Interacting diffusions as marked Gibbs point processes

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Random point processes in statistical physics
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PART ONE

EXISTENCE

1.1 THE FREE SYSTEM – CONFIGURATIONS

LET $(\mathcal{S}, \|\cdot\|)$ BE A GENERAL NORMED SPACE.

THE STATE SPACE FOR THE SYSTEM IS $\mathcal{E} := \mathbb{R}^d \times \mathcal{S}$

($\mathbb{R}^d$: LOCATION SPACE; $\mathcal{S}$: MARK SPACE)

WE CONSIDER SIMPLE POINT CONFIGURATIONS ON $\mathcal{E}$, I.E. LOC. FINITE RADON MEASURES OF THE FORM

$$\gamma = \sum_i \delta_{\underline{x}_i}, \quad \underline{x}_i = (x_i, m_i) \in \mathbb{R}^d \times \mathcal{S}.$$

($\underline{x}_i$: MARKED POINT)

LET $\mathcal{M}$ BE THE SET OF ALL SUCH CONFIGURATIONS.
WE IDENTIFY A MEASURE γ WITH ITS SUPPORT:

$$\gamma = \sum_{i=1}^N \delta_{\underline{x}_i} \equiv \{\underline{x}_1, \ldots, \underline{x}_N\} \subseteq \mathcal{E}.$$

NOTATION: $\Lambda \subseteq \mathbb{R}^d$, $\gamma_\Lambda = \gamma \cap (\Lambda \times \mathcal{S})$.

$\mathcal{M}_\Lambda$ = set of all config. supported on $\Lambda \times \mathcal{S}$.

$\mathcal{M}_f$ = set of finite config. $|\gamma| < +\infty$.

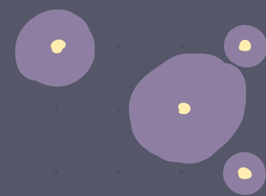
$\gamma, \xi \in \mathcal{M}$, $\gamma\xi := \gamma \cup \xi$.

$F: \mathcal{M} \to \mathbb{R}$ IS LOCAL IF $\exists \Delta \subset\subset \mathbb{R}^d$:
$F(\gamma) = F(\xi)$ WHEN $\gamma_\Delta \equiv \xi_\Delta$.

F IS TAME IF $|F(\gamma)| \le c \sum_{(x,m)\in\gamma_\Delta} (1 + \|m\|^{d+\delta})$.
$\exists \delta\ \exists c\ \exists \Delta$:

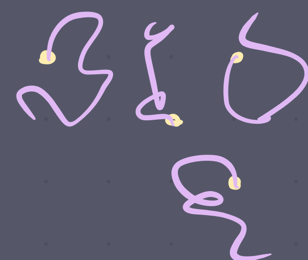
δ-TAME TOPOLOGY, I.E. THE SMALLEST TOPOLOGY S.T.
$P \mapsto \int F\,dP$, $P \in \mathcal{P}(\mathcal{M})$,
IS CONTINUOUS.

EXAMPLE 1. $\mathcal{S} = \mathbb{R}_+$



$\underline{x} = (x, m) \in \mathbb{R}^d \times \mathcal{S}$

EXAMPLE 2. $\mathcal{S} = C_0[0,1]$



$(x + m(s))_{s \in [0,1]}$

$\equiv$

$\underline{x} = (x, m)$

1.2 THE FREE SYSTEM – POISSON POINT PROCESS

LET $\mathcal{P}(\mathcal{M})$ DENOTE THE SYSTEM OF PROBABILITY MEASURES (POINT PROCESSES) ON $\mathcal{M}$.

CONSIDER A PROBABILITY MEASURE $R(dm)$ ON $\mathcal{S}$.

