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Disagreement coupling of finite Gibbs processes

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1. Gibbs processes

Setting

1. $(\mathbb{X}, \mathcal{X})$ is a Polish space equipped with a locally finite measure λ.
2. $\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$ is the set of all locally finite counting measures on $\mathbb{X}$ equipped with the standard σ-field $\mathcal{N}(\mathbb{X}) \equiv \mathcal{N}$.
3. The restriction of a measure ν on $\mathbb{X}$ to a set $B \in \mathcal{X}$ is denoted by $\nu_B := \nu(B \cap \cdot)$.
4. A point process is a random element ξ of $\mathbf{N}(\mathbb{X})$ defined over a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
5. For a locally finite measure ν on $\mathbb{X}$ let Π_ν denote the distribution of a Poisson process with intensity measure ν.

Setting

$\kappa\colon \mathbb{X} \times \mathbf{N} \to [0, \infty)$ is a measurable function.

Definition

A point process ξ on $\mathbb{X}$ is a Gibbs process with Papangelou intensity κ if

$$\mathbb{E} \int f(x, \xi)\, \xi(dx) = \mathbb{E} \int f(x, \xi + \delta_x) \kappa(x, \xi)\, \lambda(dx),$$

for each measurable $f\colon \mathbb{X} \times \mathbf{N} \to [0, \infty)$.

Definition

For $m \in \mathbb{N}$ define $\kappa_m \colon \mathbf{N} \times \mathbb{X}^m$ by

$$\kappa_m(x_1, \dots, x_m, \xi) \\ := \kappa(x_1, \xi)\kappa(x_2, \xi + \delta_{x_1}) \cdots \kappa(x_m, \xi + \delta_{x_1} + \cdots + \delta_{x_{m-1}}).$$

Theorem

Suppose that ξ is a Gibbs process with Papangelou intensity κ and let $m \in \mathbb{N}$. Then

$$\mathbb{E} \int f(\mathbf{x}, \xi)\, \xi^{(m)}(d\mathbf{x}) \\ = \mathbb{E} \int f(\mathbf{x}, \xi + \delta_{x_1} + \cdots + \delta_{x_m})\kappa_m(\mathbf{x}, \xi)\, \lambda^m(d\mathbf{x}),$$

for each measurable $f \colon \mathbb{X}^m \times \mathbf{N} \to [0, \infty)$. Here $\xi^{(m)}$ is the m-th factorial measure of ξ.

Definition

Let ξ be a Gibbs process with PI κ. Let $m \in \mathbb{N}$. Then

$$\rho_m(x_1, \ldots, x_m) := \mathbb{E}\kappa_m(x_1, \ldots, x_m, \xi)$$

is the m-th correlation function of ξ.

Definition

The Hamiltonian $H\colon \mathbf{N} \times \mathbf{N} \to (-\infty, \infty]$ is defined by

$$H(\psi, \mu) := \begin{cases} 0, & \text{if } \psi(\mathbb{X}) = 0, \\ -\log \kappa_m(x_1, \cdots, x_m, \mu), & \text{if } \psi = \delta_{x_1} + \cdots + \delta_{x_m}, \\ \infty, & \text{if } \psi(\mathbb{X}) = \infty. \end{cases}$$

For $B \in \mathcal{X}_b$ the partition function $Z_B\colon \mathbf{N} \to (0, \infty]$ is defined by

$$Z_B(\mu) := \int e^{-H(\psi,\mu)} \, \Pi_{\lambda_B}(d\psi), \quad \mu \in \mathbf{N}.$$

Theorem (Nguyen, Zessin '79, Matthes, Warmuth, Mecke '79)

Suppose that ξ is a Gibbs process with Papangelou intensity κ. Let $B \in \mathcal{X}_b$. Then

$$\mathbb{P}(Z_B(\xi_{B^c}) < \infty) = 1$$

and, for each measurable $f\colon \mathbf{N} \to [0, \infty)$, we have the DLR-equations

$$\mathbb{E}[f(\xi_B) \mid \xi_{B^c}] = Z_B(\xi_{B^c})^{-1} \int f(\psi) e^{-H(\psi,\xi_{B^c})} \Pi_{\lambda_B}(d\psi).$$

2. Interaction potentials

Setting

$U\colon \mathbf{N}_{<\infty} \to (-\infty, \infty]$ is a measurable function with $U(0) = 0$, where $\mathbf{N}_{<\infty} := \{\psi \in \mathbf{N} : \psi(\mathbb{X}) < \infty\}$.

