Condensation in Interacting Particle Systems

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Phase transitions in spatial particle systems

Berlin July 31-Aug 2, 2023

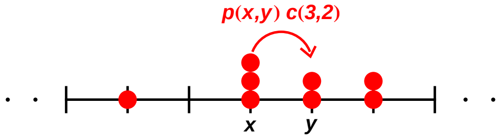
Part I - Stationary results

Stochastic particle systems

lattice Λ of size $|\Lambda| = L$

state space $E_L = \{0, 1, ..\}^\Lambda$

$$\eta = (\eta_x)_{x\in\Lambda}$$



conserved quantitiy $\Sigma_L(\eta) := \sum_{x\in\Lambda} \eta_x$, $E_{L,N} = \big\{\eta \in E_L : \Sigma_L(\eta) = N\big\}$

we consider IPS as CTMC with **generator**

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} r(\eta, x, y)\big(f(\eta^{x,y}) - f(\eta)\big)$$

Jump rates $r(\eta, x, y) = p(x, y)\, c(\eta_x, \eta_y)$ with $c(k, l) = 0 \;\Leftrightarrow\; k = 0$

p irreducible and **homogeneous** $\sum_{y\in\Lambda} \big(p(x, y) - p(y, x)\big) = 0$, $x \in \Lambda$

[Spitzer (1970); Liggett (1985); Cocozza-Thivent (1985)]

Stochastic particle systems

Zero-range process (ZRP) $c(k,\ell) = g(k) = \mathbb{1}_{k>0}(1 + b/k)\ , \quad b \geq 0$

condensation for $b > 2$

[Spitzer (1970); Andjel (1982); Drouffe, Godréche, Camia (1998); Evans (2000); Jeon, March, Pittel (2000)]

Inclusion process (IP) $c(k,\ell) = k(d + \ell)\ , \quad d \geq 0$

multispecies Moran model with mutation rate d, condensation for $d = d_L \to 0$

[Giardiná, Kurchan, Redig (2007); G., Redig, Vafayi (2011,13); Bianchi, Dommers, Giardiná (2017); Kim, Seo (2021)]

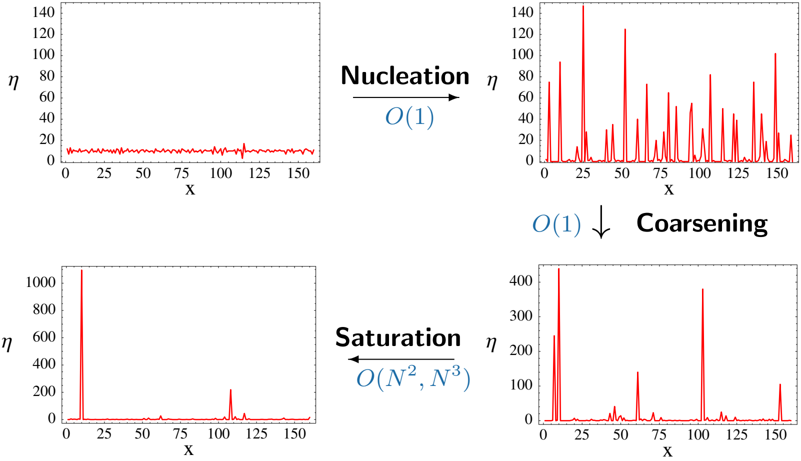
Exchange-driven growth / Explosive condensation (EDG)

$$c(k,\ell) = k^\gamma(d + \ell^\gamma)\ , \quad d \geq 0,\ \gamma > 0$$

condensation for $\gamma > 2$ [Ben Naim, Krapivsky (2003); Waclaw, Evans (2012-15)]

Dynamics of condensation

ZRP with $g(k) = 1 + b/k$, $b = 4$, $\rho_c = 1/(b-2) = 0.5$, $\rho = 10$



hydrodynamics $O(N, N^2)$ [subcritical, Stamatakis (2015)]

stationary dynamics of condensate $O(N^{1+b})$

Condensation

canonical stationary **measures** $\pi_{L,N}$ on $E_{L,N}$

spatially homogeneous $\pi_{L,N}(\eta_x) = \sum_{n\geq 1} n\pi_{L,N}[\eta_x = n] = N/L \quad x \in \Lambda$

thermodynamic limit $L, N = N_L \to \infty\ , \quad N/L \to \rho \geq 0$

single-site marginals assume that $\mu_\rho := \lim_{L,N} \pi_{L,N}[\eta_x \in .]$

exists as a weak limit on $\mathcal{M}_1(\mathbb{N}_0)$, i.e. $\pi_{L,N}(f) \to \mu_\rho(f)\ , \quad f \in C_b(\mathbb{N}_0)$

background density $\rho_b := \mu_\rho(\eta_x) \quad \leq \rho = \lim_{L,N} N/L$ (Fatou's Lemma)

Definition

A spatially homogeneous IPS exhibits condensation with background density $\rho_b \geq 0$, if μ_ρ exists and $\rho_b < \rho$.

The IPS exhibits a condensation transition with critical density $\rho_c \geq 0$,

if μ_ρ exists for all $\rho \geq 0$, and $\rho_b \begin{cases} = \rho\ ,\ \rho < \rho_c & \text{(fluid state)} \\ < \rho\ ,\ \rho > \rho_c & \text{(condensed state)} \end{cases}$.

Condensation

Phase separation

A finite fraction of mass concentrates on a vanishing volume fraction (condensate/condensed phase).

The background/bulk phase has single-site marginal μ_ρ.

