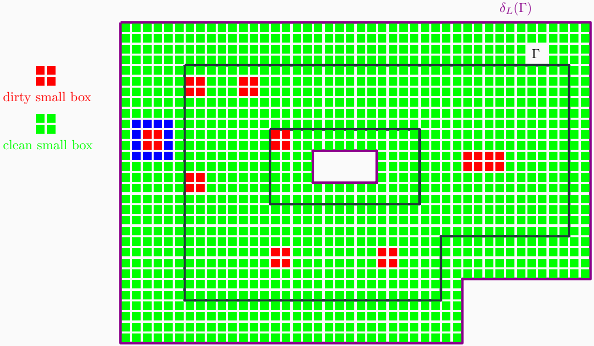
Ideas of the proof: surgery on the boundary of a contour

Modification 1:

- there exists a thickened region around $\mathcal{M}^{\pm}$ with constant sign $sign(e_1 \cdot \sigma_x)$ and such that σ is close to $\pm e_1$ on the outer boundary
- flip configuration on $\mathcal{M}^{\pm}$ if it has sign $\mp$
- energy gain: $-\mathcal{H}_{\Lambda_N}(\sigma^{(1)}) \geq -\mathcal{H}_{\Lambda_N}(\sigma)$

Ideas of the proof: surgery on the boundary of a contour

Modification 2:



- $\sigma_x^{(2)} = \sigma_x^{(1)}$ outside thickened region around dirty boxes
- $\sigma_x^{(2)} =$ gradient descent in blue region to $\sigma_x^{(1)}$
- $\sigma_x^{(2)} = \sigma_x^{(1)}$ otherwise
- energy gain: $| - \mathcal{H}_{\Lambda_N}(\sigma^{(2)}|e_1) + \mathcal{H}_{\Lambda_N}(\sigma^{(1)}|e_1)| \leq \epsilon^2 |\log(\epsilon)|^{\delta''} |\Gamma|$

Ideas of the proof: approximate Hamiltonian

- replace g_x^N by the massive field

$$g_{x,Q}^{\lambda,D} = \epsilon(-\Delta_Q^D + \lambda)^{-1}\alpha_x$$

- perform the c.o.v. $\phi_x = \theta_x - \cos(\theta_x) g_{x,Q}^{\lambda,D}$ and obtain

$$-\mathcal{K}_R(\phi|\tau) = \sum_{x\sim y} \cos(\phi_x - \phi_y) - 1 + \frac{1}{4}\sum_x m_x \cos^2(\phi_x)$$

and $m_x = \sum_{y\sim x}(\nabla_e g_Q^{\lambda,D})^2 \sim \epsilon^2|\log(\epsilon)|$

- then we have

$$-\mathcal{H}_R(\phi|\tau) \approx -\mathcal{K}_R(\phi|\tau)$$

- error small if $\mathcal{E}_R \leq C\epsilon^2|\log(\epsilon)||R|$ and α_x is typical inside R
- need energy cost of average angle to be of same order as before, choose $\lambda = \epsilon^2|\log(\epsilon)|^{1+\eta}$

Ideas of the proof: ground states for auxiliary Hamiltonian

- think of maximizers of $-\mathcal{K}$ that satisfy

$$-\sum_{y\sim x} C_{xy}(\phi_x - \phi_y) + V_x\phi_x = 0$$

with $C_{xy} = \frac{\sin(\phi_x - \phi_y)}{\phi_x - \phi_y}$ and $V_x = -\frac{\sin(\phi_x)\cos(\phi_x)m_x}{2\phi_x}$

- we can write the above equation as

$$(-L_C + V)\phi = 0$$

and interpret it as discrete elliptic PDE with random mass

- prove that maximizer is unique and uniformly close to $\pm e_1$ for points inside boundary layer $\mathcal{O}(L)$

Ideas of the proof: surgery on the boundary of a contour

Modification 3:

- construct $\sigma^{(3)}$ on $\mathcal{M}^{\pm} \setminus D^{\pm}$ by replacing $\sigma^{(2)}$ with optimizer of $-\mathcal{K}$
- angle in bulk configuration is inversion of c.o.v. of minimizers
- energy gain: $|-\mathcal{H}_{\Lambda_N}(\sigma^{(3)}|e_1) + \mathcal{H}_{\Lambda_N}(\sigma^{(2)}|e_1)| \leq C\epsilon^2 |\log(\epsilon)|^{\delta''} |\Gamma|$

Modification 4:

match $\sigma^{(3)}$ with $\overline{\sigma}$ inside the "middle strip", by forcing $\sigma^{(3)}$ towards $\pm e_1$

Glue together: flip interior components of Γ

Ideas of the proof: Peierls argument

- after all modifications we obtain

$$-\mathcal{H}_{\Lambda_N}(S(\sigma)|e_1) + \mathcal{H}_{\Lambda_N}(\sigma|e_1) \geq C\epsilon^2 |\log(\epsilon)|^{1-\delta}$$

- then

$$\begin{aligned}
&\mu_{\Lambda_N}^{\omega,e_1}(\Psi_x(\sigma) \neq 1) \\
&\leq \sum_{\Gamma} 1(\Gamma\text{regular}) \sum_{(sp(\Gamma),\Psi(\Gamma))} \mu_{\Lambda_N}^{\omega,e_1}(\Psi_x(\sigma) \neq 1, \Gamma \text{ is largest surr. } Q_L) \\
&\leq C \sum_{r=1}^{\infty} C_1^r C_2^{rL^2/\ell^2} e^{-c(\epsilon)\beta} < e^{-\frac{1}{2}c(\epsilon)\beta} \text{ for } \beta \gg 1
\end{aligned}$$

where C_1^r is the upper bound on $\sharp$ connected sets Γ surrounding Q_L with r boxes of size L^2 and $C_2^{|\Gamma|/\ell^2}$ is $\sharp$ of $\Psi(\Gamma)$ for given $\delta(\Gamma)$

Discussion

- skeleton of contour argument robust
- implementation of modifications differs from $d = 3$ (patching of almost ground states)
- random field behaves worse in $d = 2$, new bounds were needed
- possible extensions are for example considering general boundary conditions or $O(N)$ spins
- interpretation in terms of connections to IVGFF?

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Thank you for your attention!