Hidden phase transitions in thinned point clouds

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Outline

Setup and background.

Thinnings of points clouds in continuous and discrete space

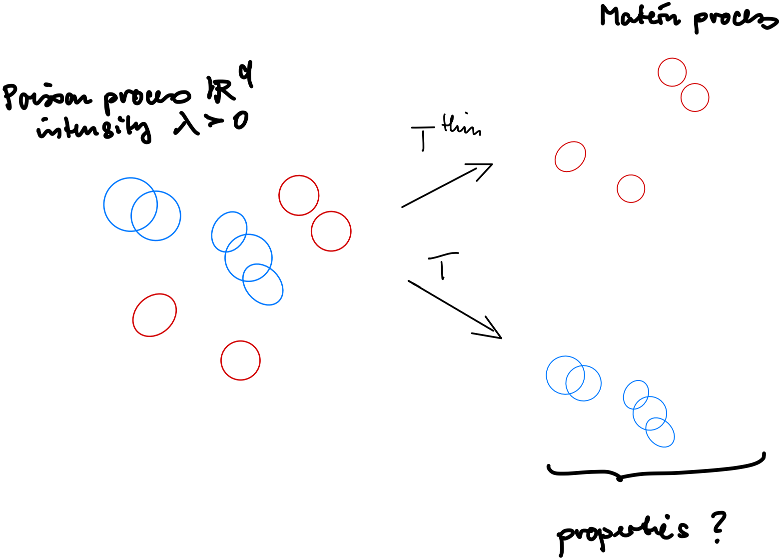
Generalities about specifications

Results on discrete thinnings, overview

Hidden phase transition and consequences for projection of Bernoulli lattice field to non-isolates - proof

What to expect in continuum?

Motivation: Thinnings of Poisson Point Process on $\mathbb{R}^d$

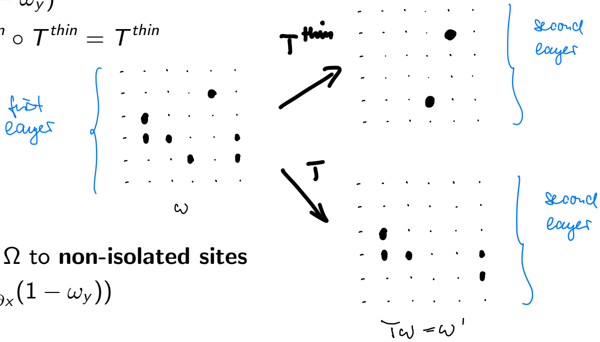


Discrete point clouds: The Bernoulli lattice field on $\mathbb{Z}^d$

$\Omega = \{0,1\}^{\mathbb{Z}^d}$ with product sigma-algebra $\mathcal{F}$
$\partial x = \{y \in \mathbb{Z}^d, \|y-x\|_1 = 1\}$ nearest neighbors
μ_p=Bern(p)$^{\otimes \mathbb{Z}^d}$ Bernoulli lattice field, site-percolation

Projection map $T^{thin} : \Omega \to \Omega$ to **isolated sites**
$(T^{thin}\omega)_x = \omega_x \prod_{y \in \partial x}(1-\omega_y)$

Projection property: $T^{thin} \circ T^{thin} = T^{thin}$



Projection map $T : \Omega \to \Omega$ to **non-isolated sites**
$(T^{thin}\omega)_x = \omega_x(1 - \prod_{y \in \partial x}(1-\omega_y))$

Definitions: (quasilocal) specification, Gibbs measure

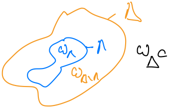
A **Specification** $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$
is a candidate system of probability kernels, for the conditional probabilities of an infinite-volume Gibbs measure μ (probability measure on Ω) to be defined by **DLR equations**

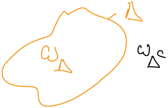
$$\mu(\gamma_\Lambda(A|\cdot)) = \mu(A) \text{ for all finite volumes } \Lambda \Subset \mathbb{Z}^d$$

Equivalent to DLR: $\mu(A|\mathcal{F}_{\Lambda^c})(\omega) = \gamma_\Lambda(A|\omega)$ μ-a.s.

Specification kernels $(\omega, A) \mapsto \gamma_\Lambda(A|\omega)$ need to satisfy

- $\mathcal{F}_{\Lambda^c}$-measurability w.r.t ω, and **properness**
- $\gamma_\Delta \gamma_\Lambda = \gamma_\Delta$ for all $\Lambda \subset \Delta \Subset \mathbb{Z}^d$





A specification γ is called **quasilocal** iff all maps

$$\omega_{\Lambda^c} \mapsto \gamma_\Lambda(\{\omega_\Lambda\}|\omega_\Lambda^c)$$

are quasilocal (continuous w.r.t product topology)

Results: nG / G for projection T to non-isolates

Theorem 1 (Non-Gibbsianness for large p)

Consider the image measure μ'_p of the Bernoulli field on $\mathbb{Z}^d$ under the map to the non-isolates in lattice dimensions $d \geq 2$.

Then, there is $p_c(d) < 1$ such that for $p \in (p_c(d), 1)$, there is **no quasilocal specification** γ' for μ'_p.

Theorem 2 (Gibbsianness for small p)

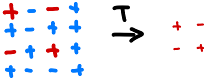
For $p < \frac{1}{2d}$ there is a quasilocal specification γ' for μ'_p.

Comments and comparison to nG via strong coupling

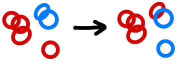
Local maps $T : (\Omega_0)^{\mathbb{Z}^d} \to (\Omega_0)^{\mathbb{Z}^d}$
can destroy the Gibbs property of a Gibbs measure μ in the image measure $T\mu$.

This was known for strongly dependent measures μ.

Renormalization group example, decimation transformation from Ising-model:
projection of a measure to sublattice $b\mathbb{Z}^d$



Other examples:
Time-evolutions, fuzzy Potts, . . .



Widom Rowlinson
under spinflip

Jahnel-K

Our example from Theorem 1 shows nG-property of $T\mu_p$,
where μ_p is even independent

Some authors: Griffiths, van Enter, Fernandez, Sokal, Maes, Haeggstrom, Schonmann, Shlosman, den Hollander, Redig, Le Ny, Verbitskiy, d'Achille, Ruszel, Iacobelli, Ermolaev, Jahnel, Kraaij, Kissel, Meissner, Henning, Bergmann, K, . . .

Results: (Failure of) continuity of conditional probabilities (Gibbs property)

Table: Bernoulli p-projections: decomposition into isolates and non-isolates

image measure	first-layer constraint model	range of p	Gibbs property of image measure	proof method
$T^{\text{thin}}\mu_p$ supported on isolated sites	non-isolation model on unfixed region	small	Gibbs	Cluster expansion
		large	Gibbs	Dobrushin uniq/disagreement perc
		mid	Gibbs?	numerical indications?
$T\mu_p$ supported on non-isolated sites	isolation model on unfixed region	small	Gibbs	Dobrushin uniq
		large	**non-Gibbs**	**hidden PT, broken transl symm**
		mid	sharp transition?	