Definition

Let $\mu \in \mathbf{N}$ and $B \in \mathcal{X}_b$. Define

$$\mathbf{N}_B(\mu) := \{\psi \in \mathbf{N}_{<\infty} : \psi \leq \mu, \psi(B) > 0\},$$
$$E_B(\mu) := \sum_{\psi \in \mathbf{N}_B(\mu)} U(\psi),$$

whenever the sum of the associated negative parts is finite. Otherwise set $E_B(\mu) := -\infty$.

Definition

For a given $B \in \mathcal{X}_b$ define the partition function $Z_B \colon \mathbf{N} \to [0, \infty]$ by

$$Z_B(\mu) := \int e^{-E_B(\psi+\mu)} \Pi_{\lambda_B}(d\psi), \quad \mu \in \mathbf{N}.$$

Definition

A Gibbs process with interaction potential U and reference measure λ is a point process ξ satisfying, for each $B \in \mathcal{X}_b$,

$$\mathbb{P}(Z_B(\xi_{B^c}) < \infty) = 1$$

and

$$\mathbb{P}(\xi_B \in \cdot \mid \xi_{B^c}) = Z_B(\xi_{B^c})^{-1} \int \mathbf{1}\{\psi \in \cdot\} e^{-E_B(\psi+\xi_{B^c})} \Pi_{\lambda_B}(d\psi).$$

Theorem (Nguyen, Zessin '79)

Suppose that ξ is a Gibbs process with interaction potential U. Then ξ is a Gibbs process with Papangelou intensity κ, given by

$$\kappa(x,\mu) = \exp[-E_{\{x\}}(\mu + \delta_x)], \quad (\mu, x) \in \mathbf{N} \times \mathbb{X}.$$

Example

Assume that U is a pair potential, that is $U(\psi) = 0$ if $\psi(\mathbb{X}) \neq 2$. Then

$$\kappa(\mu, x) = \exp\left[-\int U(\delta_x + \delta_y)\, \mu(dy) \right], \quad (\mu, x) \in \mathbf{N} \times \mathbb{X}.$$

3. Poisson thinning

Setting

1. λ is a diffuse and finite measure on $\mathbb{X}$.
2. $\kappa \colon \mathbb{X} \times \mathbf{N} \to \mathbb{R}_+$ is a measurable function satisfying for all $(x, y, \mu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbf{N}$ the cocycle condition
$$\kappa(x, \mu)\kappa(y, \mu + \delta_x) = \kappa(y, \mu)\kappa(x, \mu + \delta_y).$$
3. Write $x \leq y$ if $\phi(x) \leq \phi(y)$, where $\phi \colon \mathbb{X} \to I$ is a Borel isomorphism between $\mathbb{X}$ and a Borel subset I of $\mathbb{R}$.
4. For $B \in \mathcal{X}$ let $Z_B \colon \mathbf{N} \to (0, \infty]$ be the partition function defined w.r.t. κ and Π_λ.

Definition

Define

$$p(x,\psi) := \kappa(x,\psi_{(-\infty,x)})\frac{Z_{(x,\infty)}(\psi_{(-\infty,x)}+\delta_x)}{Z_{(x,\infty)}(\psi_{(-\infty,x)})}, \quad (x,\psi)\in\mathbb{X}\times\mathbf{N}.$$

Definition

κ is said to be stable if there exists a measurable $\alpha\colon\mathbb{X}\to[0,\infty)$ such that $\int\alpha\,d\lambda<\infty$ and

$$\kappa(x,\psi)\le\alpha(x), \quad (x,\psi)\in\mathbb{X}\times\mathbf{N}.$$

Lemma

If κ is stable, then $p(x,\psi)\le\alpha(x)$ for all $(x,\psi)\in\mathbb{X}\times\mathbf{N}$.