DEFINITION 1. THE MARKED POISSON POINT PROCESS $\pi^z \in \mathcal{P}(\mathcal{M})$ WITH INTENSITY MEASURE $z\,dx \otimes R(dm)$, $z>0$, IS THE PROBABILITY MEASURE ON $\mathcal{M}$ SUCH THAT:

FOR ANY BOUNDED $\Lambda \subset \mathbb{R}^d$,

- THE DISTRIBUTION OF THE NUMBER OF POINTS IN Λ UNDER π^z IS

$$|\gamma_\Lambda| \sim \mathrm{Pois}(z|\Lambda|).$$

- GIVEN THE NUMBER OF POINTS IN Λ, THESE ARE UNIF. DISTRIBUTED IN Λ.
- EACH POINT $x_i \in \Lambda$ IS ASSIGNED, IN AN I.I.D. MANNER, A MARK $m_i \sim R(dm)$.

WE WRITE π^z_Λ FOR THE RESTRICTION OF π^z TO $\mathcal{M}_\Lambda$.

$$\lambda(dx, dm) = z\,dx \otimes R(dm).$$

ASSUMPTION 1 (INTEGRABILITY). THERE EXISTS $\delta>0$ SUCH THAT

$$\int \exp\{\|m\|^{d+2\delta}\}\, R(dm) < +\infty.$$

2. GIBBSIAN INTERACTION

WE CONSIDER AN ENERGY FUNCTIONAL

$$H : \mathcal{M}_f \to \mathbb{R} \cup \{+\infty\}.$$

ASSUMPTION 2 (STABILITY). THERE EXISTS $c > 0$ SUCH THAT

$$\forall \gamma \in \mathcal{M}_f, \quad H(\gamma) \geq -c \sum_{(x,m)\in\gamma} (1 + \|m\|^{d+\delta}).$$

$$H(\gamma) \geq -c|\gamma|$$

FIX $\Lambda \subset\subset \mathbb{R}^d$.

DEFINITION 2. THE FINITE-VOLUME MARKED GIBBS POINT PROCESS P_Λ ON Λ WITH ACTIVITY $z > 0$ AND INVERSE TEMPERATURE $\beta > 0$ IS GIVEN BY

$$P_\Lambda(d\gamma) = \frac{1}{Z_\Lambda} e^{-\beta H(\gamma)} \pi_\Lambda^z(d\gamma).$$

Z_Λ → PARTITION FUNCTION

REMARK. THANKS TO ASSUMPTION 2,

$$0 < Z_\Lambda := \int_{\mathcal{M}_\Lambda} \exp\{-\beta H(\gamma)\} \pi_\Lambda^z(d\gamma) < +\infty.$$

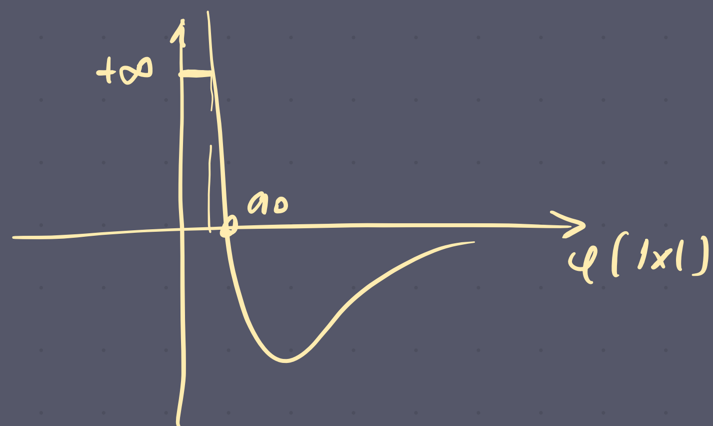
EXAMPLE 1.

$$H(\gamma) = \pm \text{Area}\Big(\bigcup_{(x,m)\in\gamma} B(x, \|m\|) \Big).$$

+ REPULSIVE, − ATTRACTIVE.