Engler-Jahnel-K, Gibbsianness of locally thinned random fields. MPRF Vol. 28 (2022)

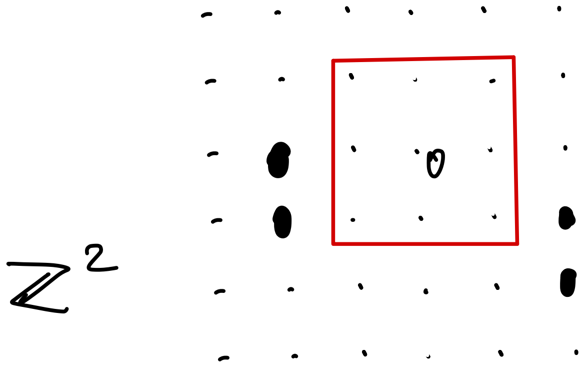
Jahnel-K, Gibbsianness and non-Gibbsianness for Bernoulli lattice fields under removal of isolated sites. arXiv:2109.13997

Proof of Theorem 1 (nG at large q on $\mathbb{Z}^d$, $d \geq 2$)

Find one non-removable point of discontinuity $\omega' \in \Omega$.

More precisely we will prove the following:

Suppose that Q is a box of side length 3 around the origin



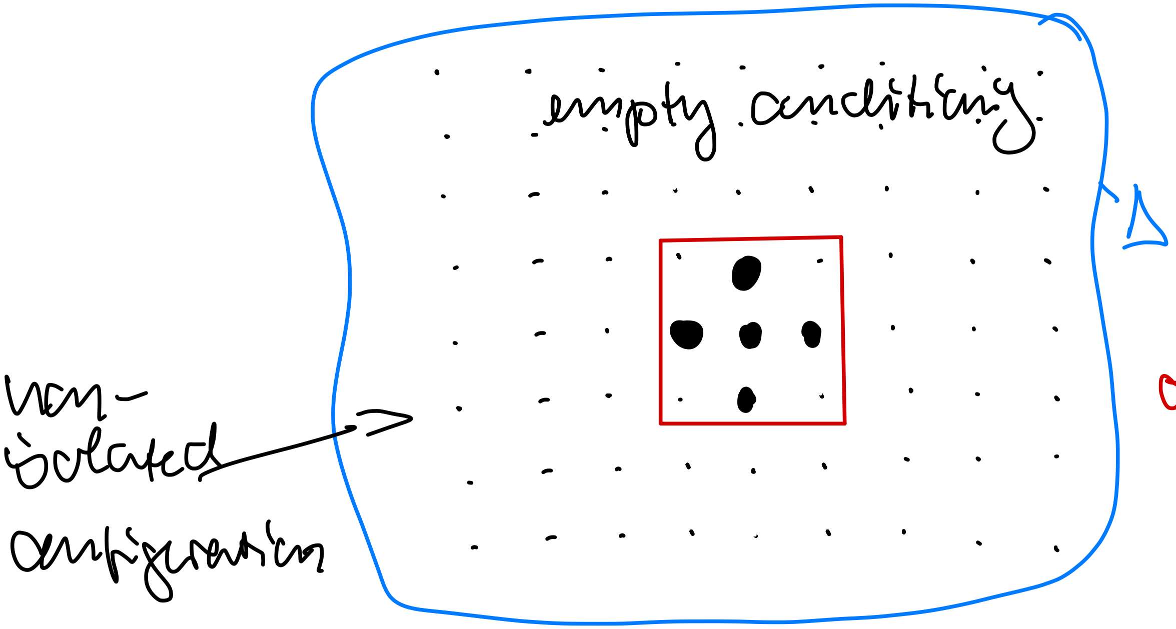
Suppose that $\gamma' = (\gamma'_\Lambda)_{\Lambda \subset\subset \mathbb{Z}^d}$ is a specification for $T\mu_p$.

Specialize to $\mathbb{Z}^2$ (only for explanation)

Claim: The map $\underset{\underset{T\Omega}{\cap}}{\omega'} \longmapsto \gamma'_Q\left(\underbrace{\begin{matrix}0&1&0\\1&1&1\\0&1&0\end{matrix}}_{\text{suitable local test-pattern}} \middle| \ \omega'_{Q^c}\right)$

non-isolated configurations

Cannot be continuous (= quasilocal) on $T\Omega$ at the fully empty conditioning $(\omega'_{Q^c})_x = 0 \quad \forall x \in Q^c$



$\omega'_{\Delta^c}, \overline{\omega}'_{\Delta^c}$ perturbations of empty conditioning outside Δ, Δ arbitrarily large, can cause jumps

To prove this Claim ("Badness of fully empty configuration"):

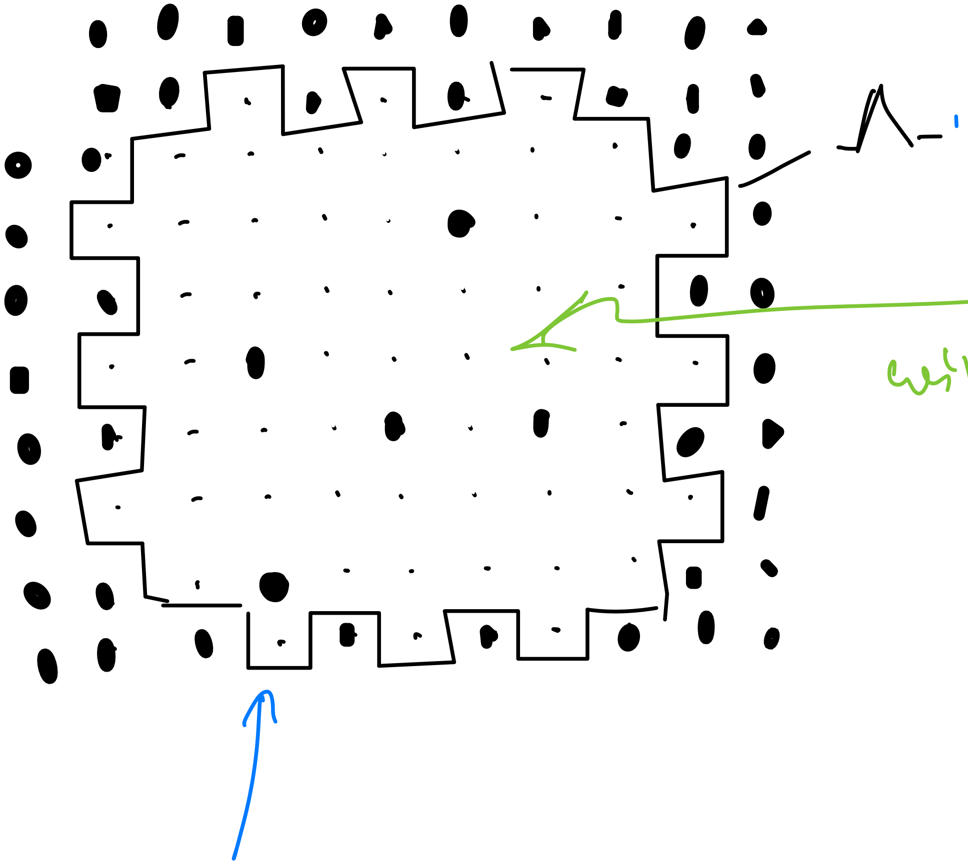
Step ① Prove Translational symmetry breaking via a Peierls argument for the conditional first layer model at bad configuration in spe:

