Delocalisation of height functions

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University of Innsbruck

joint work with Piet Lammers

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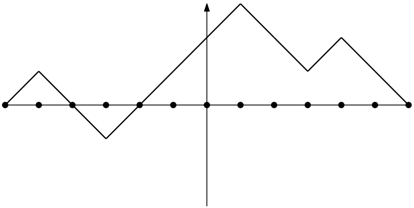
– Phase transitions in spatial particle systems –
Berlin

Delocalisation

1D time:
Random Walk → Brownian bridge
$\mathrm{Var}_n(h(0)) \sim n$.



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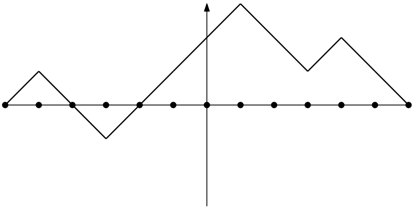

2D time: graph homomorphisms $\mathbb{Z}^2 \to \mathbb{Z}$

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- $\mathbb{P}(h)$ =uniform;
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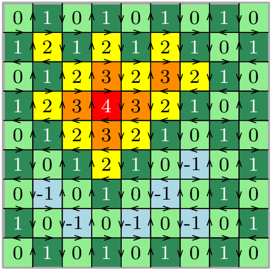

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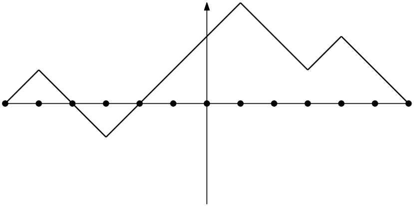

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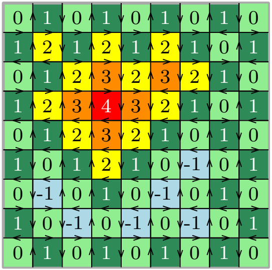

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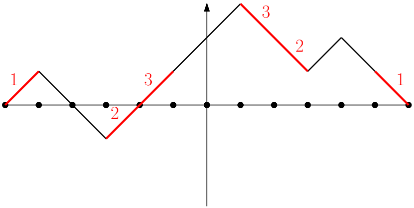

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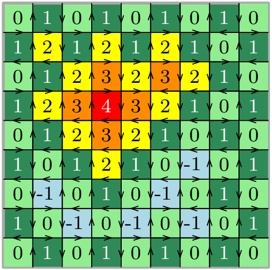

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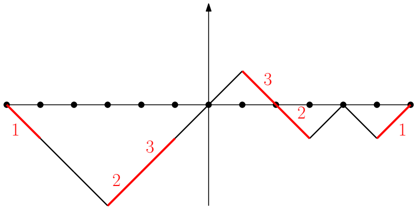

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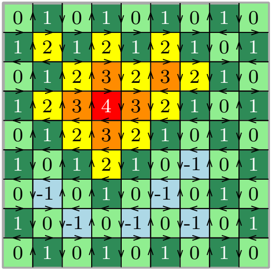

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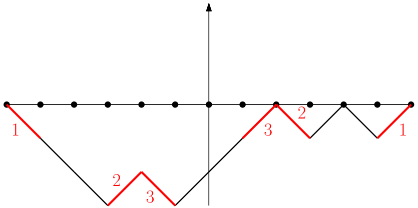

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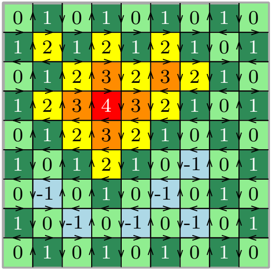

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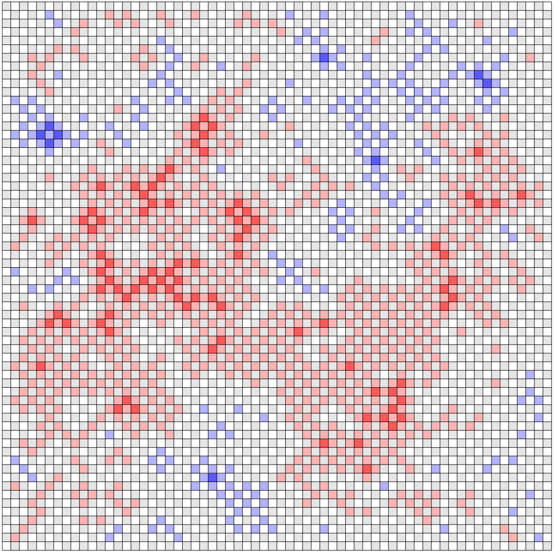

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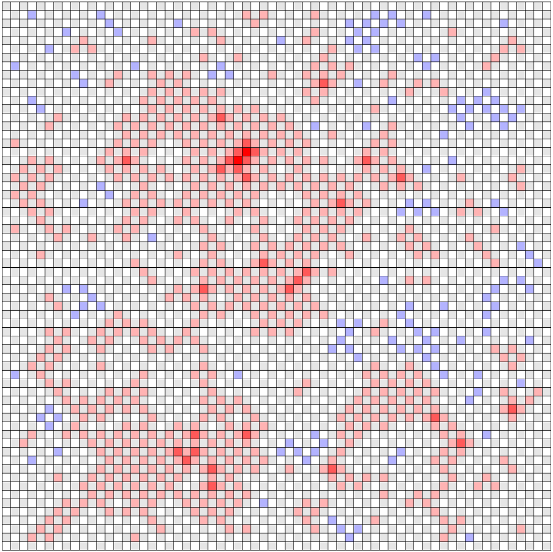