EXAMPLE 2.

$$H(\gamma) = \sum_{\{x,y\}\in\gamma} \left\{ \int_0^1 \varphi\left(|x + m_x(s) - y - m_y(s)|\right) ds \right\} \mathbb{1}_{\{|x-y| \leq a_0 + \|m_x\| + \|m_y\|\}}.$$



3. INFINITE VOLUME

WE WANT TO DESCRIBE A PROBABILITY MEASURE ON $\mathcal{M}$ CORRESPONDING TO THE CASE "$\Lambda = \mathbb{R}^d$". $\Lambda_n = [-n, n)^d$, $n \to \infty$.

ISSUE NUMBER ONE: NO MEANING FOR THE ENERGY OF AN INFINITE CONFIGURATION.

ISSUE NUMBER TWO: THE FAMILY OF MARGINALS $(P_{\Lambda_n})_{n \geq 1}$ IS NOT CONSISTENT IN THE KOLMOGOROV SENSE.

AN INCOMPLETE HISTORICAL NOTE

- UNMARKED GIBBS PP

  RUELLE (1969), MINLOS-POGHOSYAN (1977),
  PECHERSKY-ZHUKOV (1999)
  DEREUDRE-VASSEUR (2020)

- MARKED GIBBS PP

  GEORGII-ZESSIN (1993),
  CONACHE-DALETSKII-KONDRATIEV-PASUREK (2017)

- IN STOCHASTIC GEOMETRY
  DEREUDRE (2009)
  DEREUDRE-DROUILHET-GEORGII (2012)
  DEREUDRE-HOUDEBERT (2019)

- GENERAL SPACES

  POGHOSYAN-ZESSIN (2021), BETSCH (2022)

4.1 CONDITIONAL ENERGY

ISSUE NUMBER ONE: NO MEANING FOR THE ENERGY OF AN INFINITE CONFIGURATION.

DEFINITION 3. FOR ANY $\gamma \in \mathcal{M}_f$ AND ANY $\Delta \subset\subset \mathbb{R}^d$, THE CONDITIONAL ENERGY OF γ IN Δ IS GIVEN BY

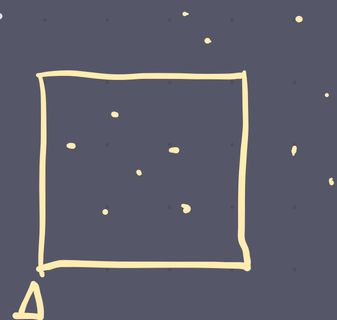
$$H_\Delta(\gamma) := H(\gamma) - H(\gamma_{\Delta^c}).$$

THIS NOTION CAN BE EXTENDED TO INFINITE CONFIGURATIONS

$$H_\Delta(\gamma) := \lim_{n \nearrow \infty} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n \setminus \Delta}), \quad \gamma \in \mathcal{M}.$$

- NOTE THAT $H_\Delta(\gamma_\Delta) = H(\gamma_\Delta)$, BUT IN GENERAL H_Δ IS NOT A LOCAL FUNCTIONAL, BECAUSE OF THE INTERACTION BTW. γ_Δ AND γ_{Δ^c}.

PROBLEM: HOW TO GUARANTEE EXISTENCE OF THE LIMIT?



4.2 TEMPERED CONFIGURATIONS AND INTERACTION RANGE

WE INTRODUCE A SUBSET $\mathcal{M}^{temp}$ OF THE CONFIGURATION SPACE $\mathcal{M}$, ON WHICH WE CAN CONTROL THE SIZE OF THE MARKS.