$\mu_{p,\Lambda} := \mathrm{Ber}(p)^{\otimes \Lambda}$ Bernoulli measure in finite volume $\Lambda \subset\subset \mathbb{Z}^d$

First layer measure

$$\gamma_\Lambda(\omega_\Lambda) := \frac{\mu_{p,\Lambda}\left(\{\omega_\Lambda\} \cap \{T(\omega_\Lambda 1_{\Lambda^c})|_\Lambda = 0_\Lambda\}\right)}{\mu_{p,\Lambda}\left(T(\omega_\Lambda 1_{\Lambda^c})|_\Lambda = 0_\Lambda\right)}$$

↳ measure conditional on isolation in Λ with fully occupied boundary condition 1_{Λ^c}.

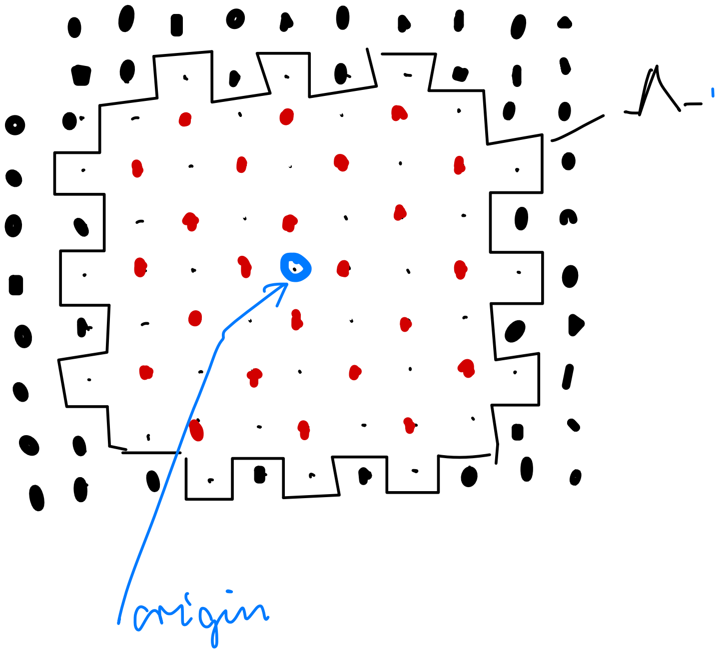


example of configuration of isolates with non-zero ν_Λ-probability

picture is typical for p small

call this shape: <u>loophole</u> boundary

Schiess scharte



maximally filled configuration :

<u>checkerboard configuration</u> ,

alternating 0 and 1 's

typical for <u>ρ very large</u>

favors many occupied sites

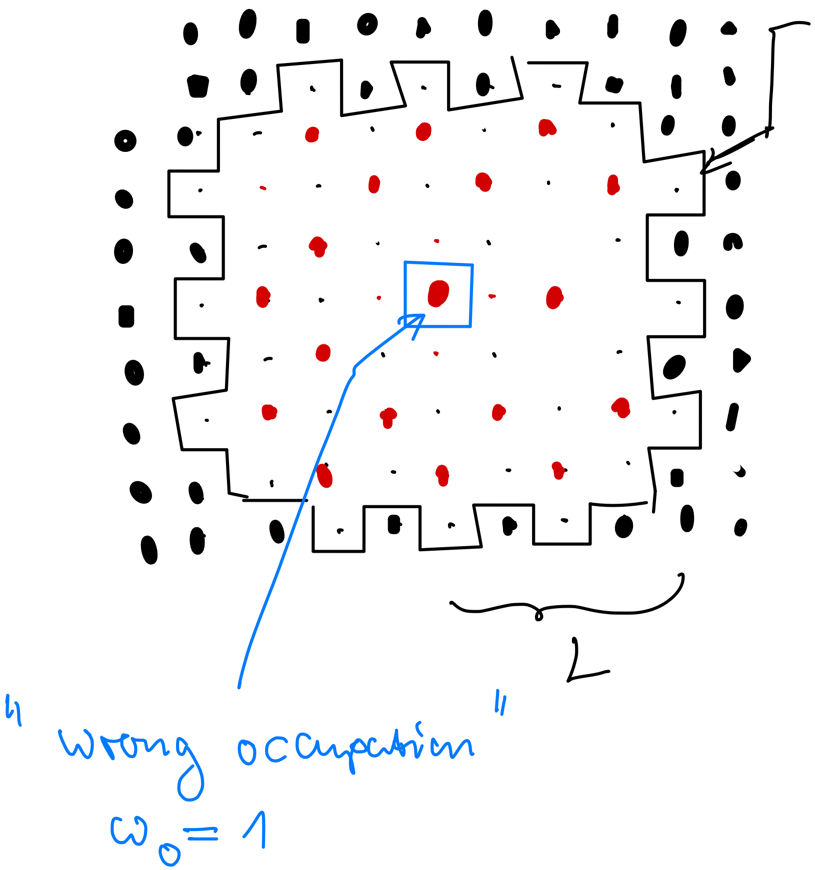
If Λ has a loophole boundary ,

and its checkerboard filling has 0 / 1

at the origin , we say Λ is

<u>a volume of type</u> 0 / 1

We will prove **uniformity** in the volume :



Proposition: Consider type-zero boxes B_L of sidelength $\sim 2L$, centered at the origin. Then

$$\sup_{L \in \mathbb{N}} \nu_{B_L}(\omega_0 = 1) \le \varepsilon(p),$$

with $\varepsilon(p) \xrightarrow[p \uparrow 1]{} 0$.

