Berlin
June 29-July 1 2022

Gibbs measures on Lattice systems, discreteness and continuity.

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I)Gibbs measures,
Continuity.
Discrete vs continuous aspects,
Generalities and characterisations.
II)Gibbs and g-measures,
local and global Markov,
SRB, Thermodynamic Formalism,
Ergodic Theory.
III)Renormalisation, coarse-graining,
thinning and decimation, (Equilibrium)
IV)stochastic evolutions (Non-equilibrium).
(Long-range examples).

Introduction:
What and why?

I)
Gibbs measures,
Markov and almost Markov
"topological continuity" properties
for lattice systems.
Discrete versus continuous symmetries,
translations and rotations.
Properties and Characterisations.

Lattices Z^d. Why?
Mark Kac: "Be wise, discretise".
Either:
Discrete space or time for convenience,
mathematical simplicity.
Or:
Physical motivations,
e.g. describe crystal lattice
where magnetic atoms are located.
Convenience:
Theory is more complete,
technically simpler than in the continuum.

Why should you be interested in discrete Lattice Systems, if you prefer Continuum?

Discrete (lattice) versus Continuum relations:

Both directions:

1) Studying Continuum systems:

Discrete models as approximation to continuum models.

Divide space into empty or non-empty(filled) square or cubic boxes.

What to expect?

Lattice results **suggestive**.

Qualitative properties,

Fast correlation decay, under strong uniqueness conditions

Existence of phase transitions at low temperatures.

2) Study (large) discrete systems.
At a large scale they look continuous.
Limit theorems:
a)CLT and related.
b)Discrete systems in some limits **approximate** continuum processes
(Here also relevant:
Convergence of lattice Gibbs measures to Poisson processes; Chayes-Klein, Ferrari-Picco, Coupier).
c) Or (almost) critical systems
Non-central limit theorems (different scalings).
approximating (Euclidean) field theories.
Remember Polya's prescription:
If you come across a problem you don't know how to solve, then there is a simpler problem you **also** can't solve.
Find it! (Lattice analogue).

Intermediate problems: Phase transitions.

1) Transfer Methods:

Example: Sometimes "Peierls contours" work in the continuum, despite being combinatorial arguments(discrete...)

2) Folklore:

Discrete transitions in $d = 2$, (finite number of phases)

Continuous transitions in $d = 3$ (infinite number of phases).

Roughly true, but....

3) Continuous spins and (un)broken continuous Symmetries.

Mermin-Wagner theorem:

" No continuous symmetry breaking in $d \leq 2$".

What it implies and doesn't imply in $d = 2$.

Possibility of and nature of possible phase transitions, crystals, soft crystals and quasicrystals...

4)Hidden and emergent continuous symmetries even in discrete spin systems.

5) Particles in the continuum carrying colours or spins (e.g. Widom-Rowlinson, ferrofluids, spin-boson systems as long-range Ising models in $d = 1$).

Why Gibbs measures?

Fundamental and **pragmatic** reasons.

1)**Fundamental**:

Thermal equilibrium description.

2)**Pragmatic**:

Gibbs measures have nice properties.

Take advantage of them.

E.g. by rephrasing other systems as Gibbs measures.

Mathematics:
a) Dynamical Systems
(SRB, deterministic dynamics.)
b) Space-time Gibbs measures,
stochastic dynamics
describing evolving large systems
(Interacting Particle Systems,
Interacting Diffusions).
Heating or cooling,
"changing the temperature".

Physics:

c) Coarse-graining (spatial) of Gibbs measures
(thinning, rescaling, Renormalising Group maps).
Critical systems,
(Euclidean) field theories.
("Wick rotation", analytic continuation
quantum field theory to stat. mech.
$\exp itH \to \exp -\beta H$).

d) Effective models for disordered systems.

i) Quenched " two-temperature" systems, spins at low temperature, particles frozen in a disordered "high-temperature" configuration.

ii) Annealed "fuzzy" models, "local coarse-graining" (e.g. at each site).

Math Questions:
Marginals of Gibbs measures,
Assumed (hoped for the above reasons)
to be Gibbsian again.
True or not?
(How) can one check?
Examples, generalities.

Discrete lattice Z^d.
Configuration space $\Omega_0^{Z^d}$.
Spins
or occupation numbers.
Interactions Φ; collection of Φ_X, for all finite $X \in Z^d$.
Φ_X are functions on Ω_0^X, the spins in X.
Translation invariant, different summability conditions.
Different interaction spaces:
$\sum_{0 \in X} f(X) ||\Phi_X|| < \infty$.
E.g. $f(X) = 1$, absolute summable. DLR
$f(X) = \exp r|X|$ for analyticity and convergent cluster expansions.
$f(X) = diam(X)$, "short-range" condition (SRB).