TEMPERED CONFIGURATIONS: $\mathcal{M}^{temp} = \bigcup_{t\geq 1} \mathcal{M}^t$, WHERE $\gamma \in \mathcal{M}^t$

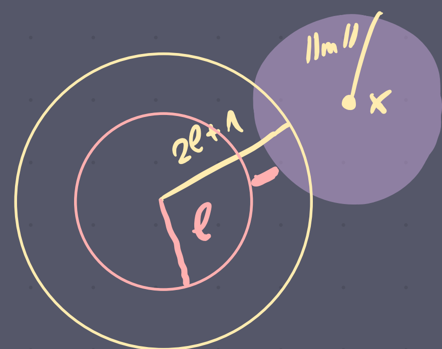
IF $\forall \ell \geq 1$, $\sum_{(x,m)\in\gamma_{B(0,\ell)}} \left(1+\|m\|^{d+\delta}\right) \leq t\,\ell^d$.

ASSUMPTION 4. FOR ANY $\Lambda \subset\subset \mathbb{R}^d$, $\xi \in \mathcal{M}^{temp}$, THERE EXISTS $c_\Lambda(\xi) > 0$ S.T.

$$H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c}) \geq -c_\Lambda(\xi) \sum_{(x,m)\in\gamma_\Lambda} \left(1+\|m\|^{d+\delta}\right).$$

LEMMA 1. FOR ANY $t \geq 1$, THERE EXISTS A CONSTANT $\ell(t)$ SUCH THAT FOR ANY $\gamma \in \mathcal{M}^t$, FOR ANY $\ell \geq \ell(t)$,

$$(x,m) \in \gamma_{B(0,2\ell+1)^c} \Rightarrow B(x,\|m\|) \cap B(0,\ell) = \emptyset.$$



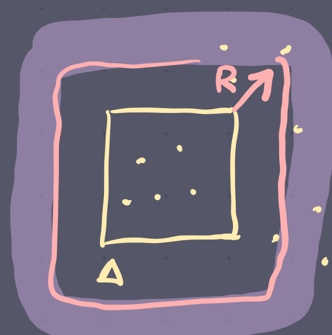
→ FOR TEMPERED CONFIGURATIONS, ONLY A FIXED NUMBER OF BALLS OF THEIR GERM-GRAIN SET

$$\Gamma(\gamma) = \bigcup_{(x,m)\in\gamma} B(x,\|m\|) \subseteq \mathbb{R}^d$$

CAN INTERSECT A GIVEN BOUNDED SUBSET OF $\mathbb{R}^d$.

ASSUMPTION 3 (RANGE). FOR ANY $\Delta \subset\subset \mathbb{R}^d$ AND ANY $\gamma \in \mathcal{M}^t$, $t \geq 1$, THERE EXISTS A FINITE NUMBER $R_\Delta(\gamma)$ SUCH THAT

$$H_\Delta(\gamma) = H(\gamma_{\Lambda \oplus B(0,R)}) - H(\gamma_{\Lambda \oplus B(0,R) \setminus \Lambda}).$$



$R_\Delta(\gamma)$ IS NON-LOCAL, BUT DEPENDS ON γ_{Δ^c} ONLY THROUGH $\ell(t)$. ITS SPECIFIC FORM CAN BE OBTAINED BY ASSUMING THAT TWO (OR MORE) POINTS INTERACT ONLY IF $B(x,\|m\|) \cap B(y,\|u\|) \neq \emptyset$.

5. DLR EQUATIONS

ISSUE NUMBER TWO: THE FAMILY OF MARGINALS $(P_{\Lambda_n})_{n\geq 1}$ IS NOT CONSISTENT IN THE KOLMOGOROV SENSE.

$(\mu_\Lambda)_\Lambda$, $\Lambda \subseteq \Delta \subset\subset \mathbb{R}^d$,

$\mu_\Lambda = \mu_\Delta \circ (\pi_\Lambda \circ \pi_\Delta^{-1})^{-1}$.