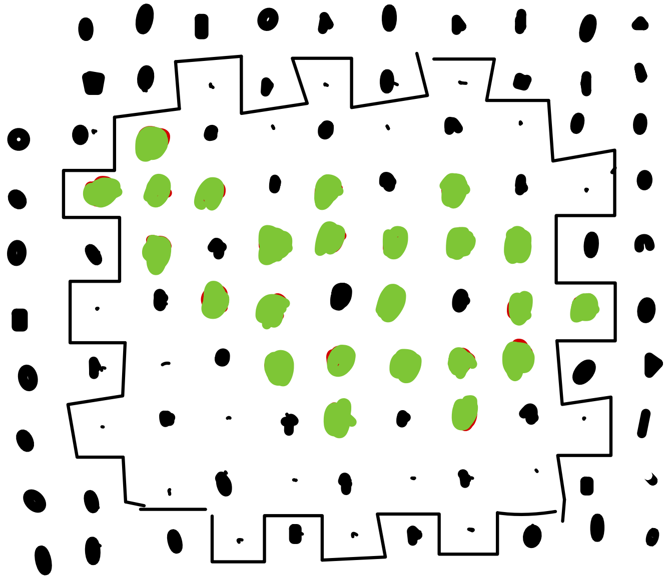
Similarly: For any lattice unit vector e:

$$\sup_{L} \gamma_{\underbrace{B_L+e}_{\text{shifted volume}}}(\omega_0=0) \leq \varepsilon(p) \quad \text{with} \quad \varepsilon(p) \xrightarrow[p\uparrow 1]{} 0$$

Proof of Proposition page 2.6

Def. For $\omega \in \{1,0\}^{\mathbb{Z}^d}$ define the set of sites

$$\Gamma(\omega) := \{ x \in \mathbb{Z}^d : \exists\, y \in \partial x \text{ such that } \omega_x = \omega_y = 0 \}$$



The connected components (in graph distance) of $\Gamma(\omega)$ are called the contours of ω.

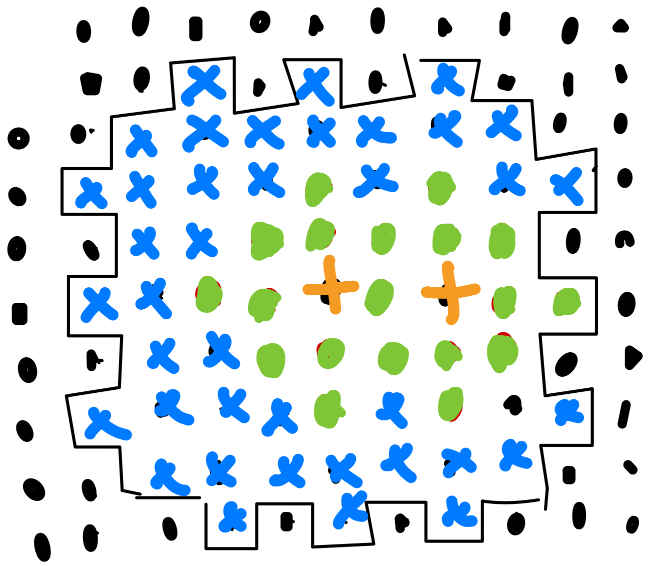
Notation for contours: γ

In the example:

two connected components γ_1, γ_2

3.2

Suppose γ is a contour in a fixed finite volume Λ. We have the volume decomposition $\Lambda = \gamma \cup V_0 \cup \bigcup_{j=1}^{k} V_j$



where V_0 is the outer connected component of $\mathbb{Z}^d \setminus \gamma$ intersected with Λ

$V_1, \dots, V_k$ are the interior connected components of γ^c

Define $\bar{\gamma} := \gamma \cup \bigcup_{j=1}^{k} V_j$

$\gamma \in \Gamma(\omega) :\Longleftrightarrow \gamma$ is a connected component of $\Gamma(\omega)$

On the way to Peierls-argument:

$$\nu_{B_L}(\omega_0 = 1)$$

$$\le \nu_{B_L}\Big(\omega \in \{0,1\}^{B_L} : \exists \gamma : \underbrace{\Gamma(\omega 1_{B_L^c})}_{\text{write shorter } =: \Gamma(\omega)} \ni \gamma \text{ and } \bar{\gamma} \ni 0\Big)$$

$$\le \sum_{\gamma : \bar{\gamma} \ni 0}' \nu_{B_L}\big(\omega : \Gamma(\omega) \ni \gamma\big)$$

Lemma (Peierls estimate)

There exists a **Peierls constant** $\tau(p) \in (0,\infty)$ with $\lim_{p \uparrow 1} \tau(p) = \infty$ such that

$$\nu_{B_L}\big(\omega : \Gamma(\omega) \ni \gamma\big) \le e^{-\tau(p)|\gamma|}$$

Difficulty: No $0 \Longleftrightarrow 1$ symmetry!

Proof : $\rho(\gamma) := (1-p)^{|\gamma|}$, γ contour

For $\omega_{\mathcal{U}} \in \{0,1\}^{\mathcal{U}}$, $\mathcal{U} \subset\subset \mathbb{Z}^d$ put

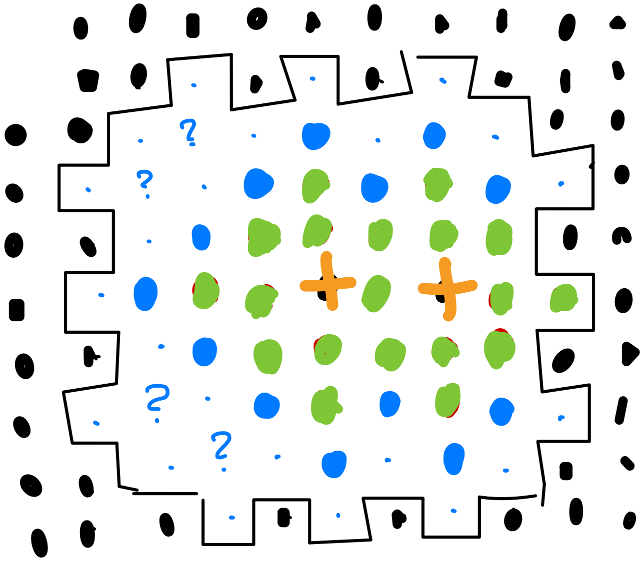
$$R(\omega_{\mathcal{U}}) := \prod_{x \in \mathcal{U}} p^{\omega_x} (1-p)^{1-\omega_x}$$ Bernoulli measure on $\mathcal{U}$

Note : $\rho(\gamma) = R(0_\gamma)$

$$\nu_{B_L}(\omega : \Gamma(\omega) \ni \gamma) = \frac{\rho(\gamma) Z_{V_0} \prod_{i=1}^{k} Z_{V_i}}{Z_{B_L}}$$ where

$$Z_{V_0} := \sum_{\underbrace{\omega_{V_0} \text{ compatible with } \gamma}_{}} R(\omega_{V_0})$$ partition function on outer component V_0.