Single-spin spaces Ω_0:
Simple or complicated.
Examples:
1) Finite (e.g. Ising, Potts or clock models).
2) Compact continuous
(e.g. continuous Ising, $n-$vector such as Classical XY,Heisenberg).
Compactness is helpful.
(In Continuum Hard-Core models have compactness properties).
3) Unbounded, discrete Z^n (e.g. SOS or discrete Gaussian models).
4) Unbounded continuous R^n. (e.g. Gaussians, $P(\Phi)$-models).
How to handle non-compact models.

Gibbs measures, and Markov-like properties:

Let G be an infinite graph,
Configuration space:
$\Omega = {\Omega_0}^{Z^d}$.
Probability measures on Ω,
labeled by **interactions**.
An interaction was a collection of functions,
$\Phi_X(\omega)$, dependent on ${\Omega_0}^X$,
where the X are subsets of G.
Let Λ be a finite subset of G.
Take Ising spins.
We write $\Omega_\Lambda = \{-,+\}^\Lambda$.

Energy (**Hamiltonian**)
$H_{\Lambda}^{\Phi,\tau}(\omega) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
Sum of interaction-energy terms.
A measure μ is *Gibbs* iff:
(A version of) the
conditional probabilities of
finite-volume configurations,
given the outside configuration, satisfies:
$\mu(\omega_\Lambda | \tau_{\Lambda^c}) = \frac{1}{Z_\Lambda^\tau} \exp - \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
for ALL configurations ω,
boundary conditions τ
and finite volumes Λ.

Gibbsian form.
Rigorous version of
"$\mu = \frac{1}{Z} \exp -H$",
Gibbs canonical ensemble.
Larger energy means
exponentially smaller probability.
Nearest-neighbour interaction means that
$\Phi(X) = 0$,
except when $X = \{i, i + e_k\}, k = 1...d$, or $X = i$,
for some $i \in Z$.
A Gibbs measure for a nearest-neighbour
model satisfies a
spatial Markov property:
$\mu(\omega_\Lambda | \tau_{\Lambda^c}) = \mu(\omega_\Lambda | \tau_{\partial\Lambda})$.
Conditioned on the border spins,
inside and *outside* are independent.

Weakening the Markov properties:
"Continuity" (=almost Markov = quasilocality).
Product topology:
Two configurations are close if they are
equal on a large enough finite region.
Topology metrisable,
metric e.g. by:
$d(\omega, \omega') = 2^{-|n|}$,
where n is the site with
minimal distance from origin such that
$\omega_n \neq \omega'_n$.

A function is continuous,
if it depends weakly on sites far away
and mostly on what happens not too far,
whatever it is (uniformly).
Warning:
Continuity in product topology.
Quasilocality.
Different notion from continuous
as in continuous symmetries
and continuous space.

Although in a formal sense we can **always** say
that the log of the probability
of a configuration
is its energy,
(which is almost trivial for finite systems,
for infinite systems
where probabilities of configurations will be zero
and their energies plus or minus infinity,
sensibly we can only consider
conditional probabilities
and then continuity becomes an issue.

When a measure lacks continuity properties it is "non-Gibbsian".
A number of measures which were
assumed to be Gibbs measures,
turned out to be not Gibbs.
Warning:
We know about compact spins.
Gibbs measures for non-compact spins
with long-range interactions
also exist;
but they don't have continuity properties,
and we understand less
about characterising them.

Remark:

Also variational characterisations.

Variational Principles:

Gibbs measures minimise (all local) free energies,
and are characterised by that property.
For translation invariant measures
there is a characterisation on Z^d
that Gibbs measures minimise free energy density
or maximise the pressure P.
Maximum (or minimum) taken over shift invariant measures.

$P(\Phi) = sup_{\mu \in I} \mu(A_\Phi) - s(\mu)$.

And all the measures realising the maxima are
precisely the translation-invariant Gibbs measures.
Entropy s is a Large-Deviation **rate function.**
The last characterisation does not always work,
e.g. not on tree graphs.

II)

The difference between
one-sided and two-sided points of view.

Two flavours:

Time, discrete.

(Dynamical Systems, asymmetric description).

Past and Future, one-sided (SRB).

Non-equilibrium

versus

Space, discrete, for the moment one-dimensional.