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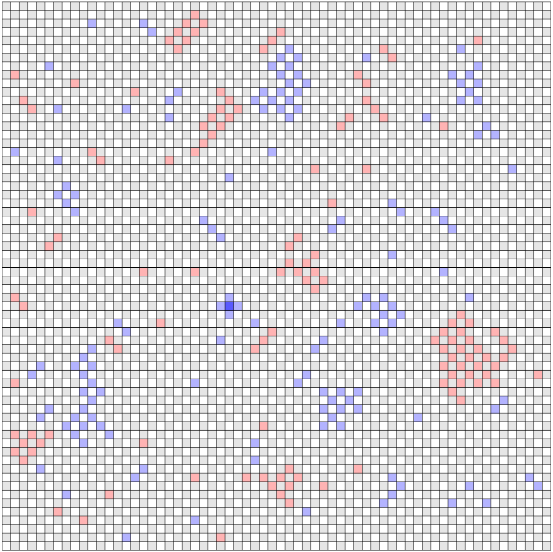

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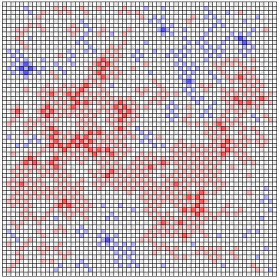

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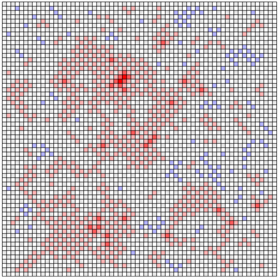

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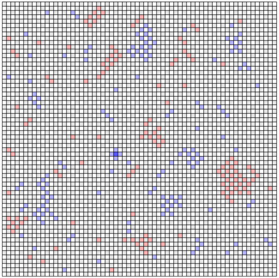

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Six-vertex model, $a, b \leq c$

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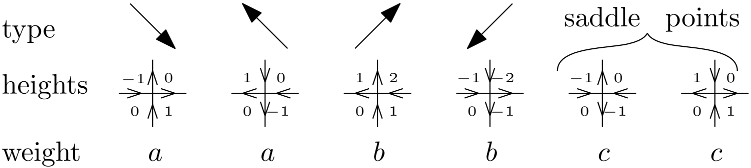

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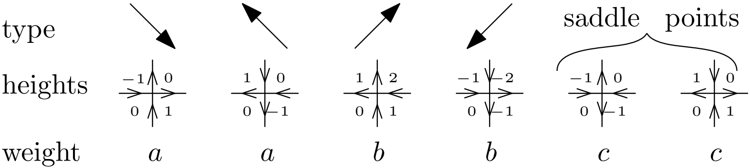

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Prop (positive association: FKG inequality)

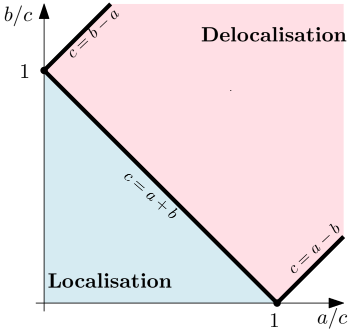
Let $a, b \leq c$. Then, for any increasing F, G,

$$\mathbb{E}(F(h) \cdot G(h)) \geq \mathbb{E}(F(h)) \cdot \mathbb{E}(G(h)).$$

[Fortuin–Kasteleyn–Ginibre'72], [Benjamini–Haggström–Mossel'00]

Background

- free energy computation
  [Yang-Yang '66], [Sutherland '67], [Lieb '67]



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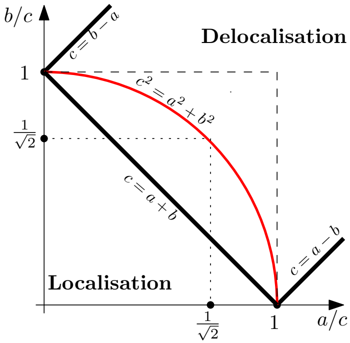

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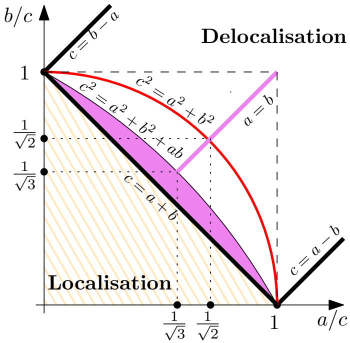
- Localisation: $a + b < c$
  [Duminil-Copin–Gagnebin–Harel–Manolescu–Tassion'16], [Ray–Spinka'19],[G.–Peled'19]
- log-Delocalisation: $a = b \leq c \leq a + b$
  [Lis'20],
  [Duminil-Copin–Karrila–Manolescu–Oulamara'20]
- Rotational invariance:
  $\sqrt{a^2 + b^2 + ab} \leq c \leq a + b$
  [Duminil-Copin–Kozlowski–Krachun–Manolescu–Oulamara'20]



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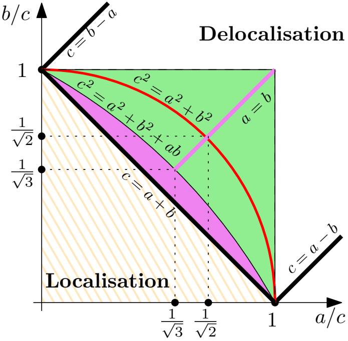

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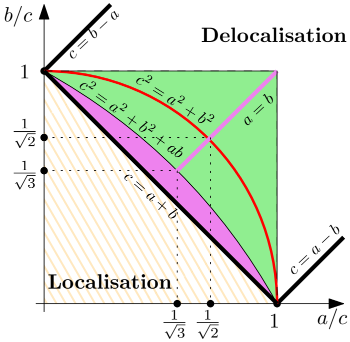

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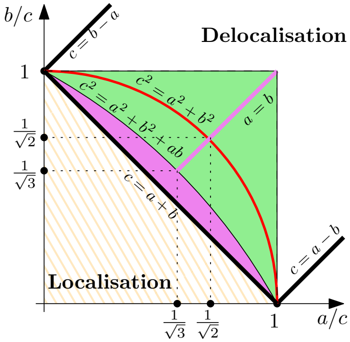

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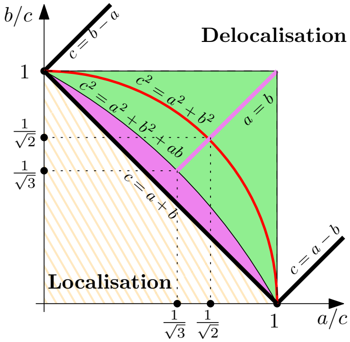

For simplicity: fix $a = b = 1$ and $c \in [1, 2]$.