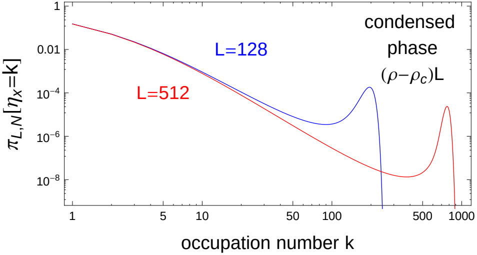
weak convergence of single-site marginals implies in the limit $\lim_{K\to\infty}\lim_{L,N}$

		condensed		bulk/background
mass fraction	$\pi_{L,N}(\eta_x)$	$= \pi_{L,N}\big(\eta_x \mathbb{1}_{\eta_x > K}\big)$	+	$\pi_{L,N}\big(\eta_x \mathbb{1}_{\eta_x \leq K}\big)$
	$\to \rho$	$\to \rho - \rho_b$		$\to \rho_b$
volume fraction	$\pi_{L,N}(1)$	$= \pi_{L,N}(\mathbb{1}_{\eta_x > K})$	+	$\pi_{L,N}(\mathbb{1}_{\eta_x \leq K})$
	$= 1$	$\to 0$		$\to 1$

Furthermore, $\pi_{L,N}\big(\eta_x f(\eta_x)\big) \to \infty$ and $\pi_{L,N}\big(\eta_x / f(\eta_x)\big) \to \mu_\rho\big(\eta_x / f(\eta_x)\big)$

as $L, N \to \infty$, $N/L \to \rho$ for all $f : \mathbb{N}_0 \to \mathbb{R}$ with $f(n) \to \infty$, $n \to \infty$

Condensation

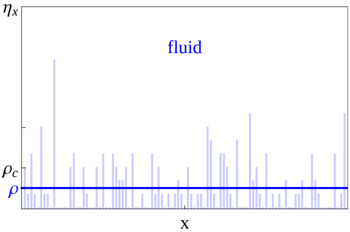


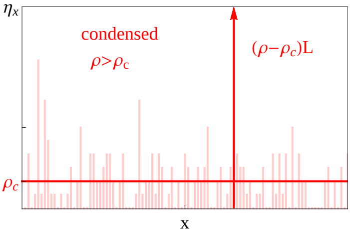
Phase separation for $\rho > \rho_c$

ZRP with rates

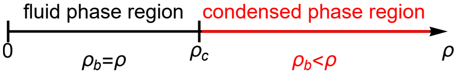
$g(k) = 1 + b/k$, $b = 4$

$\rho_c = 0.5$, $\rho = N/L = 2$





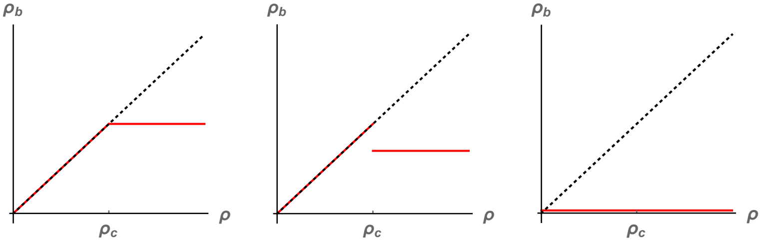
Phase diagram



Condensation

- Condensation is a **phase transition** with order parameter $\rho_b(\rho)$.
- No **grand-canonical** (Gibbs-)measures used for the definition, characterized only through single-site marginals.
- Divergence of higher order **moments**, sometimes used as order parameter

e.g. $\lim_{N/L\to\rho} \frac{1}{L}\pi_{L,N}(\eta_x^2) \begin{cases} = 0\ , & \text{fluid state} \\ > 0\ , & \text{condensed state} \end{cases}$ [O'Loan, Evans, Cates (1998)]

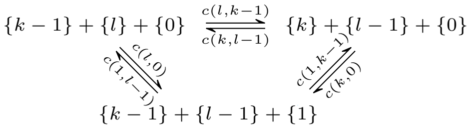


Part I - Stationary results

Grand-canonical product measures

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} p(x,y)c(\eta_x,\eta_y)\big(f(\eta^{x,y}) - f(\eta)\big)$$

curl-free condition $\quad \dfrac{c(l,k-1)}{c(k,l-1)} = \dfrac{c(l,0)}{c(1,l-1)}\dfrac{c(1,k-1)}{c(k,0)}$



[Schlichting (2018)]

and $\quad p(x,y)=p(y,x) \quad$ or $\quad c(k,l)-c(l,k)=c(k,0)-c(l,0)$

GC product measures

[Cocozza-Thivent (1985); Fajfrová, Gobron, Saada (2016)]

A spatially homogeneous IPS as above has **stationary product measures**

$$\nu_\phi^L[d\eta] = \prod_{x\in\Lambda} \frac{1}{z(\phi)} w(\eta_x)\phi^{\eta_x} d\eta \quad \text{with} \quad w(n) = \prod_{k=1}^{n} \frac{c(1,k-1)}{c(k,0)}$$

for all **fugacities** $\phi \in \mathcal{D}_{gc} \subset [0,\infty)$ such that $\quad z(\phi) := \sum_{n\geq 0} w(n)\phi^n < \infty$.

Grand-canonical product measures

Proof. $\nu_\phi^L(\mathcal{L}f) = \sum_{\eta\in E_L} \sum_{xy\in\Lambda} p(x,y)c(\eta_x,\eta_y)\big(f(\eta^{xy}) - f(\eta)\big)\nu_\phi^L[\eta]$

change of variable for fixed $x, y \in \Lambda$

$$\sum_{\eta\in E_L} c(\eta_x,\eta_y)\nu_\phi^L[\eta]f(\eta^{xy}) = \sum_{\eta\in E_L} \underbrace{c(\eta_x{+}1,\eta_y{-}1)\nu_\phi^L[\eta^{yx}]}_{=c(\eta_y,\eta_x)\nu_\phi[\eta]} f(\eta)$$

In general: $\mathcal{D}_{gc} = [0,\phi_c)$ or $[0,\phi_c]$ where $\phi_c \in [0,\infty]$ is r.o.c. of $z(\phi)$

Examples

- **ZRP**: $g(k) = k$ (independent particles)

$$w(n) = \prod_{k=1}^{n} \frac{1}{g(k)} = \frac{1}{n!} \ , \quad \nu_\phi^1[\eta_x = n] = \frac{\phi^n}{n!}\, e^{-\phi} \ , \quad \mathcal{D}_{gc} = [0,\infty)$$