PROPOSITION 2. FOR ANY $\Lambda \subseteq \Delta \subset\subset \mathbb{R}^d$, FOR P_Δ-A.S. ALL $\xi \in \mathcal{M}$,

$$P_\Delta(d\gamma_\Lambda \mid \xi_{\Lambda^c}) = \frac{1}{Z_\Lambda(\xi_{\Lambda^c})} e^{-\beta H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})} \pi_\Lambda^z(d\gamma_\Lambda). \quad \text{(DLR)}$$

[DOBRUSHIN 1968, LANFORD & RUELLE 1969] CONDITIONAL PROBABILITIES YIELD A CONSISTENT FAMILY.

$$\Xi_\Lambda(\xi, d\gamma) := \frac{1}{Z_\Lambda(\xi)} e^{-\beta H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})} \pi_\Lambda^z(d\gamma_\Lambda).$$

COMPATIBILITY: FOR ANY $\Lambda \subseteq \Delta$,

$$\int_{\mathcal{M}_{\Delta \setminus \Lambda}} \Xi_\Lambda(\eta_{\Delta\setminus\Lambda}\xi_{\Delta^c}, d\gamma_\Lambda)\, \Xi_\Delta(\xi_{\Delta^c}, d\eta_{\Delta\setminus\Lambda}) = \Xi_\Delta(\xi_{\Delta^c}, d(\gamma_\Lambda \eta_{\Delta\setminus\Lambda})).$$

→ ALL FINITE-VOLUME GIBBS MEASURES SATISFY THE ABOVE EQUATIONS.

THE IDEA IS THEN TO USE THEM AS THE DEFINITION OF AN INFINITE-VOLUME GIBBS POINT PROCESS.

DEFINITION 4. A PROBABILITY MEASURE P ON $\mathcal{M}$ IS SAID TO BE A MARKED INFINITE-VOLUME GIBBS POINT PROCESS IF, FOR ANY $\Lambda \subset\subset \mathbb{R}^d$,

$$\int_{\mathcal{M}} F(\gamma)\, P(d\gamma) = \int_{\mathcal{M}} \int_{\mathcal{M}_\Lambda} F(\gamma_\Lambda \xi_{\Lambda^c})\, \Xi_\Lambda(\xi, d\gamma)\, P(d\xi)$$

FOR ANY $F: \mathcal{M} \to \mathbb{R}$ BOUNDED, LOCAL, MEASURABLE.

6. EXISTENCE VIA THE ENTROPY METHOD

DEFINITION. LET Q, Q' BE TWO PROB. MEASURES ON $\mathcal{M}$, $\Lambda \subseteq \mathbb{R}^d$ BOUNDED. THE RELATIVE ENTROPY OF Q' W.R.T. Q ON Λ IS

$$I_\Lambda(Q|Q') := \int \log \frac{dQ_\Lambda}{dQ'_\Lambda} \, dQ_\Lambda$$

IF $Q_\Lambda \preceq Q'_\Lambda$, $+\infty$ OTHERWISE.

THE SPECIFIC ENTROPY OF Q W.R.T. Q' IS

$$\mathcal{I}(Q|Q') := \lim_{n\to\infty} \frac{1}{|\Lambda_n|} I_{\Lambda_n}(Q|Q').$$

PROPOSITION 3. (H.-O. GEORGII, H. ZESSIN - 1993). FOR ANY $a>0$, THE LEVEL SETS

$$\{Q \in \mathcal{P}(\mathcal{M}) \text{ stationary under } \mathbb{Z}^d : \mathcal{I}(Q|\pi^z) \leq a\}$$

ARE SEQ. COMPACT UNDER τ_L.

$P_n \to P$

THEOREM (A.Z. 2022). UNDER THE ABOVE ASSUMPTIONS,

FOR ANY $z, \beta > 0$, THERE EXISTS AT LEAST ONE INFINITE VOLUME GIBBS POINT PROCESS P.

SKETCH.