$(\omega_{V_0} 0_\gamma 1_{B_L^c})$ obeys the isolation constraint on V_0 with boundary conditions to V_0



Note: outer boundary of γ consists of occupied ● ● ●

Similarly

$$Z_{V_i} := \sum'_{\substack{\omega_{V_i} \text{ compatible} \\ \text{with } \gamma}} R(\omega_{V_i})$$

$$Z_{B_L} := \sum'_{\substack{\omega_{B_L} \text{ compatible} \\ \text{with } 1_{(B_L)^c}}} R(\omega_{B_L}) .$$

Def. Call γ a contour of type $\left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right.$ iff

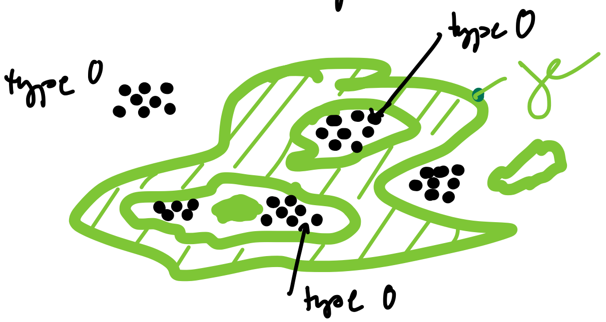
γ can be continued by the $\left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right.$ -type

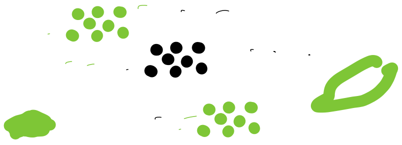
checkerboard configuration $\left\{ \begin{matrix} \omega^0 \\ \omega^1 \end{matrix} \right.$ to $\mathbb{Z}^d \setminus \bar{\gamma}$.

Assume without loss of generality γ is type 0 - contour.

<u>Case 1.</u> All interior components V_i, $i=1,\dots,k$ are also of type 0 (they can be filled with ω^0)

<u>Then</u>: $\forall \omega$ with $\Gamma(\omega) \ni \gamma$ there is the reference configuration $\omega_{\Lambda\gamma}$ ω^0_γ which also obeys the isolation constraint.



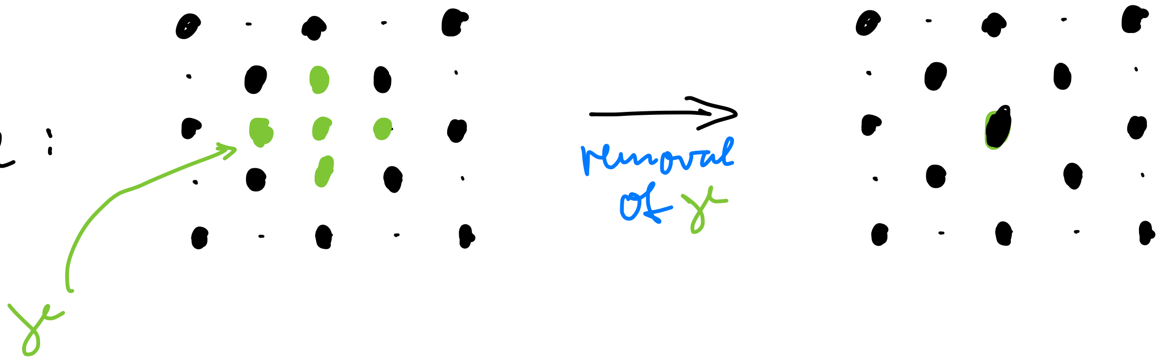


$$\Rightarrow \quad Z_{B_L} \overset{\circledast}{\geq} R(\omega^0_\gamma)\, Z_{V_0} \prod_{i=1}^{k} Z_{V_i}$$

$$g(\gamma) = R(\omega^0_\gamma)\left(\frac{1-p}{p}\right)^{N^{\text{repl}}} \quad \text{with}$$

$$N^{\underset{\uparrow \text{replacements}}{\text{repl}}} := \#\{ i \in \gamma,\ \omega^0_i = 1 \} \geq \frac{|\gamma|}{2d+1} \quad \text{(exercise)}$$

example :



$N^{\text{repl}} = 1 \geq \frac{5}{2d+1} = 1$

$$\Rightarrow \gamma_{B_L}(\omega : \Gamma(\omega) \ni \gamma) = \frac{g(\gamma)\, Z_{V_0} \prod_{i=1}^{k} Z_{V_i}}{Z_{B_L}} \overset{\circledast}{\leq} \Bigg[\underbrace{\left(\frac{1-p}{p}\right)^{\frac{1}{2d+1}}}_{=: \exp(-\tau_1(p))} \Bigg]^{|\gamma|}$$

Case 2 γ also has 'wrong' interior volumes of type 1. Denote these by $(W_j)_{j=1,\dots,\ell}$

Denote the interior volumes of type 0 by $(V_i)_{i=\ell+1,\dots,k}$

Then: $$\nu_{B_L}(\omega : \Gamma(\omega) \ni \gamma) = \frac{\rho(\gamma) Z_{V_0} \prod_{i=\ell+1}^{k} Z_{V_i} \prod_{j=1}^{\ell} Z_{W_j}}{Z_{B_L}}$$

Fix e lattice unit vector

$$\gamma_e := \Big(\gamma \setminus \bigcup_{j=1}^{\ell} (W_j + e) \Big) \cup \bigcup_{j=1}^{\ell} \Big(W_j \setminus (W_j + e) \Big)$$ replacement set of contour γ

Define for ω with $\Gamma(\omega) \ni \gamma$ the <u>reference configuration</u>

$$\Big(\omega^0_{\gamma_e} \,|\, \omega_{V_0} ,\ \omega_{\bigcup_{i=\ell+1}^{k} V_i} ,\ (\Theta_e \omega)_{\bigcup_{j=1}^{\ell} (W_j + e)} \Big), \quad (\Theta_e \omega)_i = \omega_{i-e}$$ shifted configuration

$\longleftrightarrow$ $e=1$

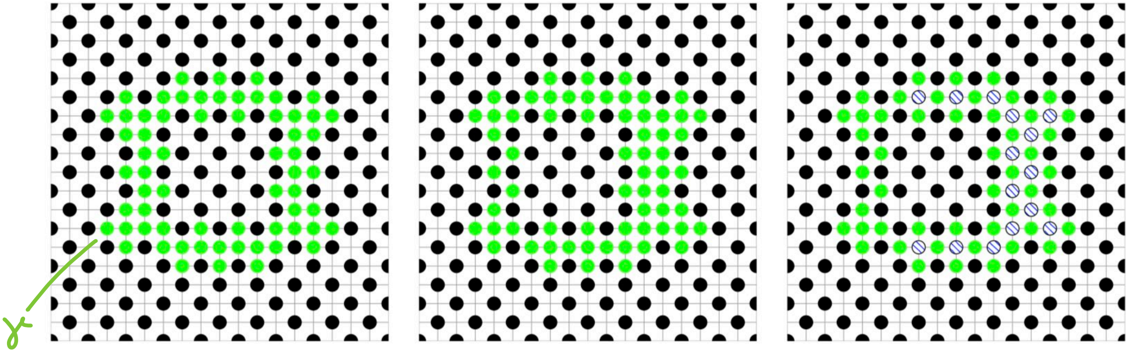


Figure 6: Illustration of a configuration with one contour γ (left in green), where the outside configuration is of type 0 and the inside configuration is of (bad) type 1. Moving the inside configuration by $-e_1$ (middle) creates a (good) configuration also inside the contour γ_e as described in (3) (middle in green), however the shift also creates isolated zeros. On the right, in the large connected component of γ_e, sites are indicated in dashed blue, which can be flipped from unoccupied to occupied, as in the configuration presented in (4), and therefore create an energetically preferable configuration.