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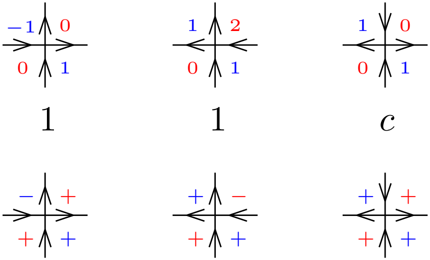
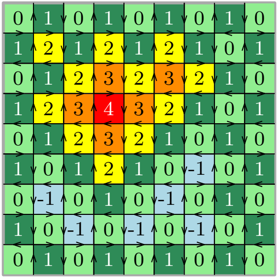
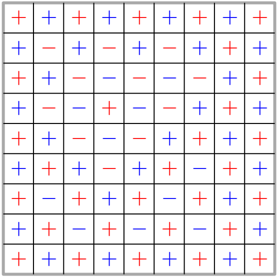
Heights mod 4 $\leftrightarrow$ spins: $0, 1 \leftrightarrow +$, $2, 3 \leftrightarrow -$.
Edge percolation $\mathrm{ES}^{\text{even}}$:







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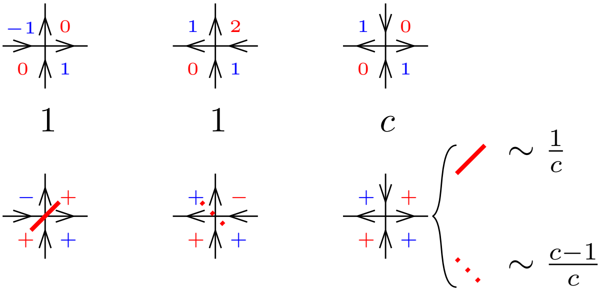

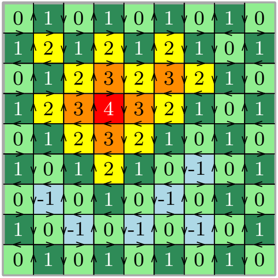

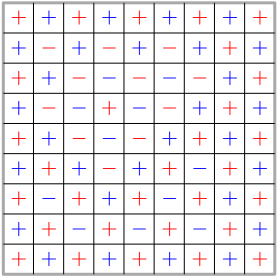

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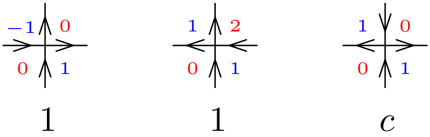

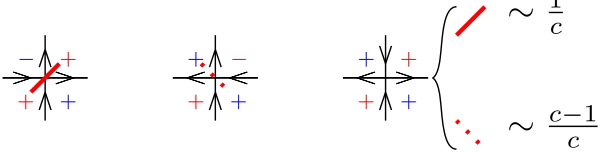

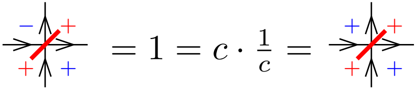

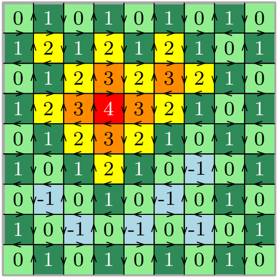

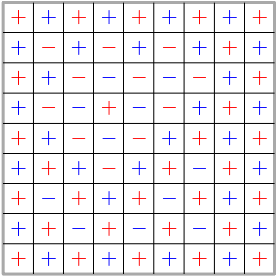

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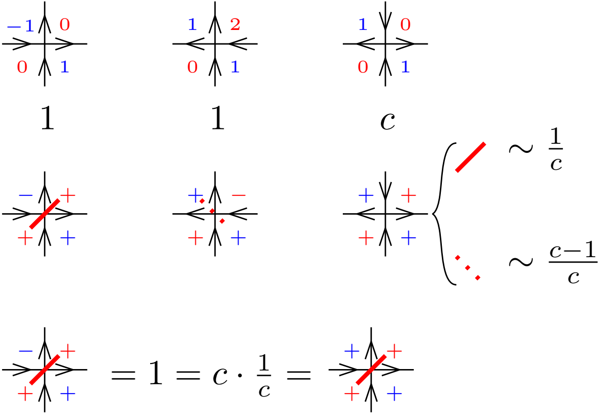

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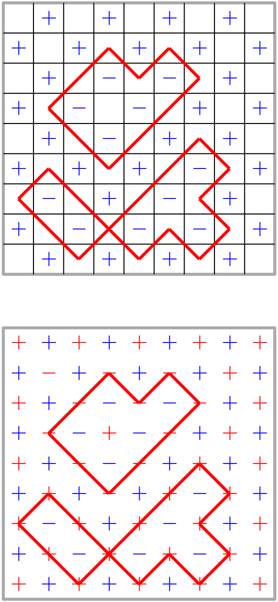