Theorem (Hofer-Temmel, Houdebert '19, L. and Otto '21)

Assume that κ is stable. Then the probability measure

$$\int \sum_{\psi \le \varphi} \mathbf{1}\{\psi \in \cdot\} \prod_{x \in \psi} \alpha(x)^{-1} p(x, \psi) \prod_{x \in \mu - \psi} \left(1 - \alpha(x)^{-1} p(x, \psi)\right) \Pi_{\alpha\mu}(d\varphi)$$

is the distribution of a Gibbs process with PI κ.

4. Poisson embedding

Setting

1. λ is a diffuse and finite measure on $\mathbb{X}$.
2. $\kappa\colon \mathbb{X} \times \mathbf{N} \to \mathbb{R}_+$ satisfies the cocycle condition.
3. Φ is a Poisson process on $\mathbb{X} \times \mathbb{R}_+$ with intensity measure $\lambda \otimes \mathrm{Leb}$.

Goal

Represent a Gibbs process with PI κ as $T(\Phi)$ for a suitable (embedding) mapping T.

Algorithm

1. Let $\mathbf{N}^*$ be a suitable space of simple counting measures ψ on $\mathbb{X} \times \mathbb{R}_+$ such that $\mathbb{P}(\Phi \in \mathbf{N}^*) = 1$.
2. For $\psi \in \mathbf{N}^*$ set

   $$x_1(\psi) := \min\{x \in \mathbb{X} : \text{there ex. } t \geq 0 \text{ s. t. } (x,t) \in \psi \text{ and } t \leq p(x,0)\}.$$

3. Define inductively,

   $$x_{n+1}(\psi) := \min\{x > x_n(\psi) : \text{there ex. } t \geq 0 \text{ s.t. } (x,t) \in \psi \text{ and } t \leq p(x, \delta_{x_1(\psi)} + \cdots + \delta_{x_n(\psi)})\}.$$

4. Set $\tau(\psi) := \sup\{n \geq 1 : x_n(\psi) \in \mathbb{X}\}$ and

   $$T(\psi) := \mathbf{1}\{\tau(\psi) < \infty\} \sum_{n=1}^{\tau(\psi)} \delta_{x_n(\psi)}.$$

Theorem (L. and Otto '21)

Assume that $Z_B(\psi_{B^c}) < \infty$ holds for Π_λ-a.e. ψ simultaneously for all $B \in \mathcal{X}$. Then $T(\Phi)$ is a Gibbs process with PI κ.

5. Disagreement coupling

Remark

Let ξ be a Gibbs process with PI κ and let $W \subset \mathbb{X}$ be a bounded Borel set. Let $\psi \in \mathbf{N}$ and define

$$\kappa_\psi(x, \mu) := \kappa(x, \psi + \mu), \quad (x, \mu) \in \mathbb{X} \times \mathbf{N}.$$

Let $\kappa_{B,\psi}$ denote the restriction of κ_ψ to $B \times \mathbf{N}(W)$. The conditional distribution $\mathbb{P}(\xi_W \in \cdot \mid \xi_{W^c})$ is almost surely a Gibbs process with PI $\kappa_{W,\xi_{W^c}}$.

Setting

- λ is diffuse and κ is a PI bounded by some $\alpha \geq 0$.
- $\sim$ is a symmetric relation on $\mathbb{X}$ such that $\{(x, y) : x \sim y\}$ is a measurable subset of $\mathbb{X}^2$.
- κ localizes w.r.t. $\sim$, that is

$$\kappa(x, \mu) = \kappa(x, C(x, \mu)), \quad (x, \mu) \in \mathbb{X} \times \mathbf{N},$$

where $C(x, \mu)$ are all points from μ which are connected via $(\mu, \sim)$ to x.