Note: $|\gamma| = |\gamma_e|$

Denote: $S^0 \subset \mathbb{Z}^d$ set of occupied sites in type 0 - groundstate

Lemma: $\underbrace{|\gamma_e \cap S^0|}_{\text{gain of occupied sites on replacement set}} \geq c_d |\gamma|$, with $c_d > 0$

$$\Rightarrow \nu_{B_L}(\omega : \Gamma(\omega) \ni \gamma) = \frac{g(\gamma)\, Z_{v_0} \prod_{i=l+1}^{k} Z_{v_i} \prod_{j=1}^{l} Z_{w_j}}{Z_{B_L}}$$

$$\leq \frac{g(\gamma)\, Z_{v_0} \prod_{i=l+1}^{k} Z_{v_i} \prod_{j=1}^{l} Z_{w_j}}{R(\omega^0_{\gamma_e})\, Z_{v_0} \prod_{i=l+1}^{k} Z_{v_i} \prod_{j=1}^{l} Z_{w_j}} \leq \left(\frac{1-p}{p}\right)^{c_d |\gamma|}$$

$\nwarrow$ shifted components

This proves the Peierls estimate

$$\nu_{B_L}(\omega : \Gamma(\omega) \ni \gamma) \leq e^{-\tau(p)|\gamma|}$$

independently of the size of the volume B_L

with

$$e^{-\tau(p)} := \max\left\{ \left(\frac{1-p}{p}\right)^{\frac{1}{\alpha+1}}, \left(\frac{1-p}{p}\right)^{c_d} \right\} .$$

■

type 0 - loophole volume

$$\nu_{B_L}(\omega_0=1) \leq \sum_{\gamma : \bar{\gamma} \ni 0} e^{-\tau(p)|\gamma|}$$

$$\leq \sum_{n \in \mathbb{N}} \underbrace{\#\{\gamma : |\gamma| = n, \bar{\gamma} \ni 0\}}_{} e^{-\tau(p) n}$$

geometric estimate $\leq n^d C^n$, $C < \infty$

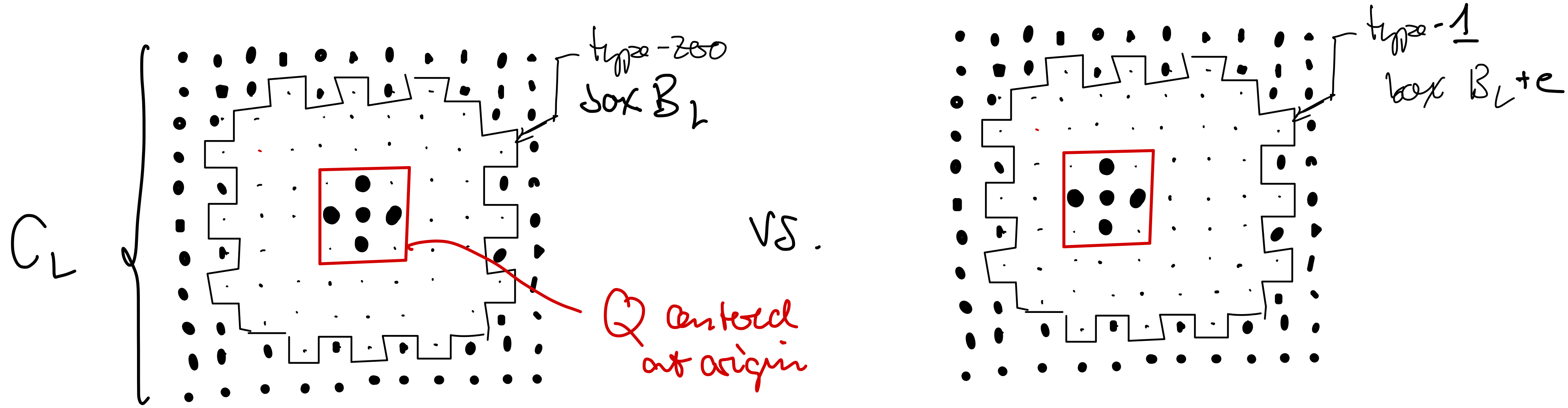
$$=: \varepsilon(p), \quad \text{finite for } \tau(p) > C$$

and $\varepsilon(p) \searrow 0$ with $p \uparrow 1$.

This proves the Proposition page 2.6 ■

Proof of Theorem 1 will show: badness of empty conditioning for any γ', $\omega_{Q^c} \mapsto \gamma'_Q\left(\substack{\bullet\\ \bullet\,\bullet\,\bullet\\ \bullet} \,\middle|\, \omega_{Q^c}\right)$

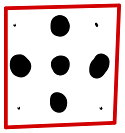
Compare on the second layer (as $L \uparrow \infty$) two conditionings:



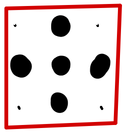
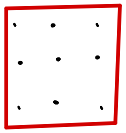
Lemma: Let B_L a type zero box. Then for $\mu'_p := T\mu_p$

$$\frac{\mu'_p\left(\boxed{\substack{\bullet\\ \bullet\,\bullet\,\bullet\\ \bullet}}' \,\middle|\, 0'_{B_L\setminus Q}\, 1'_{C_L\setminus B_L}\right)}{\mu'_p\left(\boxed{\substack{\cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot}}' \,\middle|\, 0'_{B_L\setminus Q}\, 1'_{C_L\setminus B_L}\right)} = \frac{p}{1-p}\, \nu_{B_L}\left(\boxed{\substack{\bullet\\ \bullet\ \cdot\ \bullet\\ \bullet}}\right)$$

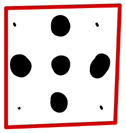
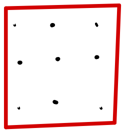
The same formula holds for the shifted volume B_L+c.