(Mathematical Physics, symmetric description).

Left and Right, two-sided (DLR).

Equilibrium.

Question:

When are both descriptions equivalent?

When not equivalent?

Introduction:
Simple background.
Markov modeling (for the short-sighted..)
Time:
Probability and Statistics (Markov chains).
Future independent of past, given the *present*.
Ergodic Theory, Dynamical Systems.
Ahistoric, forget history.
(Henry Ford: All history is bunk...)
First Emperor Qin Shi Huang: "Erase history."

Space:

Statistical (Mathematical) Physics.

Markov: Inside independent of outside,
given the *border*.

(Take control of your borders).

Japanese Sakoku policy:

Isolate your country from everywhere else.

(If only....)

2-state Markov chains

-timelike-

versus

1-dimensional, nearest neighbour, spin
(e.g. Ising) models

-spacelike-.

Probability measures on e.g.
two-symbol sequences;
configuration space $\Omega = \{-,+\}^Z$.

Theorem:
(well-known, see e.g.
Wikipedia lemma "Markov Property",
see further Georgii).
Stationary Markov chains, i.e.
invariant Markov measures on histories,
and n.n. Gibbs measures,
in dimension 1,
are the **same** mathematical objects.
(Brascamp,Spitzer,...)

Warning: This is about objects (measures)
on *infinite* time/space.

Question:
If we try to be more far-sighted
does this sameness stay true?
Case 1:
General Ergodic Processes.
Answer: NO!
Gurevich, Ornstein-Weiss, in the 1970's
constructed examples which were
one-sided random,
two-sided deterministic.
Thus one-sided entropy (Kolmogorov-Sinai) *positive*
different from two-sided entropy which is *zero*,
and one-sided tail (*trivial*) different
from two-sided tail (*contains everything*).
Remark: Ergodic (invariants) are one-sided objects.
Ergodic Theory is a one-sided theory.

Case 2:
Continuous (Quasilocal) Ergodic Processes.
What if we change independent of Markov to
weakly dependent
(continuous, quasilocal, almost Markov),
does this sameness
between one-sided and two-sided
remain true?
Then we exclude the Gurevich and Ornstein-Weiss examples,
which are not continuous (in the product topology).
Various aspects studied by various people.
(Fernández, Gallo, Maillard, Verbitskiy, Redig,
Pollicott, Walters, den Hollander-Steif, Tempelman...)

Answers:

Sameness with extra regularity conditions: Yes.

One-sided equals two-sided.

(SRB, Thermodynamic Formalism..).

Without those: NO!

Neither class includes the other.

One direction known (since 2011),

(Fernández, Gallo, Maillard).

Also new example:

"Schonmann projection",

One-dimensional restriction of pure

two-dimensional n. n. Gibbs measures at low T.

Recent work with Shlosman.

Following Bethuelsen and Conache.

Other direction:

New (with Bissacot,Endo, Le Ny).

Time version:
Class of Stochastic Processes,
rediscovered repeatedly,
under a variety of names:
(g-measures=
Chains of Infinite Order=
Chains with Complete Connections=
Uniform Martingales/Random Markov
Processes)=
SCUM (Stochastic Chains with Unbounded Memory).
(Keane 70's, Harris 50's,
Onicescu-Mihoc and Doeblin-Fortet 30's,
Kalikow 90's).
Studied in
Ergodic Theory, Probability.

Spatial version:
Gibbs (=DLR) measures=
Gibbs fields=
" almost" Markov random fields.
Discovered independently,
in East (mathematics)
and West (physics),
(Dobrushin, Lanford-Ruelle 60's).
Mathematical Physics.
Here two-state -Bernoulli- variables,
(= **Ising** spins:)
$\omega_i = \pm$, for all $i \in Z$.
(Can be much more general.)
Warning:
DLR Gibbs $\neq$ SRB Gibbs.

Gibbs measures:

Let G be an infinite graph, here Z.

Configuration space:

Space of sequences: $\Omega = \{-, +\}^G$.

Probability measures on Ω,

labeleled by **interactions**.

An interaction is a collection of functions,

$\Phi_X(\omega)$, dependent on $\{-, +\}^X$,

where the X are subsets of G.

Let Λ be a finite subset of G.

We write $\Omega_\Lambda = \{-, +\}^\Lambda$.