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PART TWO

UNIQUENESS

(PAIR POTENTIAL)

AN INCOMPLETE HISTORICAL NOTE

- CLUSTER EXPANSION

  KOTECKÝ-PREISS (1986)
  RUELLE (1969), POGHOSYAN-VELTSCHI (2009)
  JANSEN (2020), POGHOSYAN-ZESSIN (2021)

- DOBRUSHIN CONTRACTION PRINCIPLE

  DOBRUSHIN (1968)
  PECHERSKY-ZHUKOV (1999), HOUDEBERT-Z. (2022)

- DISAGREEMENT PERCOLATION

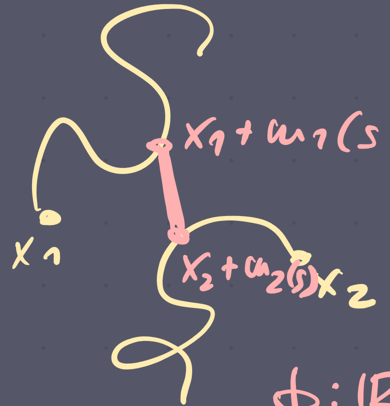
  VAN DEN BERG-MAES (1994)
  DE MASI, PRESUTTI, MEROLA, VIGNAUD (2008)
  HOFER-TEMMEL (2019), BETSCH, LAST, OTTO (2022)
  HOUDEBERT

1. N-POINT CORRELATION FUNCTIONS

$$\underline{x} = (x, m)$$

LET $H(\gamma) = \sum_{\{\underline{x}_1, \underline{x}_2\} \subseteq \gamma} \Phi(\underline{x}_1, \underline{x}_2)$.

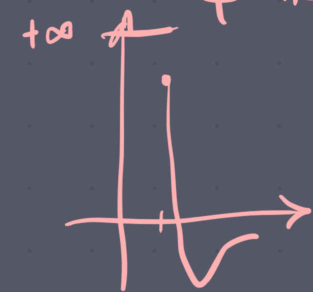
$$\Phi(\underline{x}_1, \underline{x}_2) = \int_0^1 \phi\big(|x_1 - x_2 + m_1(s) - m_2(s)|\big)\, ds$$



DEFINITION 1. THE N-POINT CORRELATION FUNCTION OF ANY $P \in \mathcal{G}(H)$ ADMITS THE FOLLOWING REPRESENTATION

$$\rho_N(x_1, \dots, x_N) = e^{-\beta \sum_{i<j} \Phi(\underline{x}_i, \underline{x}_j)} \int e^{-\beta \sum_{i=1}^N \sum_{\underline{y} \in \xi} \Phi(\underline{x}_i, \underline{y})}\, P(d\xi).$$

$$\phi : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$$



THE KEY POINT OF THIS SECTION IS TO SHOW THAT THE CORRELATIONS FUNCTIONS $(\rho_N)_N$ OF ANY GIBBS P.P. SOLVE, FOR ANY $N \geq 1$ AND A.A. $(\underline{x}_0, \dots, \underline{x}_N) \in (\mathbb{R}^d \times \Omega)^{N+1}$, THE SEQUENCE OF KIRKWOOD - SALSBURG EQUATIONS

$$\rho_{N+1}(\underline{x}_0, \dots, \underline{x}_N) = e^{-\beta \sum_{i=1}^N \Phi(\underline{x}_0, \underline{x}_i)} \Big[\rho_N(x_1, \dots, x_N) + \sum_{k \geq 1} \frac{z^k}{k!} \int \prod_{j=1}^k \Big(e^{-\beta \Phi(\underline{x}_0, \underline{y}_j)} - 1\Big)\, \rho_{N+k}(\underline{x}_1, \dots, x_N, \underline{y}_1, \dots, \underline{y}_k)\, \sigma^{\otimes k}(\underline{y}_1, \dots, \underline{y}_k) \Big].$$

$$\sigma(dx, dm) = e^{-\psi(x)} z(dx) \otimes R(dm).$$

$$\rho_0 \equiv 1.$$

2. THE KIRKWOOD-SALSBURG OPERATOR

LET $c: \mathcal{E} \to \mathbb{R}_{\geq 0}$ AND CONSIDER THE BANACH SPACE $\mathbb{X}_c$ OF SEQUENCES

$r = (r_N)_N$ s.t.