Here we write $0'$, $1'$, $'$, … for values of <u>second</u> layer configurations

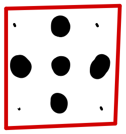
<u>Proof</u>:

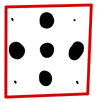
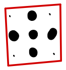
l.h.s. = $\mu_p'\Big($ $'$, $0'_{B_L\setminus Q}\,1'_{C_L\setminus B_L}\Big)$ / $\mu_p'\Big($ $'$, $0'_{B_L\setminus Q}\,1'_{C_L\setminus B_L}\Big)$

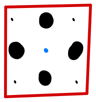
= (Def. μ_p')

$\mu_p\Big(T\sigma_{C_L} = \big($ $'$, $0'_{B_L\setminus Q}, 1'_{C_L\setminus B_L}\big)\Big)$ / $\mu_p\Big(T\sigma_{C_L} = \big($ $'$, $0'_{B_L\setminus Q}, 1'_{C_L\setminus B_L}\big)\Big)$

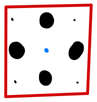
Rewrite numerator:

$\mu_p \big(T\sigma_{C_L} = \big($  $', 0'_{B_L\setminus Q}, 1'_{C_L\setminus B_L} \big) \big)$

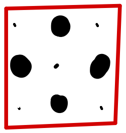
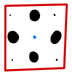
$= \mu_p(\sigma_Q =$  $)$ $\mu_{p, C_L\setminus Q}\Big(T\big($  $, \sigma_{C_L\setminus Q}\big)\Big|_{C_L\setminus Q} = 0'_{B_L\setminus Q} 1'_{C_L\setminus B_L} \Big)$

$\frac{p}{1-p} \mu_p(\sigma_Q =$  $)$

$\downarrow$



Same condition on $C_L\setminus Q$

$\overset{(**)}{=} \frac{p}{1-p} \mu_{p, C_L}\Big($  $, T\big($  $, \sigma_{C_L\setminus Q}\big)\Big|_{C_L\setminus Q} = 0'_{B_L\setminus Q} 1'_{C_L\setminus B_L} \Big)$

numerator in Def of ν_p page 2.3

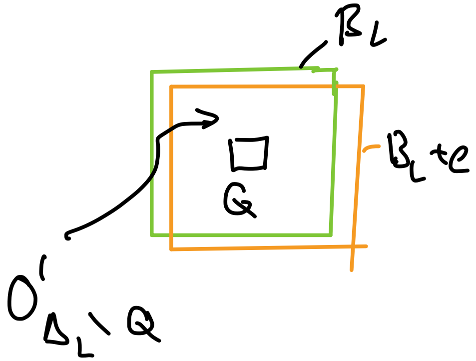
(*), (~~**~~) $\Rightarrow$ Lemma 4.1 ●

Note By Peierls Estimate

$$\nu_{B_L}\Big(\underbrace{\begin{smallmatrix} \cdot & \bullet & \cdot \\ \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \end{smallmatrix}}_{\text{good}}\Big) \geq 1 - \underbrace{\gamma_{B_L}\big(\exists \gamma : \bar{\gamma} \cap Q \neq \emptyset\big)}_{\leq |Q|\, \varepsilon(p)}$$

$$\nu_{B_L+e}\Big(\underbrace{\begin{smallmatrix} \cdot & \bullet & \cdot \\ \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \end{smallmatrix}}_{\text{wrong}}\Big) \leq \nu_{B_L+e}(\sigma_0 = 0) \leq \varepsilon(p)$$

Finishing the proof: $\Delta_L := B_L \cap B_{L+e}$



$$\mu'_p\big(\omega'_Q \mid O'_{B_L\setminus Q} \wedge 1'_{C_L \setminus B_L}\big)$$

$$= \int \gamma'_Q\big(\omega'_Q \mid \underbrace{O'_{B_L\setminus Q}}_{\supset \Delta_L \setminus Q} \wedge 1'_{C_L\setminus B_L}\, \tilde{\omega}_{\Omega_L^c}\big)\, \underbrace{\mu'_p\big(d\tilde{\omega}_{C_L^c} \mid O'_{B_L\setminus Q} \wedge 1'_{C_L\setminus Q}\big)}_{\text{un explicit}}$$

total probability

↑ any specification

$$\geq \inf_{\omega'_{\Delta_L^c}} \gamma'_Q(\omega'_Q \mid \mathcal{O}'_{\Delta_L \setminus Q}\ \omega'_{\Delta_L^c}) =: a_L(\omega'_Q)$$

Similarly $\mu'_p(\omega'_Q \mid \mathcal{O}'_{B_L+\underline{e} \setminus Q}\ 1'_{C_L \setminus (B_L+e)})$

$$\leq \sup_{\omega'_{\Delta_L^c}} \gamma'_Q(\omega'_Q \mid \mathcal{O}'_{\Delta_L \setminus Q}\ \omega'_{\Delta_L^c}) =: b_L(\omega'_Q)$$

Assume: γ' quasilocal. Then $\dfrac{a_L(\cdot)}{b_L(\cdot)} \xrightarrow[L \uparrow \infty]{} 1$

$\Downarrow$

Continuity at $\mathcal{O}'$-conditioning and $$\frac{a_L\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}{b_L\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}} \xrightarrow[L\uparrow\infty]{} 1$$

But

(1) $$\frac{p}{1-p}\underbrace{\nu_{B_L}\begin{pmatrix} \cdot & \bullet & \cdot \\ \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \end{pmatrix}}_{\geq 1-|Q|\varepsilon(p)} = \frac{\mu_p'\left(\begin{matrix} \cdot & \bullet & \cdot \\ \bullet & \bullet & \bullet \\ \cdot & \bullet & \cdot \end{matrix} \,\middle|\, 0'_{B_L\setminus Q}1'_{C_L\setminus B_L}\right)}{\mu_p'\left(\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \,\middle|\, 0'_{B_L\setminus Q}1'_{C_L\setminus B_L}\right)} \leq \frac{b_L\begin{pmatrix} \cdot & \bullet & \cdot \\ \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \end{pmatrix}}{a_L\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}$$

and

(2) $$\frac{p}{1-p}\underbrace{\nu_{B_L+e}\begin{pmatrix} \cdot & \bullet & \cdot \\ \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \end{pmatrix}}_{\leq \varepsilon(p)} \geq \frac{a_L\begin{pmatrix} \cdot & \bullet & \cdot \\ \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \end{pmatrix}}{b_L\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}$$