- **ZRP**: $g(k) \equiv g > 0$

$$w(n) = g^{-n} \ , \quad \nu_\phi^1[\eta_x = n] = (1-\phi/g)(\phi/g)^n \ , \quad \mathcal{D}_{gc} = [0,g)$$

Grand-canonical product measures

Examples (with $\phi_c = 1$)

- **ZRP**: $g(k) = \mathbb{1}_{k>0}(1 + b/k)$

$$w(n) = \prod_{k=1}^{n} \frac{1}{g(k)} = \prod_{k=1}^{n} \frac{k}{k+b} = \frac{n!\Gamma(b+1)}{\Gamma(b+1+n)} \simeq \Gamma(b+1)n^{-b}$$

- **EDG**: $c(k,l) = k^\gamma(d + l^\gamma)$

$$w(n) = \prod_{k=1}^{n} \frac{d + (k-1)^\gamma}{k^\gamma} \sim n^{-\gamma}$$

Density $R(\phi) := \nu_\phi^1(\eta_x) = \sum_{n\geq 1} n \frac{w(n)\phi^n}{z(\phi)} = \phi\,\partial_\phi \log z(\phi)$, $\phi \in \mathcal{D}_{gc}$

- $R(\phi) = \partial_{\log\phi} z(\phi)$, $\mu = \log\phi$ **chemical potential**
- $\partial^2_{\log\phi} z(\phi) = \nu_\phi^1(\eta_x^2) - \nu_\phi^1(\eta_x)^2 > 0$

  so $\log z(\phi)$ is a **strictly convex function** of $\mu = \log\phi$
- $\Rightarrow$ $R(\phi) \uparrow$ on $[0, \phi_c)$. Also, $R(\phi) \uparrow \infty$ if $\phi_c = \infty$.

Grand-canonical product measures

Examples (with $\phi_c = 1$)

- **ZRP**: $g(k) = \mathbb{1}_{k>0}(1 + b/k)$

$$w(n) = \prod_{k=1}^{n} \frac{1}{g(k)} = \prod_{k=1}^{n} \frac{k}{k+b} = \frac{n!\Gamma(b+1)}{\Gamma(b+1+n)} \simeq \Gamma(b+1) n^{-b}$$

- **EDG**: $c(k,l) = k^\gamma (d + l^\gamma)$

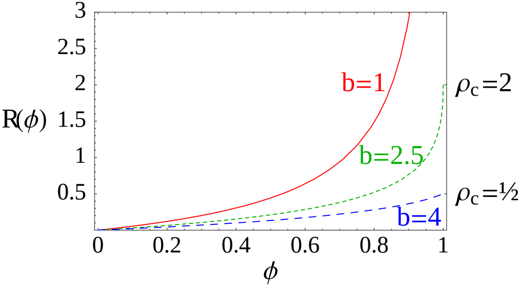
$$w(n) = \prod_{k=1}^{n} \frac{d + (k-1)^\gamma}{k^\gamma} \sim n^{-\gamma}$$

ZRP $w(n) \sim n^{-b}$

$\mathcal{D}_{gc} = [0,1)$, $b \in [0,1]$

$\mathcal{D}_{gc} = [0,1]$, $b > 1$

$\rho_c = R(1) < \infty$, $b > 2$



Conserved quantities

$$f\big(\eta(t)\big) - f\big(\eta(0)\big) = \int_0^t \mathcal{L}f\big(\eta(s)\big)ds + \mathcal{M}_f(t)$$

with martingale $\big(\mathcal{M}_f(t) : t \geq 0\big)$ with (predictable) **quadratic variation**

$$\langle \mathcal{M}_f \rangle(t) = \int_0^t \big(\mathcal{L}f^2 - 2f\mathcal{L}f\big)\big(\eta(s)\big)ds$$

- $\mathcal{L}f(\eta) = 0$, $\eta \in E \quad \Leftrightarrow \quad \big(f(\eta(t)) : t \geq 0\big)$ is a **martingale**
- $\mathcal{L}f(\eta), \mathcal{L}f^2(\eta) = 0$, $\eta \in E \quad \Leftrightarrow \quad f(\eta(t)) \equiv f(\eta(0))$, $t \geq 0$

  and $f : E \to \mathbb{R}$ is **conserved** $\quad \big(\Rightarrow\ g(f)$ conserved for any $g : \mathbb{R} \to \mathbb{R}\big)$
- ν_ϕ is stationary on E_L, Σ_L and thus $\mathbb{1}_{E_{L,N}}$ is conserved for any $N \geq 0$

$$\nu_\phi^L\big[.\big|\Sigma_L = N\big] = \frac{\mathbb{1}_{E_{L,N}}}{\nu_\phi^L[\Sigma_L = N]}\,\nu_\phi^L$$

  is stationary on $E_{L,N}$, and by **uniqueness** we have

$$\pi_{L,N} = \nu_\phi^L\big[.\big|\Sigma_L = N\big] \quad \text{for any } \phi \in \mathcal{D}_{gc}$$

Part I - Stationary results

Equivalence of ensembles

- **canonical ensemble** $\{\pi_{L,N} : L \geq 1,\ N \geq 0\}$
- **grand-canonical ensemble** $\{\nu_\phi^L : L \geq 1,\ \phi \in \mathcal{D}_{gc}\}$

Specific relative entropy $\pi_{L,N} \ll \nu_\phi^L$ for all $\phi \in \mathcal{D}_{gc}$

$$h_{L,N}(\phi) := \frac{1}{L} H\Big(\pi_{L,N}\,;\,\nu_\phi^L\Big) = \frac{1}{L} \sum_{\eta \in E_{L,N}} \pi_{L,N}(\eta) \log \frac{\pi_{L,N}(\eta)}{\nu_\phi^L(\eta)}$$