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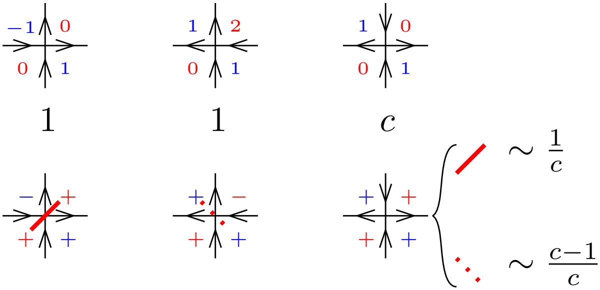
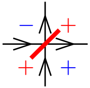
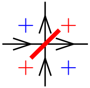


 $= 1 = c \cdot \frac{1}{c} =$ 

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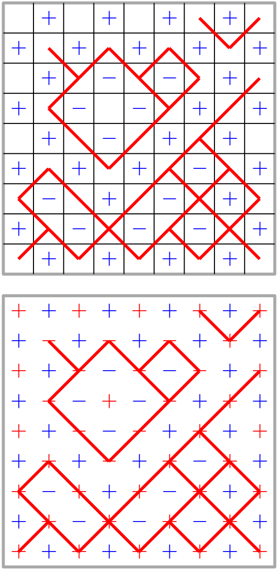

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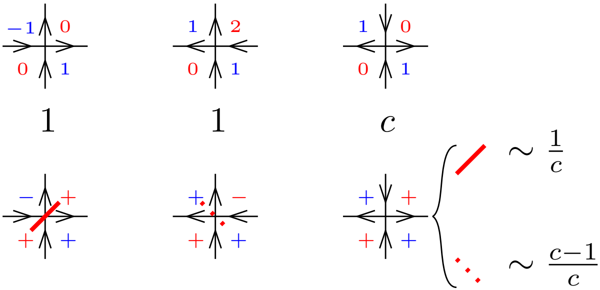
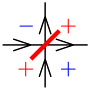
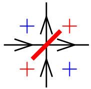


 $= 1 = c \cdot \frac{1}{c} =$ 

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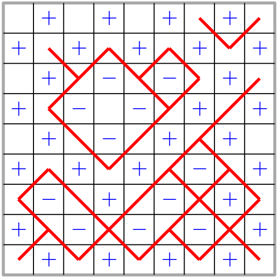

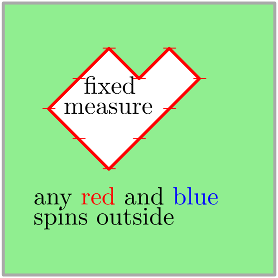

Step 2: joint FKG property

Spins σ^{even} and edges $\mathrm{ES}^{\text{even}}$: $\sigma^{\text{even}} \equiv$ const on clusters of $\mathrm{ES}^{\text{even}}$.
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Prop (G.–Lammers '23)

Let $a, b \leq c$. The triplet $(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}+}, -\mathrm{ES}^{\text{even}-})$ satisfies the FKG inequality:

$$\mathbb{E}[F(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}}) \cdot G(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}})] \geq \mathbb{E}[F(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}})] \cdot \mathbb{E}[G(\sigma^{\text{even}}, \mathrm{ES}^{\text{even}})],$$

for any F, G increasing in σ^{even} and $\mathrm{ES}^{\text{even}+}$ and decreasing in $\mathrm{ES}^{\text{even}-}$.

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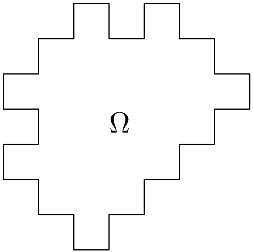
Proof.

1. FKG for $(\mathrm{ES}^{\text{even}+}, -\mathrm{ES}^{\text{even}-})$ is satisfied by $\mathbb{P}^\sigma := \mathbb{P}(\cdot \,|\, \sigma^{\text{even}} = \sigma)$;
2. the law of $(\mathrm{ES}^{\text{even}+}, -\mathrm{ES}^{\text{even}-})$ under $\mathbb{P}^\sigma$ is $\nearrow$ in σ;
3. $\mathbb{E}[F \cdot G] = \int \mathbb{P}^\sigma(F \cdot G) \geq \int \mathbb{P}^\sigma(F) \cdot \mathbb{P}^\sigma(G) \geq \int \mathbb{P}^\sigma(F) \cdot \int \mathbb{P}^\sigma(G) = \mathbb{E}[F] \cdot \mathbb{E}[G]$.

□

Step 3: Ergodicity for σ^{even}

Take μ_Ω^+: marginal of $\mathbb{P}_\Omega^+$ on σ^{even} .
What are the **maximal boundary conditions** for σ^{even}?



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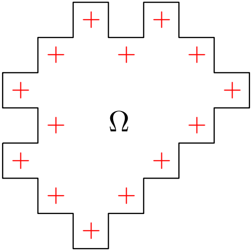

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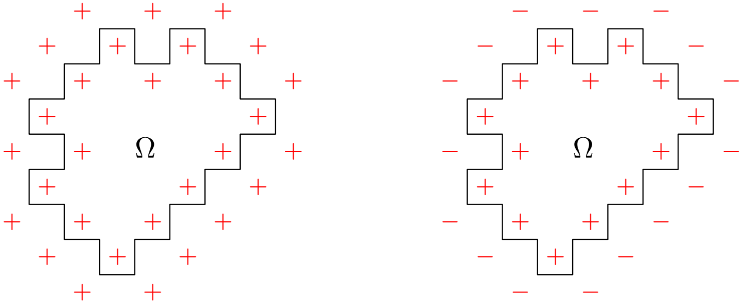