Theorem (van den Berg, Maes ’94, Hofer-Temmel, Houdebert ’19, L. and Otto ’21)

Let the preceding assumptions be satisfied, let $W \in \mathcal{X}$ be bounded and let $\psi, \psi' \in \mathbf{N}(W^c)$. Then there exist point processes ξ, ξ' and η on W (defined on the same probability space) with the following properties.

- *ξ is a Gibbs process with PI $\kappa_{W,\psi}$ and ξ' is a Gibbs process with PI $\kappa_{W,\psi'}$.*
- *η is a Poisson process with intensity measure $\alpha\lambda_W$.*
- *Every point in $|\xi - \xi'|$ is connected via $\xi + \xi'$ to $\psi + \psi'$.*
- *The support of $\xi + \xi'$ is contained in the support of η.*

6. Decorrelation in a subcritical regime

Setting

- λ is diffuse and κ is a PI bounded by some $\alpha > 0$.
- $\sim$ is a symmetric measurable relation on $\mathbb{X}$.
- κ localizes w.r.t. $\sim$.
- $w \colon [0, \infty) \to [0, \infty)$ is a continuous decreasing function with $\lim_{r \to \infty} w(r) = 0$.

Definition (Blaszczyszyn, Yogeshwaran and Yukich '19)

A point process ξ with correlation functions ρ_n, $n \in \mathbb{N}$, *w-decorrelates* if there exist for all $k, m \in \mathbb{N}$ a $c_{k,m} \geq 0$ such that

$$\begin{aligned} &\rho_{k+m}(x_1, \dots, x_{k+m}) - \rho_k(x_1, \dots, x_k)\rho_m(x_{k+1}, \dots, x_{k+m}) \\ &\leq c(k,m) \cdot w(d(\{x_1, \dots, x_k\}, \{x_{k+1}, \dots, x_{k+m}\})) \end{aligned}$$

for λ^{k+m}-a.e. $(x_1, \dots, x_{k+m})$.

Theorem (Benes et. al. '19, Betsch and L. '22)

Let ξ be a Gibbs process with PI κ bounded by $\alpha \geq 0$. Assume that for all bounded $B, W \in \mathcal{X}$ with $B \subset W$

$$\Pi_{\alpha\lambda}(\nu \in \mathbf{N} : C(x, \nu + \mu)(W^c) > 0) \leq w(d(B, W^c))$$

for λ-a.e. $x \in B$ and all finite $\mu \in \mathbf{N}$ with $\mu(W) = 0$. Then ξ w-decorrelates with $c(k, m) := 2\alpha^{k+m} \min\{k, m\}$.

Remark

Under the assumption of the theorem it can be proved that there exists exactly one Gibbs process with PI κ.

Definition

Let ξ be a point process on $\mathbb{X}$ and $s, t, r \geq 0$. The β-mixing coefficient of a point process ξ is defined by

$$m_{s,t}(r) := \sup_{B,C} \|\mathbb{P}_{(\xi_B,\xi_C)} - \mathbb{P}_{\xi_B} \otimes \mathbb{P}_{\xi_C}\|_{\mathrm{TV}},$$

where the supremum is taken over all bounded $B, C \in \mathcal{X}$, such that $\lambda(B) \leq s$, $\lambda(C) \leq t$ and $\mathrm{dist}(B, C) > r$.

Theorem (Betsch and L. '22)

Under the assumptions of the previous theorem,

$$m_{s,t}(r) \leq 4\alpha e^{2\alpha(s+t)} \min\{s, t\} w(r), \quad r, s, t \geq 0.$$

Lemma (Poinas '19, Betsch and L. '22)

Let ξ be a point process admitting correlation functions ρ_n, $n \in \mathbb{N}$. Let $B, C \in \mathcal{X}$ be bounded and disjoint and assume that $\mathbb{E}3^{\xi(B \cup C)} < \infty$. Then

$$\begin{aligned} &\|\mathbb{P}_{(\xi_B,\xi_C)} - \mathbb{P}_{\xi_B} \otimes \mathbb{P}_{\xi_C}\|_{\mathrm{TV}} \\ &\quad \le \sum_{k,m=1}^{\infty} \frac{2^{k+m}}{k!m!} \int_{B^k \times C^m} |\rho_{k+m}(\mathbf{x},\mathbf{y}) - \rho_k(\mathbf{x})\rho_m(\mathbf{y})| \lambda^{k+m}(d(\mathbf{x},\mathbf{y})). \end{aligned}$$