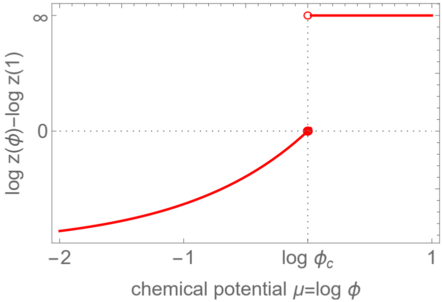
$$h_{L,N}(\phi) = -\frac{1}{L} \log \nu_\phi^L\big[\Sigma_L = N\big]$$

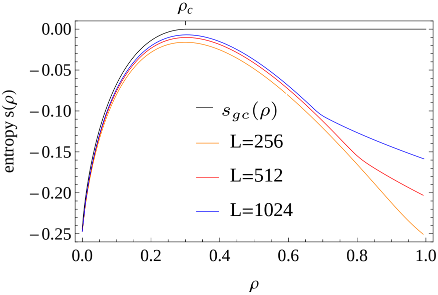
[Csiszár, Körner (1981), Lewis et al (1994)]

$$\inf_{\phi\in\mathcal{D}_{gc}} h_{L,N}(\phi) = \underbrace{\inf_{\phi\in\mathcal{D}_{gc}} \Big[\log z(\phi) - \frac{N}{L}\log\phi\Big]}_{\text{entropy } s_{gc}(N/L)} - \frac{1}{L}\log Z_{L,N} \to s_{gc}(\rho) - s_c(\rho)$$

minimizer $\phi = \Phi(\rho) := \begin{cases} R^{-1}(\rho)\ , & \rho \leq \rho_c \\ \phi_c\ , & \rho \geq \rho_c \end{cases}$ as $N/L \to \rho$

Equivalence of ensembles





thermodynamic pressure $\lim_{L\to\infty} \frac{1}{L} \log z^L(\phi) = \log z(\phi)$

convex function of $\log \phi \in \mathbb{R}$

gc entropy/free energy $s_{gc}(\rho) = \log z\big(\Phi(\rho)\big) - \rho \log \Phi(\rho)$ concave

Equivalence of ensembles

Theorem. [G, Schütz, Spohn (2003); Chleboun, G (2014)]

Consider an IPS with grand-canonical ensemble $\{\nu_\phi^L\}$ with $\phi_c \in [0,\infty)$ and let $\rho_c := R(\phi_c) \in [0,\infty]$. Then in the thermodynamic limit $N/L \to \rho$

$$h_{L,N}(\phi) \to 0 \quad \text{provided that} \begin{cases} R(\phi) = \rho \ , & \rho < \rho_c \\ \phi = \phi_c \ , & \rho \geq \rho_c \end{cases} .$$

If $\rho_c < \infty$ we have a condensation transition with $\rho_b = \rho_c$.

Corollaries

- **subadditivity** of relative entropy: take $\Delta \subset\subset \Lambda$ [Csiszar (1984)]

$$H\big(\pi_{L,N}^\Delta ; \nu_\phi^\Delta\big) \leq C|\Delta|\, h_{L,N}(\phi)$$

- **Pinsker's inequality** [Pinsker (1960)]

$$d_{TV}\big(\pi_{L,N}^\Delta, \nu_\phi^\Delta\big) = \frac{1}{2}\pi_{L,N}^\Delta\Big(\Big|\frac{\pi_{L,N}^\Delta}{\nu_\phi^\Delta} - 1\Big|\Big) \leq \sqrt{2H\big(\pi_{L,N}^\Delta ; \nu_\phi^\Delta\big)}$$

Corollaries on weak convergence on $E = \mathbb{N}_0^{\mathbb{N}}$

Equivalence of ensembles

Proof idea.

- **local limit theorem** [Davis, McDonald (1995)]

$$h_{L,N}(\phi) = -\frac{1}{L} \log \underbrace{\nu_\phi^L \big[\Sigma_L = N \big]}_{\sim 1/\sqrt{L}} \sim \frac{\log L}{L} \to 0$$

  provided that $R(\phi) = \rho \le \rho_c$ **(subcritical case)**

- **supercricital case:** a simple **large deviation estimate**

$$h_{L,N}(\phi_c) \le -\frac{1}{L} \log \nu_{\phi_c}^{L-1} \big[\Sigma_{L-1} = \rho_c (L-1) \big] - \frac{1}{L} \log \nu_{\phi_c}^{1} \big[\eta_1 = N - \rho_c (L-1) \big]$$

  [Nagaev (1979); Doney (2001)]

- $\nu_{\phi_c}^1$ is **subexponential** since ϕ_c is maximal radius of convergence, i.e.

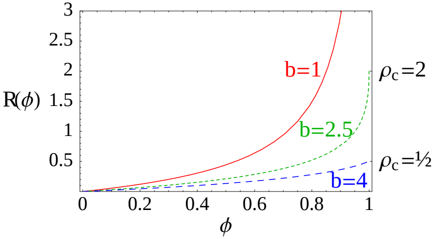
$$\frac{1}{n} \log \nu_{\phi_c}^1 [\eta = n] \to 0 \quad \text{as} \quad n \to \infty \, .$$

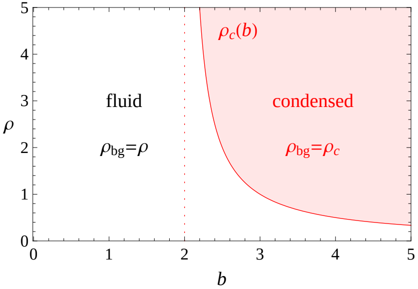
Examples

- **power law** $w(n) \sim n^{-b}$, $\rho_c < \infty \Leftrightarrow b > 2$

  e.g. ZRP with $g(k) = \mathbb{1}_{k>0}(1 + b/k)$

  or EDG with $c(k, l) = k^\gamma (d + l^\gamma)$





- **stretched exponential** $w(n) \sim e^{-bn^{1-\gamma}/(1-\gamma)}$, $\gamma \in (0, 1)$

  e.g. ZRP with $g(k) = \mathbb{1}_{k>0}(1 + b/k^\gamma)$

LLN and CLT for the condensate

Maximum $M_L(\eta) = \max_{x\in\Lambda} \eta_x$

condition on its location $\tilde{\pi}_{L,N} = \pi_{L,N}\big[\,.\big|\eta_1 = M_L\big]$