Energy (**Hamiltonian**)
$H_{\Lambda}^{\Phi,\tau}(\omega) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
Sum of interaction-energy terms.
A measure μ is *Gibbs* iff:
(A version of) the
conditional probabilities of
finite-volume configurations,
given the outside configuration, satisfies:
$\mu(\omega_\Lambda | \tau_{\Lambda^c}) = \frac{1}{Z_\Lambda^\tau} \exp - \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_\Lambda \tau_{\Lambda^c})$.
for ALL configurations ω,
boundary conditions τ
and finite volumes Λ.

Gibbsian form.
Rigorous version of
"$\mu = \frac{1}{Z} \exp -H$",
Gibbs canonical ensemble.
Larger energy means
exponentially smaller probability.
Nearest-neighbour interaction means that
$\Phi(X) = 0$,
except when $X = \{i, i+1\}$ or $X = i$,
for some $i \in Z$.
A Gibbs measure for a nearest-neighbour
model satisfies a
spatial Markov property:
$\mu(\omega_{\{1,\dots,n\}} | \tau_{\{1\dots n\}^c}) = \mu(\omega_{\{1\dots n\}} | \tau_0 \tau_{n+1})$.
Conditioned on the border spins,
at 0 and $n+1$,
inside and *outside* are independent.

A two-state Markov chain is again
a measure on the same sequence space Ω.
Now it has to satisfy the " ordinary"
(timelike) Markov property:
$\mu(\omega_{\{1...n\}}|\tau_{\{-\infty,....,0\}}) = \mu(\omega_{\{1...n\}}|\tau_0)$.
One can describe this via a product
of 2-by-2 stochastic matrices P
with non-zero entries:
$P(k,l) = P(\omega_i = k \to \omega_{i+1} = l)$.
Here $k, l = \pm$ and i is any site (=time) in Z.
There is a one-to-one connection between
stationary (time-invariant)
2-state Markov Chains
and (space-translation-invariant) nearest-neighbor
Ising Gibbs measures.

Continuity (=almost Markov = quasilocality).
Product topology:
Two sequences are close if they are
equal on a large enough finite interval.
Topology metrisable,
metric e.g. by:
$d(\omega, \omega') = 2^{-|n|}$,
where n is the site with
minimal distance from origin such that
$\omega_n \neq \omega'_n$.
A function is continuous,
if it depends weakly on sites far away
and mostly on what happens not too far,
(or not too long ago)
whatever it is.

Processes (time):
$\mu(\sigma_0 = \omega_0 | \omega_{Z-}) = g(\omega_0 \omega_{Z-})$,
with g-function continuous.
Probability of getting ω_0, given the past.
Continuous dependence on the **past**.
Continuity studied since the 30's
(Doeblin-Fortet).
Claim!?:
Continuity implies uniqueness (Harris(50's)).
Mistake in proof pointed out by Keane (70's).
Counterexamples due to Bramson-Kalikow (90's).
Sharper criterion Berger-Hoffman-Sidoravicius (2003-2017).

Gibbs measures:

Continuity of conditional probabilities
corresponds to summability of interactions.
$\sum_{0 \in X} ||\Phi_X|| < \infty$.
Continuous dependence on **outside**
beyond the border.
(Quasilocality).
No action at a distance.
(No observable influence from behind the moon)
Plus: "non-nullness".
Any finite change in the -infinite- system
costs a finite amount of energy.
Any configuration in finite domain
occurs with finite probability,
whatever is happening outside.
Gibbs measures satisfy (equivalently) a
finite-energy condition.
Equivalence holds (Kozlov-Sullivan; Barbieri et al):
Finite-energy + continuity = Gibbs.

Our Counterexample:
(Gibbs, non-g-measure).
Gibbs measures for Dyson models.
Low temperatures.
Long-range Ising models.
Ferromagnetic pair interactions.
$\Phi_{i,j}(\omega) = -J|i-j|^{-\alpha}\omega_i\omega_j$.
Interesting regime $1 < \alpha \leq 2$.
Phase transition for large J,
at low temperatures:
There exist then **two** different
Gibbs measures, for the same interaction,
called μ^+ and μ^-, for such Φ.

Microscopic interfaces **don't exist**.
(One-dimensional Aizenman-Higuchi)
Spatially continuous conditional probabilities.
Warning:
Impossible for Markov Chains or Fields,
always uniqueness in $d = 1$.

(Digression, Question in Continuum:
Can interfaces in 3d continuous space be excluded?)

Back to $d = 1$.

Claim:
At low T and for $\alpha^* < \alpha < 2$
Dyson Gibbs measures are not g-measures.
Here technical condition $\alpha^* = 3 - \frac{\ln 3}{\ln 2}$.
Proof uses technically rather hard,
largely Italian, Input,
perturbative, cluster expansions, from others,
giving the α^* condition,
plus three simple Observations.