Example

Let $\mathbb{X}$ be space of all non-empty compact subsets of a Polish space equipped with the Hausdorff metric. Define the relation $\sim$ by $K \sim L$ if $K \cap L = \emptyset$. Assume that ξ is Gibbs with a PI κ which is bounded and localizes w.r.t. $\sim$. Let λ be a diffuse measure on $\mathbb{X}$ such that the diameter of K is smaller than some constant for λ-a.e. K. If the Boolean model with intensity measure λ does not percolate, then ξ is decorrelating and β-mixing.

8. A uniqueness result for repulsive pair potentials

Setting

- λ is locally finite and κ is a PI defined by a pair potential $U\colon \mathbb{X} \times \mathbb{X} \to [0, \infty)$.
- $\varphi\colon \mathbb{X} \times \mathbb{X} \to [0, 1]$ is defined by

$$\varphi(x, y) := 1 - e^{-U(x,y)}, \quad x, y \in \mathbb{X}.$$

- We assume that

$$\int \varphi(x, y)\, \lambda(dy) < \infty, \quad \lambda\text{-a.e. } x \in \mathbb{X}.$$

- Φ is a Poisson process with intensity measure λ.

Definition

The random connection model (RCM) Γ based on Φ is a random graph with vertices from Φ. Edges arises by connecting every pair $x, y \in \Phi$ with probability $\varphi(x, y)$, independently for different pairs.

Definition

Let $x \in \mathbb{X}$ and $\Phi_x := \Phi + \delta_x$. Then C_x denotes the cluster of a RCM based on Φ_x, interpreted as a point process on $\mathbb{X}$.

Definition

The RCM is subcritical if

$$\mathbb{P}\big(C_x(\mathbb{X}) < \infty\big) = 1, \quad \lambda\text{-a.e. } x \in \mathbb{X}.$$

Theorem (Betsch and L. '21)

Assume that (U, λ) is subcritical. Then there is exactly only one distribution of a Gibbs process with pair potential U.

Idea of the proof:

- Reduce to diffuse intensity measure.
- Consider the product $\mathbb{X} \times \mathbb{M}$, where $\mathbb{M}$ is a suitable mark space equipped with a probability measure $\mathbb{Q}$.
- Given an approximation parameter $\delta > 0$ define a relation $\sim_\delta$ on $\mathbb{X} \times \mathbb{M}$ in terms of the connection function $\varphi = 1 - e^{-U}$
- Consider locally defined Gibbs processes ξ_δ which are hard core w.r.t. $\sim_\delta$, depending on boundary conditions.

- The projection ξ_δ onto $\mathbb{X}$ approximates a (locally) defined Gibbs process with PI κ with the same boundary condition.
- Consider a Poisson process Ψ on $\mathbb{X} \times \mathbb{M}$ with intensity measure $\lambda \otimes \mathbb{Q}$ and the resulting random graph based on $\sim_\delta$.
- As $\delta \to 0$ the clusters of this graph approximate the clusters of a RCM based on $\Psi(\cdot \times \mathbb{M})$ and φ (and the boundary conditions).
- Perform a disagreement coupling $(\xi_\delta, \xi'_\delta, \Psi)$, where ξ_δ, ξ'_δ are Gibbs processes with (different) boundary conditions.
- Project the coupling on $\mathbb{X}$ and let $\delta \to 0$, such that the dependence on the boundary conditions is controlled by the clusters of a RCM driven by $\Psi(\cdot \times \mathbb{M})$ and φ.
- Use the DLR-equations to localize two (infinite) Gibbs processes with pair potential *U* and show that their distributions coincide on local events.

8. References

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Thank You!