Theorem. [Armendáriz, Loulakis (2008); Armendáriz, G, Loulakis (2013); Xu (2020)]

Consider a condensing IPS with stationary product measures and $w(n) \sim n^{-b}$, $b > 2$ or $w(n) \sim e^{-\alpha n^{1-\gamma}}$, $\gamma \in (0,1)$. Then

$$d_{TV}\big(\tilde{\pi}_{L,N}^{\Lambda\setminus 1}, \nu_{\phi_c}^{L-1}\big) \to 0 \quad \text{as } N/L \to \rho > \rho_c\ .$$

In particular, $\frac{M_L}{L} \xrightarrow{\pi_{L,N}} \rho - \rho_c$, and we have the **CLT**

$$\frac{M_L - (N - \rho_c L)}{\sigma_L} \xrightarrow{\pi_{L,N}} \begin{cases} \mathcal{L}_{b-1}\ , & 2 < b < 3 \\ \mathcal{N}_{0,1}\ , & \text{otherwise} \end{cases}\ , \quad \sigma_L \sim \begin{cases} L^{1/(b-1)}\ , & 2 < b < 3 \\ \sqrt{L\log L}\ , & b = 3 \\ \sqrt{L}\ , & \text{otherwise} \end{cases}$$

[Denisov, Dieker, Shneer (2008); Armendáriz, Loulakis (2011)]

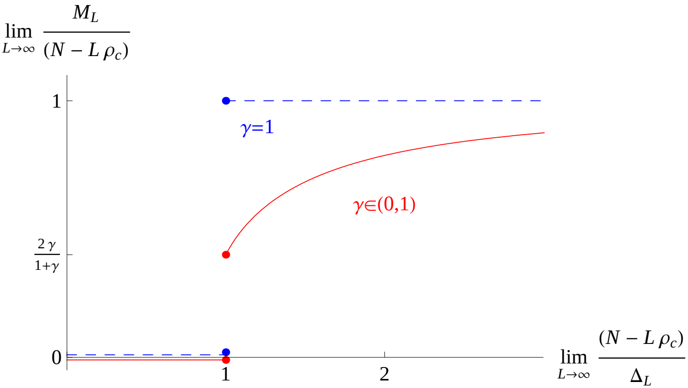
Part I - Stationary results

At the critical point

IPS with $w(n) \sim n^{-b}$, $b > 2$ or $w(n) \sim e^{-\alpha n^{1-\gamma}}$, $\gamma \in (0,1)$

condensation transition with critical density $\rho_c \in (0,\infty)$

thermodynamic limit $N/L \to \rho_c \in (0,\infty)$, **excess mass** $N - \rho_c L \to \infty$



Δ_L is the **critical scale** for the excess mass.

At the critical point

LLN for the maximum. [Armendáriz, G, Loulakis (2013); Berger, Birkner, Yuan (2023)]

Assume $w(n) \sim n^{-b}$, $b > 3$ and let $\Delta_L = \sigma\sqrt{(b-3)L\log L}$. Then

$$\frac{M_L}{N-\rho_c L} \xrightarrow{\pi_{L,N}} \begin{cases} 0 & , \\ Be(p) & , \\ 1 & , \end{cases} \quad \text{if} \quad \lim \frac{N-\rho_c L}{\Delta_L} \begin{cases} < 1 \\ 1 \\ > 1 \end{cases}$$

with $p \in (0,1)$ explicit, depending on $\lim N - \rho_c L - \Delta_L$.

Assume $w(n) \sim e^{-\alpha n^{1-\gamma}}$, $\gamma \in (0,1)$ and let $\Delta_L = c_\gamma (\sigma^2 L)^{1/(1+\gamma)}$. Then

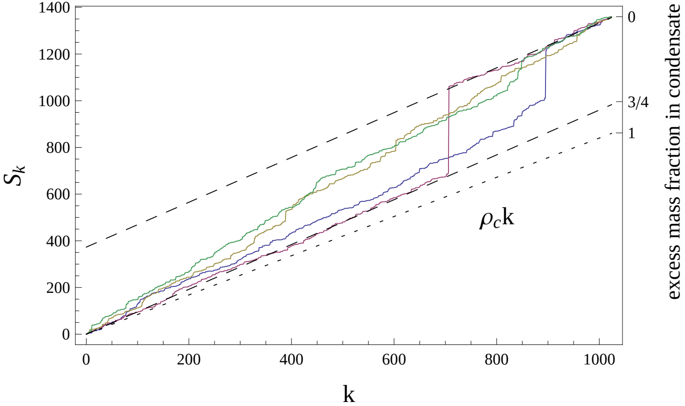
$$\frac{M_L}{N-\rho_c L} \xrightarrow{\pi_{L,N}} \begin{cases} 0 & , \\ \frac{2\gamma}{1+\gamma} Be(p) & , \\ a(t) & , \end{cases} \quad \text{if} \quad t = \lim \frac{N-\rho_c L}{\Delta_L} \begin{cases} < 1 \\ 1 \\ > 1 \end{cases}$$

with $p \in (0,1)$ explicit, depending on $\lim N - \rho_c L - \Delta_L$,

$a(t)$ implicit, $a(1) = \frac{2\gamma}{1+\gamma}$, $a(t) \nearrow 1$ as $t \to \infty$.