$\exists\, b_r > 0: \forall N \geq 1,$

$$|r_N(x_1, \dots, x_N)| \leq b_r \prod_{i=1}^{N} c(\underline{x}_i). \quad \left(\leq b_r\, a^N\right)$$

THE ABOVE EQUATIONS CAN BE REWRITTEN AS A FIXED POINT PROBLEM ON $\mathbb{X}_c$:

$$r = K_z r + 1_z,$$

WITH $1_z = (1_{z,N})_N$ GIVEN BY

$$1_{z,1}(\underline{x_1}) = 1, \quad 1_{z,N} \equiv 0 \quad \forall N \geq 2.$$

3. UNIQUENESS STRATEGY

i) HAVE AN ESTIMATE OF THE FORM

$$\rho_N(\underline{x}_1, \dots, \underline{x}_N) \leq \prod_{i=1}^{N} C(\underline{x}_i)$$

ii) PROVE THAT THE OPERATOR K_z IS A CONTRACTION

iii) SHOW THAT $(\rho_N)_N$ IS A SOLUTION OF THE FIXED POINT PROBLEM.

4. ASSUMPTIONS

ASSUMPTION 1. $\exists\, b : \mathcal{E} \to \mathbb{R}_+$ S.T.

$$\sum_{i=1}^{N} \Phi(x_0, x_i) \geq -b(x_0).$$

ASSUMPTION 2. $\exists\, a : \mathcal{E} \to [a, +\infty)$ AND $z_{crit}(\beta) > 0$ S.T.

S.T.

$$z_{crit} \int_{\mathcal{E}} \exp\{a(\underline{y}) + b(\underline{y})\} \left| e^{-\beta \Phi(\underline{x}, \underline{y})} - 1 \right| \sigma(d\underline{y}) \leq a(\underline{x})$$

ASSUMPTION 3. FOR ANY $P \in \mathcal{G}(\Phi)$, FOR ANY $N \geq 1$,

FOR $\sigma^{\otimes N}$- A.A. $\{\underline{x}_1, \ldots, \underline{x}_N\} \in \mathcal{E}$,

$$\left| e_N^{(P)}(x_1, \ldots, x_N) \right| \leq \prod_{i=1}^{N} e^{a(\underline{x}_i) + b(\underline{x}_i)} .$$

5. PROOF IN A SIMPLIFIED CASE

ASSUME THAT

(1) $\sum_{1 \le i < j \le N} \Phi(\underline{x}_i, \underline{x}_j) \ge -B_\phi N, \; B_\phi > 0.$

(2) A UNIFORM RUELLE BOUND HOLDS

$$\forall N, \; |\rho^N| \le c^N, \; c > 0.$$

(3) $C(\beta) := \sup_{\underline{x} \in \mathcal{E}} \int |e^{-\beta \Phi(\underline{x}, \underline{y})} - 1| \, \sigma(d\underline{y}) < +\infty.$

COMPUTING THE NORM OF THE K-S OPERATOR YIELDS

$$|(K_z r)_{N+1}(\underline{x}_0, \_, \underline{x}_N)| \le c^{N+1} c^{-1} e^{2\beta B_\phi + z C(\beta) c},$$

SO THAT K_z IS A CONTRACTION IF

$$c^{-1} e^{2\beta B_\phi + z C(\beta) c} < 1.$$

→ IF $c = e^{3\beta B\phi}$, THEN

$$z_{crit}(\beta) = \beta B_\phi \left(C(\beta) e^{3\beta B_\phi} \right)^{-1}.$$

THEOREM (7. 2022).

$\forall z < z_{crit}(\beta)$ (UNDER ASSUMPT.) $\exists!$ GPP P.

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