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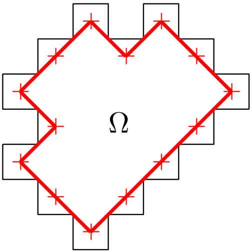
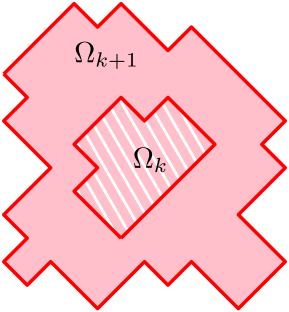
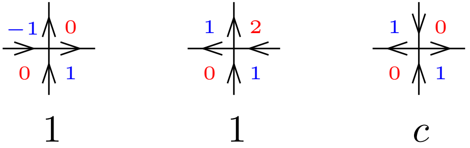
Not enough: the measure on Ω also depends on σ^{even} outside.
Solution: augment randomness and take $\mathrm{ES}^{\text{even}+} \equiv 1$ on $\partial\Omega$.
$\Rightarrow$ weak limit $\mu_\Omega^+ \searrow \mu^+$ exists, is **ergodic** and **tail trivial**.
Define $\mathbb{P}^+$: assign $\pm$ to clusters of $(\mathrm{ES}^{\text{even}})^* \sim 1/2$ independently.

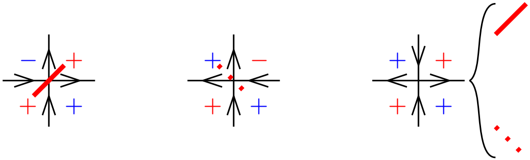




Step 4: Coupled even & odd edges: $\mathrm{ES}^{\text{even}}$ and ES^{odd}

Coupled $\mathrm{ES}^{\text{even}}$ and ES^{odd} edge configurations **[Lis'19]**.





$\sim \frac{1}{c}$

$\sim \frac{c-1}{c}$

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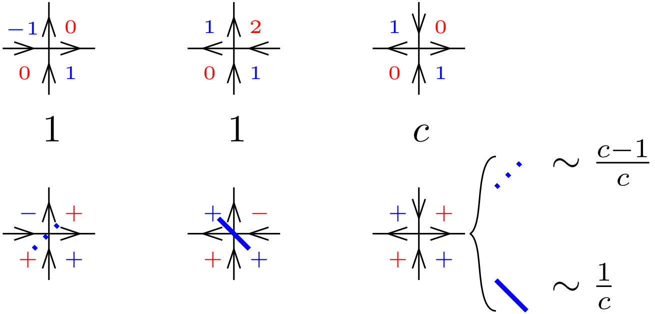

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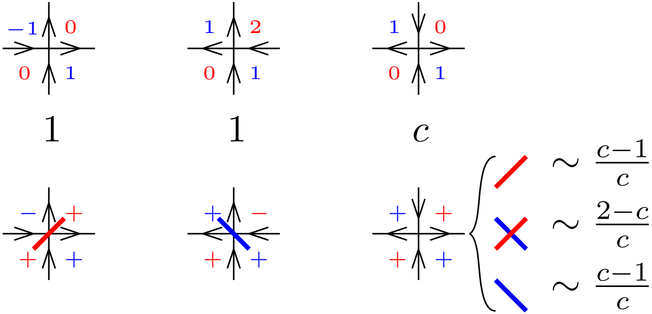

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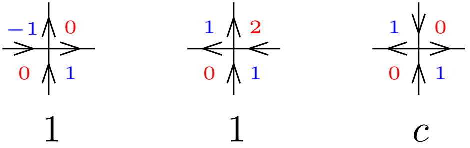
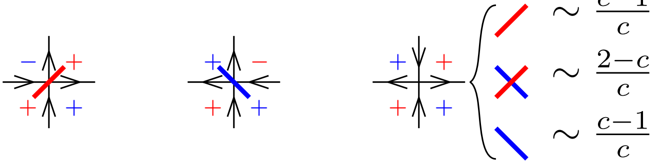


Super-duality

$$(\mathrm{ES}^{\text{even}})^* \subseteq \mathrm{ES}^{\text{odd}}$$



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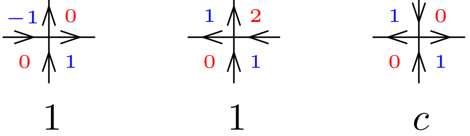

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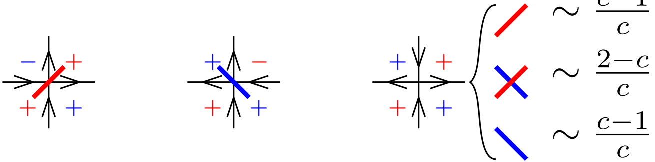

Sample **a part** of $(\mathrm{ES}^{\text{even}}, \mathrm{ES}^{\text{odd}})$:

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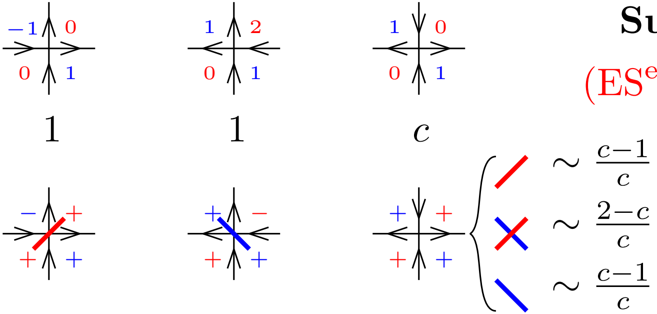

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Conditioned on this, $h(0) \sim$ **Simple Random Walk on the clusters**.
Goal: No infinite clusters in ($\mathrm{ES}^{\text{even}}$ and ES^{odd}).

Step 5: Non-coexistence + super-duality $\Rightarrow$ full ergodicity

No ∞ cluster in $(\mathrm{ES}^{\mathsf{even}})^*$ $\Rightarrow$ $\mathbb{P}^+$ is ergodic $\Rightarrow$ no ∞ cluster in $\mathrm{ES}^{\mathsf{odd}}$.