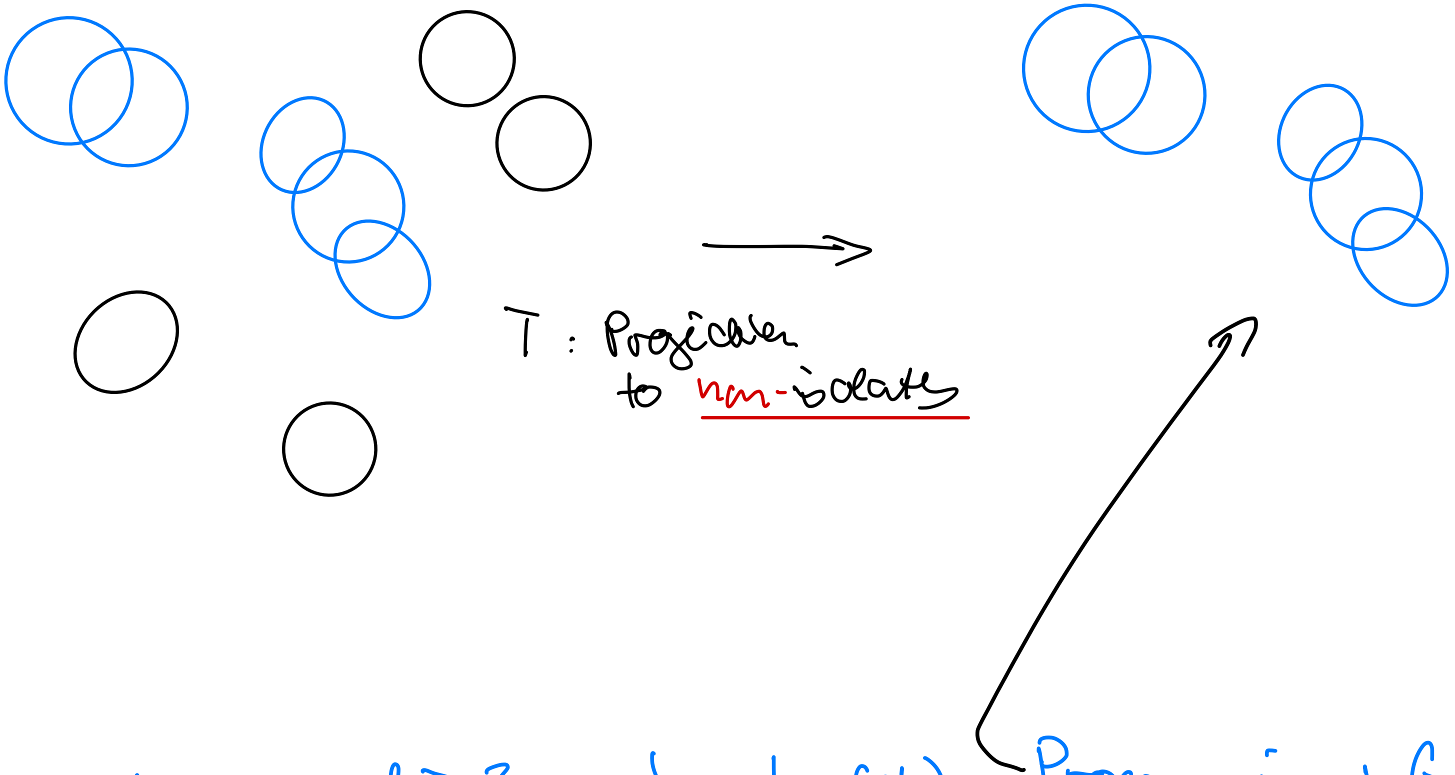
$$\frac{(2)}{(1)} : \quad \frac{\varepsilon(p)}{1-|Q|\,\varepsilon(p)} \geq \frac{a_2\begin{pmatrix} - & \bullet & \cdot \\ \bullet & & \bullet \\ - & \bullet & \cdot \end{pmatrix}}{b_L\begin{pmatrix} - & - & - \\ - & \cdot & - \\ - & - & - \end{pmatrix}} \; \frac{a_L\begin{pmatrix} - & - & - \\ - & \cdot & - \\ - & - & - \end{pmatrix}}{b_2\begin{pmatrix} - & \bullet & \cdot \\ \bullet & & \bullet \\ - & \bullet & \cdot \end{pmatrix}} \xrightarrow[L\uparrow\infty]{} 1$$

which is a contradiction to the continuity of γ'_Q at $0'$

Hence μ'_p is a non-Gibbsian measure and Theorem 1 is proved ■

Back to Poisson Point Processes in $\mathbb{R}^d$?

Intensity $\lambda \in (0,\infty)$, Radius 1



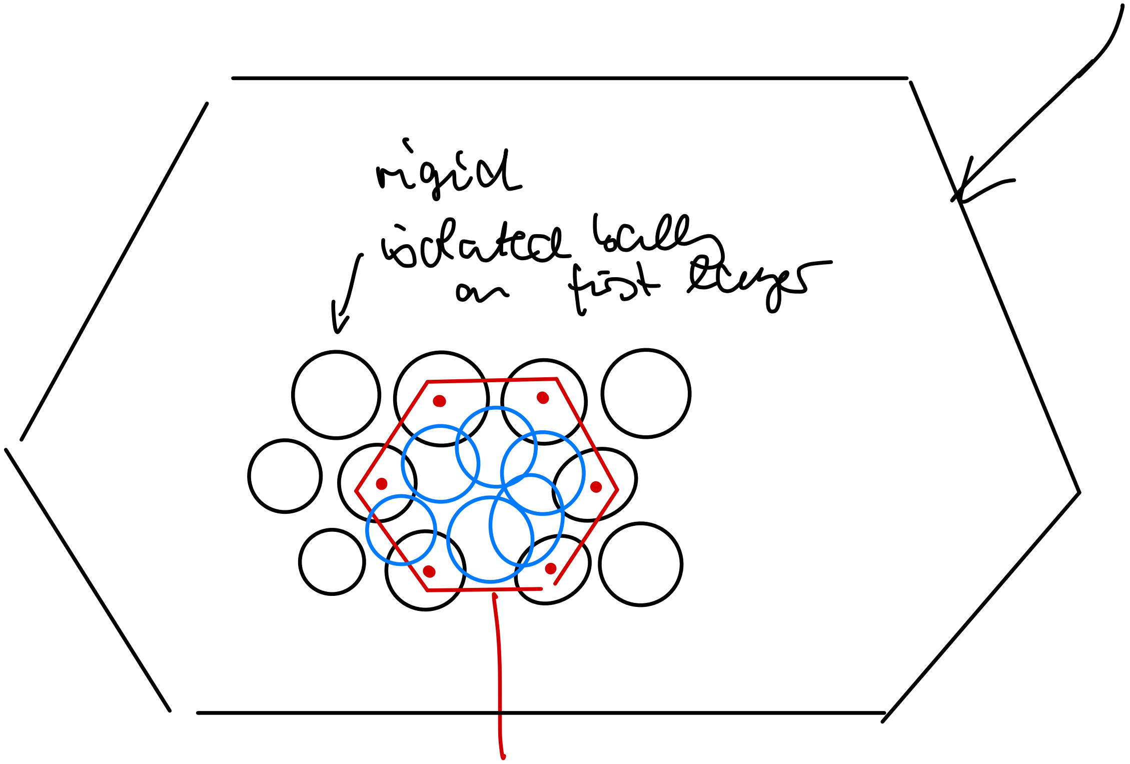
Conjecture: $d \geq 3$, $\lambda \geq \lambda_c(d)$ Process is nG
with empty configuration bad
for all possible specifications γ'.

Try analogous proof in $\mathbb{R}^d$.

This would need: provable long-range order in hard spheres model (analogue of ν_p see pages 2.3 / 2.4)

indicatable by shape of volume B_L

Theil Jansen Mohl Rolles ?



now: rotate B_L ↶

or translate B_L ↔

observation window Q with local non-isolated test-pattern

Thank you !