Input:

Interface result for Dyson models
(Cassandro, Merola, Picco, Rozikov).
Take interval $[-L, +L]$,
all spins to the left are minus,
all spins to the right are plus.
Then there is an interface point **IF**, such that:
1) To the left of the interface
we are in the minus phase (μ^-),
to the right of the interface
we are in the plus phase (μ^+).
2) With overwhelming probability the location
of the interface is at most $O(L^{\frac{\alpha}{2}})$ from the center.
$\ldots - - - - - m \ldots . |$**IF**$| + m \ldots .. | + + + + + \ldots$

Observation 1:
If I change all spins to the left of a length-N interval of minuses, the effect from the left on the central O(L) interval is bounded by $O(LN^{1-\alpha})$, thus small for N large.

Consequence:
A large interval of minuses (size N) will have a moderately large (size L) interval of minus phase on both sides. Interfaces are pushed away.

Observation 2:
If I decouple a comparatively small interval,
of size $L_1 = o(L)$,
inside the beginning (left side)
of my minus-phase interval,
this hardly changes the interface location.
(Cost of **IF** shift by εL is larger, namely $O(L^{2-\alpha})$.
Shown by Cassandro et al.)

Observation 3:
If I make in this L_1 interval
an alternating configuration
+-+-+-+-+-...
then the total energy (influence)
on its complement
is bounded by the double sum
$\sum_{i=1...,L_1, j>L_1}(|j-i|^{-\alpha} - |j+1-i|^{-\alpha})=$
$\sum_{i=1...,L_1, j>L_1}(O(|j-i|^{-(\alpha+1)})=$
$\sum_{i=1...,L_1} O(|i|^{-\alpha})$
which is bounded, uniformly in L_1.
Therefore finite, small effect.

Remark:
Effect only at positive temperature.
Entropic Repulsion.
A large alternating interval,
preceded by a MUCH
larger interval of minuses,
cannot shield the influence
of this homogeneous minus interval.
But this means precisely that
the conditional probability of finding a plus (or a minus),
at a given site, conditioned on an alternating past,
is not continuous.
Thus two-sided continuity
occurring at the same time
as one-sided discontinuity.

Alternating configuration is
discontinuity point,
(**bad** point)
due to cancellations of pluses and minuses.
Set of discontinuity points may be atypical
have measure zero, but nonremovable.
... $- - - - - + - + - + - X$ (- N, alt_L intervals)
versus
... $+ + + + - + - + - + - X$ (+N, alt_L intervals)
Expected value of X differs,
by more than cst,
uniformly in L and N(L).
Direct influence from Deep Past.

Analogies with higher-dimensional
Gibbs measures.
Analogy g-measures:
Global Markov property.
Conditioning on infinite-volume
(like half-line) events.
There are Markov fields
which are not Globally Markov.
Other analogy:
There are Markov fields which depend
discontinuously on lexicographic past.

Schonmann projection.
One-dimensional
marginal of 2d Ising measures.
Entropic repulsion in transversal
but not in longitudinal direction.
Non-Gibbs, but g-measure.

-|++++++++++++|-+-+-+-?......
μ_-, + interval N, alternating n ?....
left of origin.
Large contour vertical size $\sqrt{N}$,
near center of interval;
but close to beginning and end
of + interval.
Stays far fromthe origin.

vs

two-sided

-|+++++++++++|-+-+-+-?+-+-+-|++++++++++|-
N,n,? n,N.
Single large contour, when $N(n)$ large enough,
vertical distance $\sqrt{N}$ from,
surrounding, origin.
Contours behave like
random walks conditioned
to be positive (or negative).
Contour analysis very low T,
due to
Ioffe,Velenik,Ott,Wachtel,Toninelli,Shlosman...
Presumably all subcritical T
by Ornstein-Zernike methods.

Conclusion:
Two-sided continuous dependence
-spacelike- does *not* imply
one-sided continuous dependence
-timelike.
But neither is it implied.
Summary:
Controlling borders is NOT the same as
control of history,
except for the shortsighted.