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Assume $\exists$ an ∞ cluster in $(\mathrm{ES}^{\text{even}})^*$. By **[Burton–Keane'89]** and $\mathrm{ES}^{\text{even}}$-ergodicity,

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Theorem (Zhang'90s; Sheffield '05)

If μ is a probability measure on $\{0,1\}^{E(\mathbb{Z}^2)}$ that is FKG and shift invariant, then

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(*) and **non-coexistence** imply $\mathbb{P}^+(\mathrm{ES}^{\text{even}}$ has an ∞ cluster$) = 0$.
Compare with (**): contradiction with the red/blue symmetry.

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Recall our goal: no ∞ cluster in $\mathrm{ES}^{\mathsf{odd}}$ (**done!**) nor in $\mathrm{ES}^{\mathsf{even}}$ (**not yet**).

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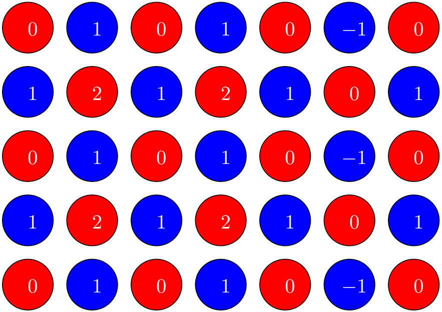

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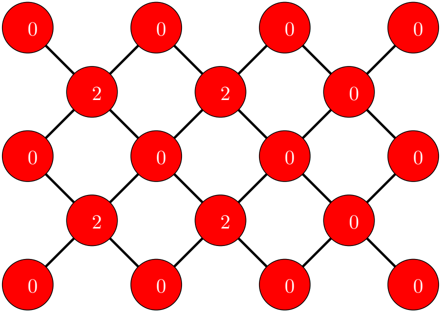

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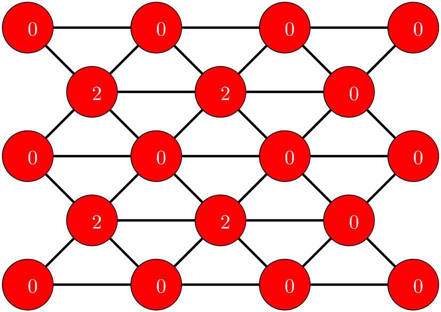

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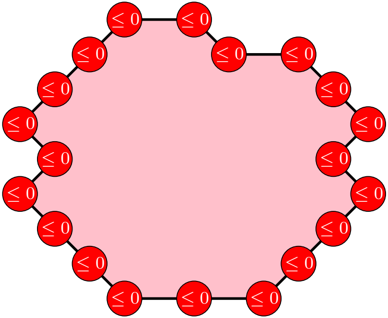

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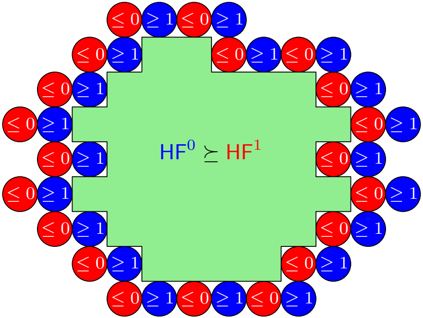

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Outside define:

$$h^0(i,j) := 1 - h^1(i-1,j) \sim \mathsf{HF}^1.$$

Conditioned on the exterior of the circuit:

$$\mathsf{HF}^0 \succeq \mathsf{HF}^1 \quad \text{in the interior.}$$

Lipschitz functions; loop $O(n)$ model at $n = 2$

$h\colon \mathrm{Faces}(\mathrm{Hex}) \to \mathbb{Z}$, so that $h(u) - h(v) \in \{0, \pm 1\}$ if $u \sim v$. Measure:

$$\mathbb{P}(h) \propto x^{\#\{u\sim v\colon h(u)\neq h(v)\}} \quad \Rightarrow \quad \mathbb{P}(\text{level lines}) \propto 2^{\#\text{loops}} x^{\#\text{edges}}$$

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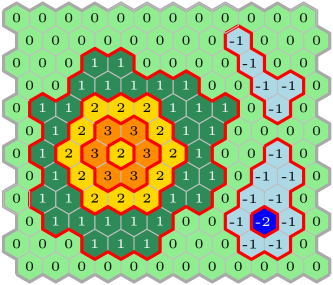

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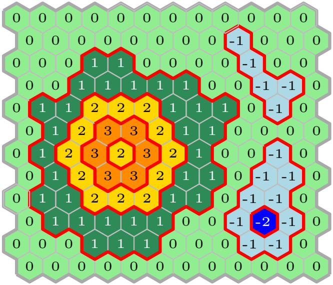

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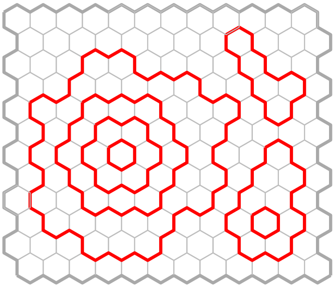

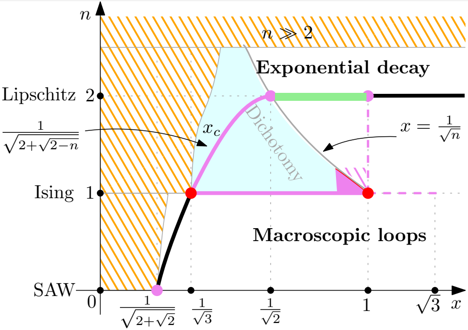

Random-cluster model: continuity of the phase transition

$\mathbb{Z}^2$, $p \in (0,1)$, $q > 0$. Box $\Lambda_n = (V, E) \subset \mathbb{L}$.
For a percolation configuration $\omega \in \{\text{closed}, \text{open}\}^E$,

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Theorem (G.–Lammers '23)

Let $1 \leq q \leq 4$. Then, $\mathbb{P}^{\text{wired}} = \mathbb{P}^{\text{free}}$. No infinite cluster at the self-dual line.

