Condensation in interacting particle systems

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Condensation in IPS has been studied in probability and statistical mechanics for the past 20 years or so [12, 13, 22]. While there exist continuous versions, we focus on models with a conserved quantity (called mass) which is discretized into individual particles moving on a regular lattice or a graph. These can represent actual mass or energy in physical applications, individuals in a biological population or vehicles in a traffic model. When the particle density exceeds a critical level, a condensing IPS separates into a bulk phase with a homogeneous distribution of mass, and a condensate where a non-zero mass fraction concentrates on a vanishing volume fraction. Classical examples which will be discussed in the lectures include zero-range [29, 13] and inclusion processes [16], and more general models of mass transport [15] or exchange-driven growth [5]. In the first lecture we will discuss results at stationarity in analogy to the classical theory of Gibbs measures and phase transitions. In the second lecture we address the dynamics of condensing IPS, for which there are various open problems.

Part 1 - Stationary results. We define condensation for a sequence of canonical measures $\pi_{L,N}$ with a single conserved quantity (total mass N) and lattice size L in the thermodynamic limit $N, L \to \infty$ with $N/L \to \rho$. The particle density $\rho \geq 0$ is the order parameter for the condensation transition which is characterized in the context of the equivalence of ensembles. We focus mostly on spatially homogeneous product measures which arise in a large class of IPS [11]. Here the condensed phase usually concentrates on a single lattice site and can be characterized by large deviations for sub-exponential random variables. When system parameters are scaled with the size L, the condensate may also exhibit a non-trivial structure, which is related to Poisson-Dirichlet distributions for inclusion processes [24, 9]. As a phase transition, condensation can be continuous or discontinuous and we will discuss several natural examples for both cases [21]. We shortly mention extensions to nearest-neighbour (or pair-factorized) stationary measures [14], spatial inhomogeneities (e.g. [10]) and more than one conserved quantity (several particle species) [23].

Part 2 - Dynamic results. The stationary dynamics of condensing IPS was the first to be understood rigorously [3], and can exhibit metastability w.r.t. the spatial location of the condensate in spatially homogeneous systems. We will discuss a combination of potential theory [7] and martingale techniques [4] that has been developed to derive rigorous scaling limits for the dynamics of the condensate location for reversible zero-range [3, 2] and inclusion processes [6], and most recently also for non-reversible models [27, 26].

Out-of-equilibrium dynamics of condensing IPS usually consists of three regimes [22, 18]: Starting from a homogeneous initial distribution at a super-critical density, particle clusters form locally (nucleation), then exchange mass leading to a decrease in the number and an increase in the typical size of clusters (coarsening). When the largest cluster size scales like the total mass N, the system saturates and converges to a typical stationary configuration on a time scale depending on L. We will present a complete heuristic picture, including a coarsening scaling law [25, 17], and a few rigorous results that have been obtained so far [20, 1], related also to propagation of chaos in mean-field models [19, 28].

All above mentioned results rely on a separation of time scales where dynamics in the condensed phase is slower than in the bulk, which is the case e.g. for zero-range dynamics. There is also another class of models related to exchange-driven growth [15, 8], where the condensate is faster than the bulk. This leads to qualitatively different dynamics for which there are essentially no rigorous results so far.

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