A.v.E. with R. Bissacot (Sao Paulo), E. Endo (Shanghai),
A. Le Ny (Paris).
arXiv 1705.03156, Comm. Math. Phys., 363, 767–788.
A.v.E. with S.B.Shlosman.
AIHP (to appear, arXiv 2102.10622
A.v.E. , A. Le Ny, F. Paccaut(Amiens)
(MPRF 27, p315– 338),
arXiv 2011.14664
S. Berghout, R. Fernández and E. Verbitskiy,
Erg.Th.Dyn.Systems 39, p3224

Further long-range questions addressed:
1) Get rid of the technical restriction on α,
and large n.n. term,
with Bissacot, Endo, Kimura, Ruszel.
(Kimura, Littin-Picco)
2) Understand $\alpha = 2$ case (open).
3) Other Dyson model questions,
a) Add possibly decaying
inhomogeneous external fields, deterministic or random.
b) Metastability,
c) Metastates for random boundary conditions.

with Endo, Le Ny, Kimura, Ruszel, Spitoni,

Littin (3a).

III)

Renormalisation and Coarse-Graining.

Divide lattice Z^d in cubic, non-overlapping blocks, size L^d, B_j, $j \in Z'^d$.
Define a "block spin" (renormalised, primed, spin)
$\omega'_j = f(\{\omega_i; i \in B_j\})$.
Consider a Gibbs measure μ^{Φ} on Ω.
Consider the marginal measure μ'
on the renormalised spins,.
Thus on configuration space Ω'.
Question:
Is there a Φ', such that $\mu' = \mu^{\Phi'}$?

Motivation:

IF there is, then the map

$R : \Phi \to \Phi' = R(\Phi)$

is a well-defined Renormalisation Group (RG) map.

Then the whole machinery and folklore

of RG theory might be invoked.

Fixed points of R, eigenvalues of linearisation,

critical exponents....

Question first addressed mathematically by

Griffiths and Pearce.

Further studies by Israel,

van Enter-Fernández-Sokal, plus....

Examples:

(Simplest Ising spins, $\omega_i = \pm 1$):

$\omega'_j = \omega_{Lj}$ (decimation).

$\omega'_j = sign(\sum_{i \in B_j)} \omega_i$ (majority, when L is odd).

With modifications:

Different spin spaces:

$\omega'_j = \sum_{i \in B_j} \omega_i$, average

(now a single-site spin ω' takes more values than ω).

More complicated spins,

such as Potts or vector-valued spins.

Randomizing,

prescribe the conditional distribution of the block spin given the spin configuration:

$\mu(\omega'_j|\omega_{i\in B_j}) = P(\omega', \omega)$.

E.g.:

Toss a coin when the total spin of an even block equals zero.

Or

Follow majority with large probability

Or

Copy, but with a probability of making mistakes.

(Block size then is trivial, $L = 1$.)

Local (on-site) coarse-graining.
Examples:
"Fuzzy Potts" (for colourblind)
reduce number of colours (merge some).
Local discretisation
Vector spins to clock spins,
discrete time units. 24 hours...

Overlapping blocks.

Usually Z^d, but also on trees, or complete graphs
Mean-field models (Külske).
Difference in detailed results, but some general rules hold.
Therefore often similar statements can be proven for different graphs.
And sometimes even in the continuum.

Strategy of proofs:
Divide the original variables in two groups:
1) Visible
(block -renormalised- spins,
visible Potts colours,
entire hours).
versus
2)Invisible
(internal spins,
indistinguishable Potts colours,
minutes, seconds.. .)
Conditional probabilities of visible spins, continuous or not?
Fix visible spins, except one.
Can this visible spin at origin be strongly dependent
on visible spins near infinity, when environment is fixed?

If so, the marginal on the visible spins is
not a Gibbs measure
and then a few things must happen.
The information about the visible spins far away
must be transmitted by invisible messengers.
But the visible spins must be able to instruct them.
Mathematical conditions:
0) Conditioning is well-defined,
even on a zero-measure configuration.
1) Conditioned on a certain visible (bad) spin configuration
there needs to be a phase transition.
2) An order parameter of this transition should be
coupled to the spin at the origin.
3) Visible boundary conditions need to be able to
select an invisible phase.

Example:
Long-range Dyson model in $d = 1$.
Decimation on $2Z$.
Remember
$H(\sigma) = \sum_{i,j} |i-j|^{-\alpha} \omega_i \omega_j, 1 < \alpha \leq 2$.
Let $\omega'_i = \omega_{2i}$.
Consider μ^+ at low T and take the marginal ν^+ on the σ'.
Claim 1: ν^+ is not a Gibbs measure. (A.v.E. with A. Le Ny).
Claim 2: At higher T it is a Gibbs measure (Kennedy).