Works also for anisotropic weights (i.e. rectangular lattice).
Not a new result:
[Duminil-Copin–Sidoravicius–Tassion'15], [Duminil-Copin–Li–Manolescu'17]
New proof:
no use of parafermionic observable, Bethe Ansatz, Yang–Baxter.

Discussion

Summary:

1. six-vertex and Lipschitz together;
2. RCM: continuity without integrability;
3. joint FKG: spins + Edwards–Sokal;
4. non-coexistence theorem;
5. $\mathbb{T}$-circuit argument;
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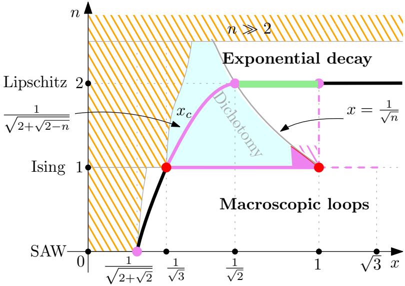

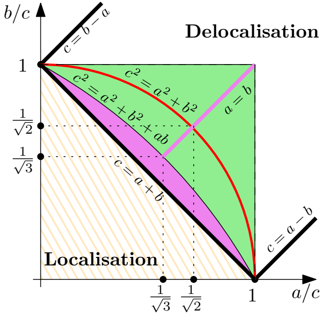

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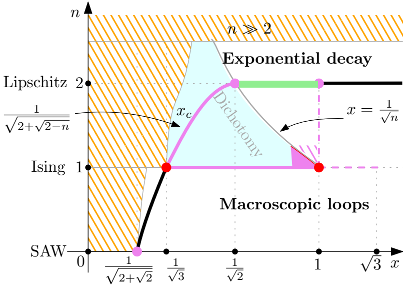
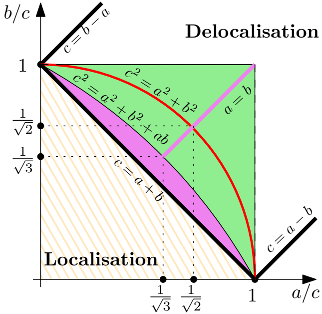
Future directions:

1. Lipschitz: localisation when $x < 1/\sqrt{2}$?
2. Lipschitz functions on $\mathbb{Z}^2$?
3. Loop $O(n)$ at $x_c(n)$ without integrability?
4. RSW/dichotomy without $pi/2$ rotations: log-deloc when $a \neq b$?
5. random-cluster model: $q < 1$?





Lipschitz functions: site percolation duality

h mod 4	0	1	2	3
red spin	+	+	−	−
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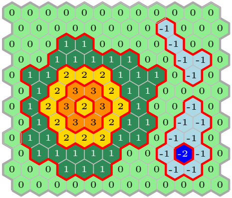

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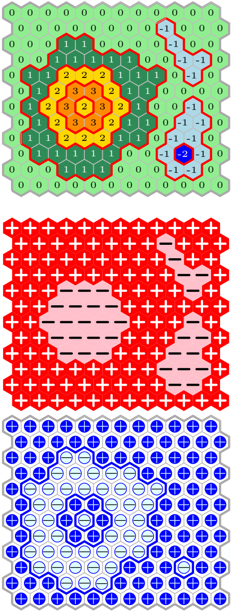

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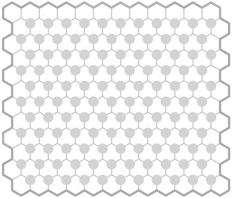

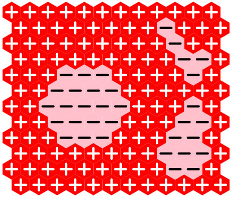

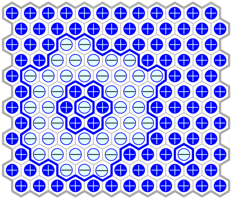

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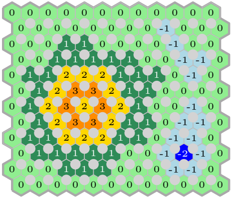

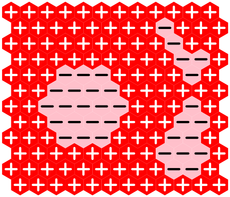

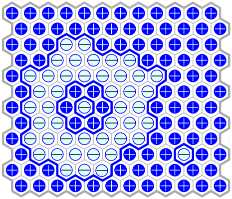

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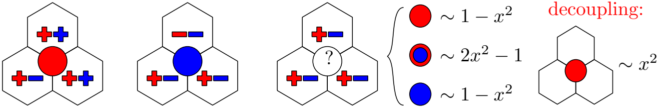
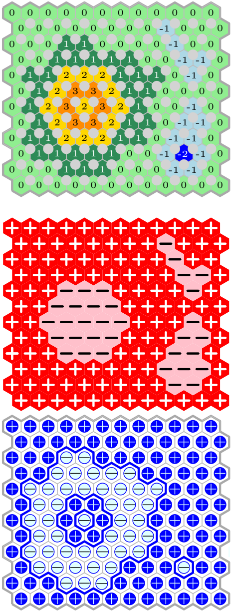


x^2 x^2 1



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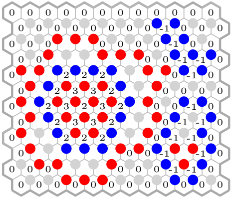

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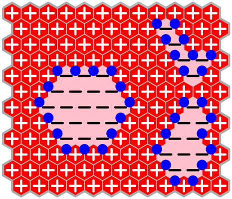