At the critical point

$$S_k = \sum_{x=1}^{k} \eta_x$$



$$L = 1024,\ N = 1360,\ \gamma = 0.6,\ b = 2$$
$$\rho_c = 0.842,\ \sigma^2 = 2.55,\ a(1) = \frac{2\gamma}{1+\gamma} = 3/4$$

At the critical point

Bulk fluctuations

[Armendáriz, G, Loulakis (2013); Berger, Birkner, Yuan (2023)]

Assume $\sigma^2 = \nu^1_{\phi_c}(\eta_x^2) < \infty$. If $\frac{M_L}{\Delta_L} \to 0$ **(subcritical regime)**

$$X_s^L := \frac{1}{\sigma\sqrt{L}} \sum_{x=1}^{[sL]} \left(\eta_x - \tfrac{N}{L}\right) \stackrel{\pi_{L,N}}{\Rightarrow} BB_s \ .$$

If $N - \rho_c L > \Delta_L$ and $\frac{M_L}{N-\rho_c L} \to \kappa > 0$ **(supercritical regime)**

$$Y_s^L := \frac{1}{\sigma\sqrt{L}} \sum_{x=1}^{[sL]} \left(\tilde{\eta}_x - \tfrac{N - a_L(N-\rho_c L)}{L}\right) \stackrel{\pi_{L,N}}{\Rightarrow} BB_s + s\Phi \ ,$$

where $\tilde{\eta}_x = \eta_x \mathbb{1}\{\eta_x \leq L^{1/4}\}$ and $\Phi \sim \mathcal{N}\big(0, 1/(1 - \tfrac{\gamma(1-a(t))}{a(t)})\big)$.

Fluctuations of the maximum switch from **Fréchet** (power-law)/ **Gumbel** (stretched exponential) to Gaussian at $N - \rho_c L = \Delta_L$.

[Evans, Majumdar (2008); Iyer, Das, Barma (2023)]

Part I - Stationary results

Finite-size scaling

stationary **current/activity**

canonical $$j_{L,N} = \frac{1}{L}\sum_{x,y\in\Lambda} p(x,y)\pi_{L,N}\big(c(\eta_x,\eta_y)\big) = C_L \pi_{L,N}\big(c(\eta_1,\eta_2)\big)$$

grand-canonical $$j_{Lgc}(\phi) = \frac{1}{L}\sum_{x,y\in\Lambda} p(x,y)\nu_\phi^2\big(c(\eta_x,\eta_y)\big) = C_L \nu_\phi^2\big(c(\eta_1,\eta_2)\big)$$

equivalence of ensembles for bounded rates $c(k,l)$

$$j_{L,N} \to j(\rho) := \begin{cases} j_{gc}(\phi) \ , \ R(\phi)=\rho\le\rho_c \\ j_{gc}(\phi_c) \ , \ \rho>\rho_c \end{cases} \quad \text{as } N/L\to\rho$$

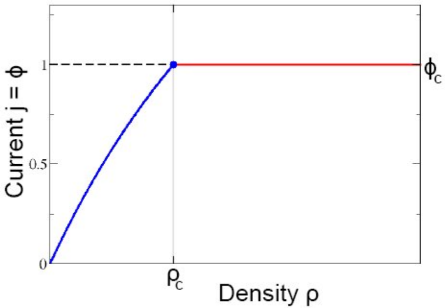
recursion with initial condition $Z_{1,k}=w(k)$, $k\ge 0$

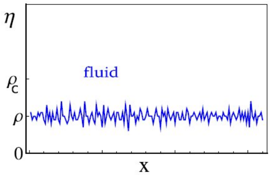
$$Z_{L,N} = \sum_{k=0}^{N} Z_{m,k} Z_{L-m,N-k} \ , \quad m\in\{1,\ldots,L-1\}$$

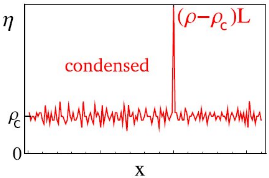
ZRP $g(k)w(k)=w(k-1) \quad \Rightarrow \quad j_{L,N} = \frac{Z_{L,N-1}}{Z_{L,N}} \ , \quad j_{gc}(\phi) = \nu_\phi^1\big(g(\eta_x)\big) = \phi$