Condition 0)
is OK, there is a well-defined
"Global Specification",
(so we can condition on configurations in infinite volume).
with infinite complement. (Fernández-Pfister).
Uses ferromagnetic character of model.
Condition 1)
Choose even spins ω_{2i} alternating.
Then each odd spin has plus and minus spin
at odd distances which cancel.
+.-.+.-.+.-.+.-.+.-.
Then odd spins form Dyson model with
weaker interaction constants $(2|i-j|)^{-\alpha}$,
same decay power,
still has has transition but at lower T.

Condition 2:

Spin at origin uncoupled, is visible
feels either plus field or minus field
from invisible phase.
If outside an alternating interval
in two large intervals left and right
we choose + visible spins

$.......|+.+.+.+.+.|-.+.-.+.-.?.-.+.-.+.-.|+.+.+.+.+.|....$

then the alternating configuration is a discontinuity point.
The sets of configurations which are arbitrary beyond
form open sets,
so in each neighborhood there are two open sets
where the function differs more than ε:

Discontinuity!

$\mu(\omega'_0|\omega'^{alt}_M\omega'^{+}_N)-$
$\mu(\omega'_0|\omega'^{alt}_M\omega'^{-}_N)$
$\geq \varepsilon.$

For each alternating interval M,
we can choose plus (or minus) intervals $N(M)$.
In a magnetic field unique Gibbs measure,
because of ferromagnetic character
so influence from beyond N dies out.
(Yang-Lee or inequalities).

Remark:
If the original model is in a small field there is an
opposing bad configuration, cancelling it.
Still the decimated measure is non-Gibbs.

Complementary regime:
If the field is too strong, or the temperature is too high,
NO choice of even spins can induce a phase transition.
Proof by Dobrushin uniqueness theorem
or cluster expansion-analyticity.
"High" includes a temperature interval
around (below) critical point (Kennedy).
Non-Gibbsianness in open set in (H, T)-plane,
around low-T coexistence interval,
but not the whole coexistence interval.
Possible:
A model has a phase transition,
but no phase transition under ANY conditioning.

Example 2:
1d long-range XY model.
Again polynomial decay,
now with two-component vector spins:
$H = \sum_{i,j \in Z} -|i-j|^{-\alpha} \overrightarrow{\omega_i} . \overrightarrow{\omega_j}$.
When $1 < \alpha < 2$ same argument as for Ising spins.
When $\alpha = 2$, different arguments.
The model itself has no transition (Mermin-Wagner).
But:
Choose double periodic configuration:
NNSSNNSS.... for visible spins.
Then invisible spins between the fixed ones
N.N.S.S.N.N.S.S......
feel periodic field in NS direction.
No continuous symmetry after conditioning,
but EW discrete symmetry.

+H 0 -H 0 +H 0 -H....
Ferromagnet tries to equalise spins,
field tries to alternate.
Ordering is a compromise,
either
NE,E,SE,E,NE,E,SE....
or
NW,W,SW,W,NW,W,SW
similar in EW direction,
partially following alternating NS field.
Spin-flop (Ruszel, Crawford...).
At low T decimated state not a Gibbs measure.
(with Le Ny, d'Achille).

Bad configurations are "atypical" (have measure zero).
Thus transformed measures almost Gibbs.
Other examples:
Roughly similar, not precisely.
Sometimes non-Gibbsianness far from transition,
sometimes it occurs for models without a transition.
Conditioning then often induces
different type of transition as occurs in the original model.
Majority rule in strong field, on Ising,
decimation 2d XY-model.
Overlapping blocks (hard core conditions),
more easily cause non-Gibbsianness.
Potts model decimation above T_c.

Other topics:
Random cluster models
Disordered models
Schonmann projection (lower dimension)
Bad points "atypical" (measure zero),
almost and (intuitively) weak Gibbs.

IV):

Dynamics.

If we change external circumstances,
the system changes.
We can model this
either
by deterministic (Hamiltonian) dynamics
or by stochastic (IPS) dynamics. (Liggett, Spitzer).

Problems with deterministic dynamics:

1)Can not be defined for discrete spin systems.

2) Energy-conserving, but cannot describe temperature changes.
(for all temperatures, the Gibbs measures are invariant).

3) Entropy is invariant, problems with 2nd law of thermodynamics.

4) Often hard to define.

Thus both problems from physics and mathematics...

If we consider a Gibbs measure for a certain Hamiltonian
it is also invariant for various stochastic dynamics
(IPS, interacting diffusions...).
Often for such a stochastic dynamics only Gibbs measures
for one interaction are invariant.
So heating and cooling are possible to model.
For Ising spins:
Flip rates.
Probability per time unit to flip,
dependent on environment.
Continuous dependence.