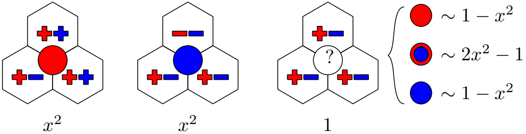

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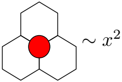

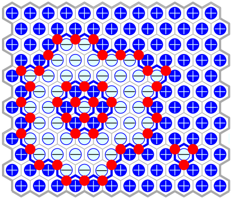

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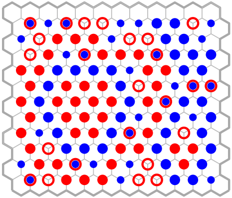

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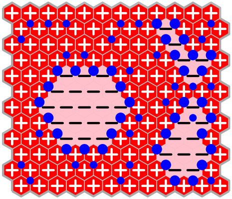

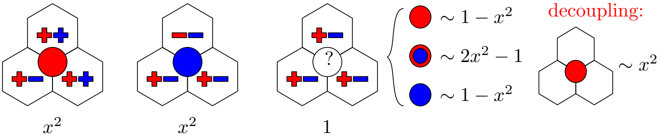

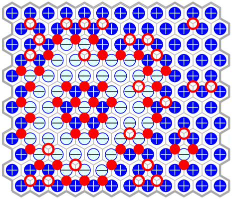

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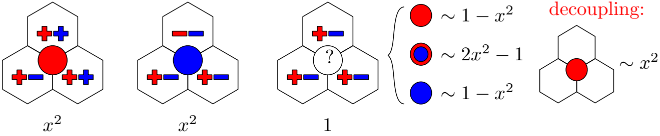
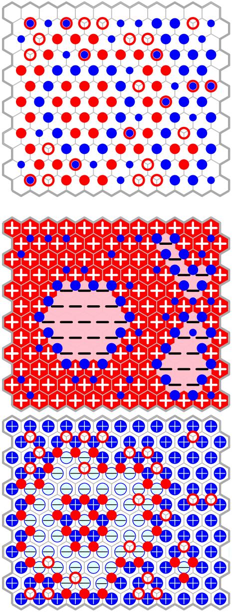


FKG for $(\sigma, \mathrm{ES}^+, -\mathrm{ES}^-)$. **Super-duality** when $x^2 \geq 1/2$.
As before: the limit $\mathbb{P}_n^+ \to \mathbb{P}^+$ is ergodic.
Non-coexistence: $\{\sigma = +\}$ and $\{\sigma = -\}$ don't percolate.
⇒ there are ∞ many blue loops ⇒ same for red loops.



Random-cluster model: BKW correspondence

[Temperley–Lieb'71], [Baxter–Kelland–Wu'76]

Symmetric: $a = b = 1$, $p = p_{\mathrm{sd}} = \frac{\sqrt{q}}{\sqrt{q}+1}$. Write $\sqrt{q} = 2\cos\lambda$.

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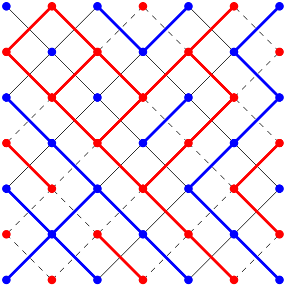

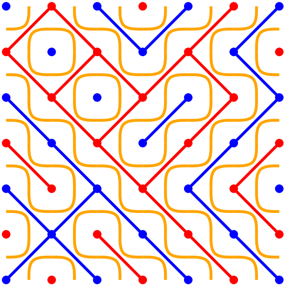

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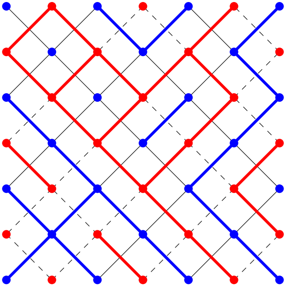

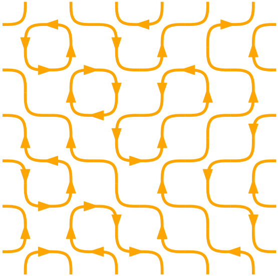

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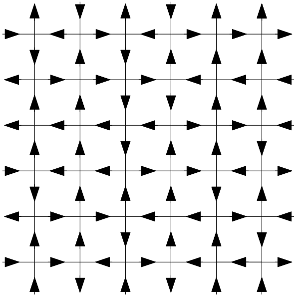

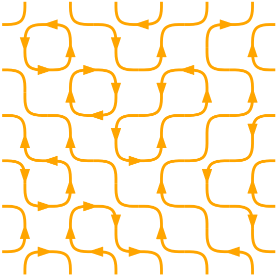

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Same holds with a defect line **[Dubédat'11]**:

$$\mathbb{E}_{6\mathrm{V}}[e^{i\alpha(h(u)-h(v))}] = \mathbb{E}_{\mathrm{RCM}}[F_{\lambda,\alpha}(\#\mathrm{loops}(u), \#\mathrm{loops}(v))],$$

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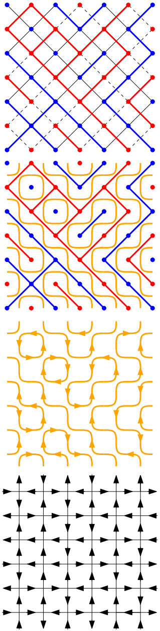

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Note: $\lambda \in [0, \pi/3]$. Fix $\alpha = \pi/8 \in (0, \pi/6)$. Then,

$$\mathbb{E}_{6\mathrm{V}}[e^{i\alpha(h(u)-h(v))}] \geq \mathbb{P}_{\mathrm{RCM}}(u \leftrightarrow v).$$

By **delocalisation**: $\mathbb{E}_{6\mathrm{V}}[e^{i\alpha(h(u)-h(v))}] \to 0$, as $|u-v| \to \infty$.