Let the spin flip rate for
flipping a spin at site x,
starting from configuration ω be $c(x, \omega)$.
Then the generator L of the dynamics is given by
$Lf = \sum_{x \in Z^d} c(x, \omega) f(\omega^x) - f(\omega)$,
and the dynamics by $\exp tL$.
Simplest example:
Independent spin flip (infinite-T Glauber dynamics).
Then $c(x, \omega)$ only depends on the value of ω at x.
Starting from a low-T Gibbs measure
it might be seen as a model
for heating up a large system.
But....

If we consider the evolved Gibbs measure.
then we can (equivalently)
consider it a copying with mistakes,
with the probability of a mistake growing with time.
Yet another way of looking at it is
considering the Ising model on two layers.
The first layer is the original Ising model,
each vertex in the second layer is only coupled
with the same vertex in the first layer.
Fakirbed graphs (bed with nails)...
The coupling along the nails starts very large,
and then weakens in time.
The marginal on the second layer (the end of the nails)
is the evolved measure.

Formally
$H = H(\omega) + \sum_{i \in Z^d} \lambda_t \omega_i \omega'_i$.
1) When λ_t is large, (short time)
for any choice of ω'
the conditioned measure is unique
(Dobrushin uniqueness theorem in strong fields).
2) When λ_t is small, (large times)
there are choices of ω',
for example alternating, sometimes random,
for which the conditioned first layer has a phase transition.
As in RG maps, the evolved measure
on the ω' becomes non-Gibbsian.
No intermediate temperature,
because NO temperature.

More to be said:
Small-temperature dynamics
like infinite-temperature dynamics.
Perturbation theory (cluster expansions).
On Z^d same behaviour
for all initial Gibbs measures.
Large-deviation description
on paths in space of measures.

Other family of examples:
Trees.
Ising models on Cayley tree graphs.
At low T the free-boundary-condition measure $\mu^{\#}$
is not the mixture of plus and minus measure.
It also does not satisfy the variational principle,
as the free energy per bond is higher
than in plus or minus measure.

If we apply infinite-temperature Glauber dynamics,
the behaviour also is different.
Whereas the plus and minus measures
we find G-NG-G regimes
for the $\mu^{\#}$ the evolved measure remains non-Gibbs,
has full measure of bad points.
Typically bad.
So we have G-NG-FNG:
Gibbs, Non-Gibbs (almost Gibbs), Fully Non-Gibbs.

In this case the conditioning in the bulk
needs to let pass influence from the boundary
to allow for a phase transition.
Now full-measure, but not open
sets of external spins.
On trees arguments saying that
boundary-over-volume terms
approach zero don't work any more.
Other difference from Z^d:

Dynamics, interpretation:
Seeing a possibly bad -say alternating-configuration after some time in a large volume can have two causes:
1) The bad configuration was there to begin with, due to thermal fluctuation in initial measure.
2) The bad configuration occurred spontaneously, due to a number of improbable spin flips.
Comparing both via a large-deviation analysis.
For short times
1) has larger probability.
Gibbsian regime.
For large times
2) has larger probability.
Non-Gibbsian regime

Widom-Rowlinson models (also in continuum)
Jahnel-Külske.
A-particles and B-particles
which cannot touch (or overlap)
different type of particles.
Hard-core, forbidden configurations.
At high density coexistence
between A-rich and B-rich Gibbs measures.

Stochastic dynamics:
Stochastic flips between A and B
(non-moving particle locations.)
Gibbs measures become
non-Gibbsian immediately.
Typical configurations then are bad.
Reasons (with starting A-measure):
Infinite A-cluster with a few B particles
is forbidden at time 0

Conclusions:

Being a Gibbs measure is useful,
but not always that robust.
A local transformation
on a (quasi)local measure
van make it nonlocal.
Can be lost by
1)Looking Globally,
2) Thinning and Coarse-graining,
3) Stochastic Dynamics.
Gibbs and non-Gibbs in different regimes,
depending on details.

Some papers:

Coarse-graining

A.v.E., R. Fernández and A.D. Sokal JSP 72, p879.

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Stochastic Dynamics

A.v.E., R.Fernández, F. den Hollander, F. Redig,
CMP 226, p 101 and MMJ 10, p 687)

A.v.E., W.M. Ruszel Stoch.Proc.Appl. 119, p1886.

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