Finite-size scaling

fundamental diagram for ZRP with $g(k) = \mathbb{1}_{k\geq 1}(1 + b/k^{\gamma})$

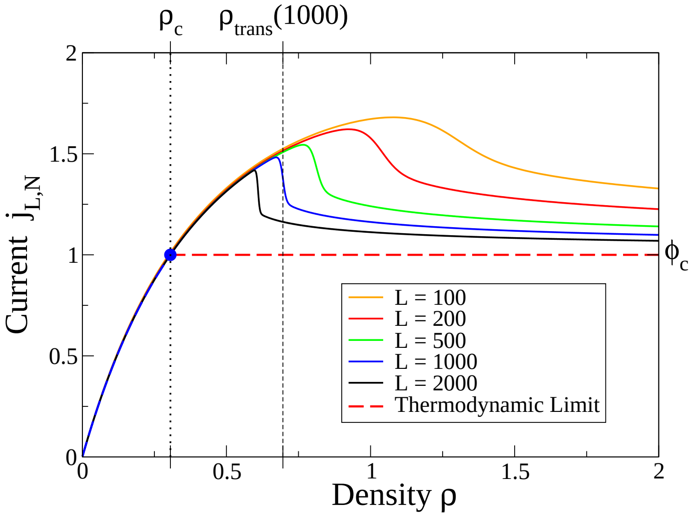




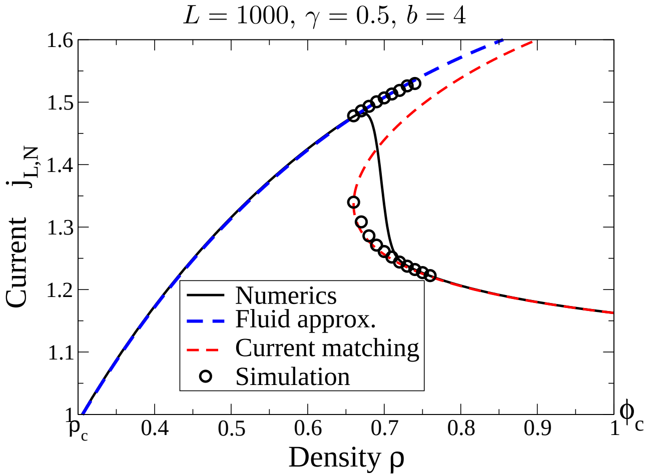


Finite-size scaling

ZRP with $g(k) = \mathbb{1}_{k\geq 1}(1 + b/k^\gamma)$, $\gamma = 0.5$, $b = 4$



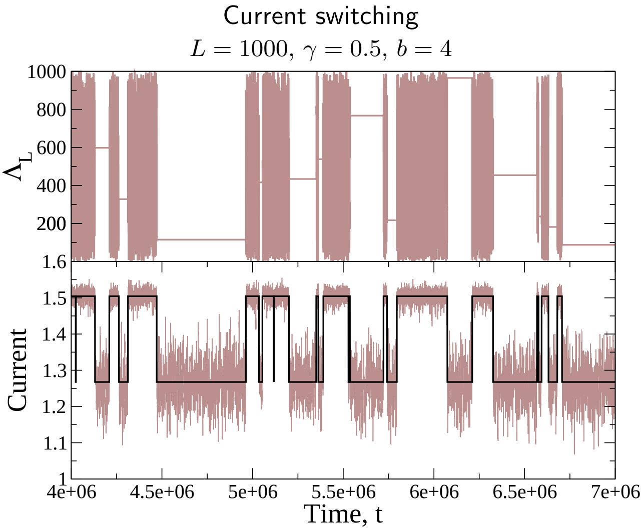
Finite-size scaling



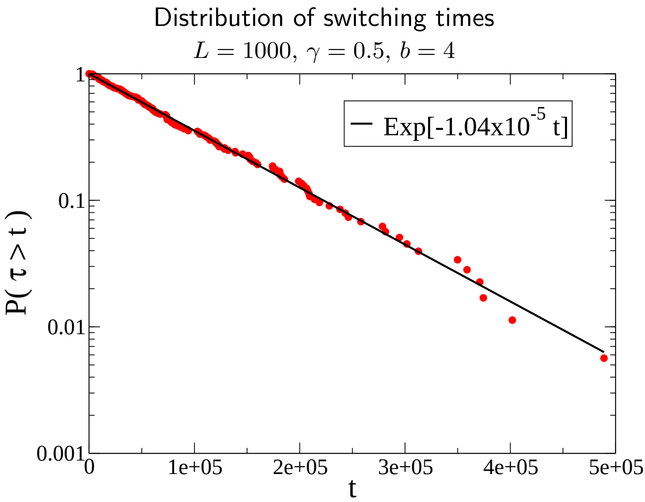
fluid approximation $R_N(\phi) := \frac{1}{z_N(\phi)} \sum_{n=1}^{N} n w(n) \phi^n \ , \quad \Phi_N(\rho) \in [0, \infty)$

current matching $\Phi_N(\rho_b) = g(m) \ , \quad \rho_b = \frac{N-m}{L-1}$

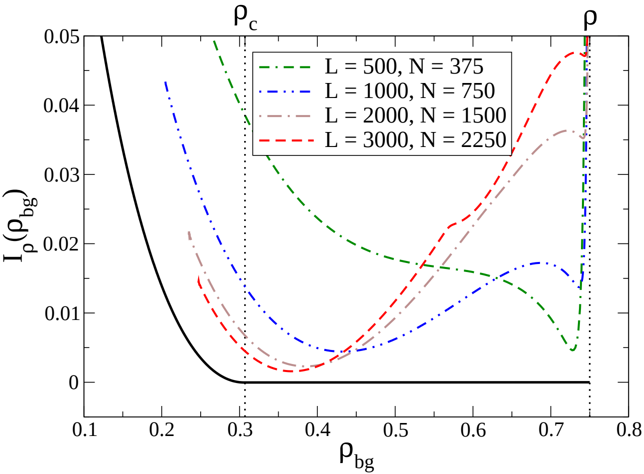
Finite-size scaling



Finite-size scaling



Finite-size scaling



rate function $\frac{1}{L}\log \pi_{L,N}[M_L = m] \;\to\; I_\rho(\rho_{bg}) = s_c(\rho) - s_c(\rho_{bg})$

as $N, L, m \to \infty\;,\quad N/L \to \rho\;,\quad \frac{N-m}{L-1} \to \rho_{bg}$

Finite-size scaling

ZRP with $g(k) = \mathbb{1}_{k\geq 1}(1 + b/k^\gamma)$, $\gamma \in (0,1)$, $b > 0$

LDP

[Chleboun, G (2010)]

$L, N, m \to \infty,\ N = \rho_c L + \delta\rho L^{1/(1+\gamma)},\ m = L^{1/(1+\gamma)}(\delta\rho - \delta\rho_{bg})$

$$-L^{(\gamma-1)/(1+\gamma)} \log \pi_{L,N}[M_L = m] \ \to\ I^{(2)}_{\delta\rho}(\delta\rho_{bg})$$

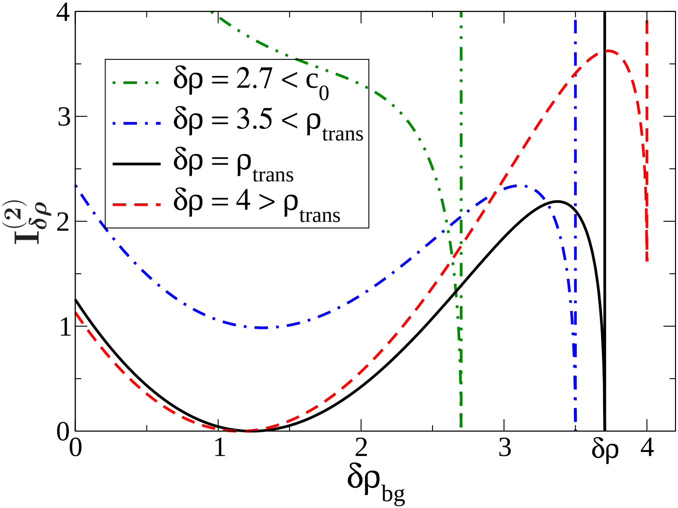
with a rate function with a double-well structure:

$$I^{(2)}_{\delta\rho}(\delta\rho_{bg}) = \frac{\delta\rho_{bg}^2}{2\sigma_c^2} + \frac{b}{1-\gamma}(\delta\rho - \delta\rho_{bg})^{1-\gamma} - \inf_{r\in(0,\delta\rho)} \left\{ \frac{r^2}{\sigma_c^2} + \frac{b}{1-\gamma}(\delta\rho - r)^{1-\gamma} \right\}$$

provided $\frac{1}{\sigma_c^2}\delta\rho_{bg} < \frac{b}{(1-\gamma)(\delta\rho-\delta\rho_{bg})^\gamma}$ where $\sigma_c^2 = \nu^1_{\phi_c}(\eta_x^2) - \rho_c^2$.

based on **LLT** for power law [Doney (2001)] , stretched exponential [Nagaev (1968)]

Finite-size scaling

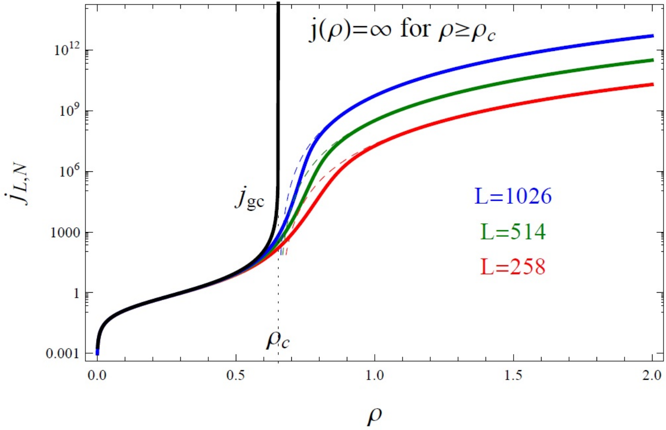


Finite-size scaling

EDG $c(k,l) = k^\gamma(d + l^\gamma)$, $\gamma > 2$, $d > 0$ with $w(n) \sim n^{-\gamma}$

rates c are **unbounded**, converge only for $\rho < \rho_c$ or $\phi < \phi_c = 1$

$\nu^2_{\phi_c}\big(c(\eta_x, \eta_y)\big) = \infty$ and $\pi_{L,N}\big(c(\eta_x, \eta_y)\big) \to \infty$ for $N/L \to \rho \geq \rho_c$



Part I - Stationary results

Several conservation laws

IPS with stationary product measures and weights $w(k^1, k^2)$ for two types

ZRP $\quad g_1(k^1,k^2) = \frac{w(k^1-1,k^2)}{w(k^1,k^2)}\ , \quad g_2(k^1,k^2) = \frac{w(k^1,k^2-1)}{w(k^1,k^2)}$

canonical measures $\quad \pi_{L,\mathbf{N}}[d\boldsymbol{\eta}] = \frac{1}{Z_{L,\mathbf{N}}} \prod_{x\in\Lambda} w(\boldsymbol{\eta}_x) d\boldsymbol{\eta}\ , \quad \mathbf{N} = (N_1, N_2)$

grand-canonical measures $\quad \nu^L_{\boldsymbol{\mu}}[d\boldsymbol{\eta}] = \prod_{x\in\Lambda} \frac{1}{z(\boldsymbol{\mu})} e^{\boldsymbol{\mu}\cdot\boldsymbol{\eta}_x} w(\boldsymbol{\eta}_x) d\boldsymbol{\eta}$

with **chemical potential** $\boldsymbol{\mu} = (\log\phi_1, \log\phi_2) \in \mathcal{D}_{\boldsymbol{\mu}} \subset \mathbb{R}^2$ convex

domain of accessible densities $\mathcal{D}_{\boldsymbol{\rho}} = \mathbf{R}(\mathcal{D}_{\boldsymbol{\mu}}) \subset (0,\infty)^2$

Equivalence of ensembles $\quad$ minimize over $\boldsymbol{\mu} \in \mathcal{D}_{\boldsymbol{\mu}}$

$$h_{L,\mathbf{N}}(\boldsymbol{\mu}) := \frac{1}{L} H(\pi_{L,\mathbf{N}}; \nu^L_{\boldsymbol{\mu}}) = \Big[\log z(\boldsymbol{\mu}) - \frac{\mathbf{N}}{L}\cdot\boldsymbol{\mu}\Big] - \frac{1}{L}\log Z_{L,\mathbf{N}}$$

$$\inf_{\boldsymbol{\mu}\in\mathcal{D}_{\boldsymbol{\mu}}} \lim_{N/L\to\rho} h_{L,\mathbf{N}}(\boldsymbol{\mu}) = \inf_{\boldsymbol{\mu}\in\mathcal{D}_{\boldsymbol{\mu}}} \Big[\log z(\boldsymbol{\mu}) - \boldsymbol{\rho}\cdot\boldsymbol{\mu}\Big] - s_c(\boldsymbol{\rho})$$

If $\boldsymbol{\rho} \notin \mathcal{D}_{\boldsymbol{\rho}}$ then we have boundary minimizer $\boldsymbol{\mu} = \mathbf{M}(\boldsymbol{\rho}) \in \partial\mathcal{D}_{\boldsymbol{\mu}}$

Several conservation laws

Bulk density $\boldsymbol{\rho}_b(\boldsymbol{\rho}) := \mathbf{R}\big(\mathbf{M}(\boldsymbol{\rho})\big)$

Phase diagram $A_i = \big\{\boldsymbol{\rho} \in (0,\infty)^2 \big| \rho_b^i(\boldsymbol{\rho}) < \rho^i\big\}$

$$PD = \big\{D_\rho, A_1 \setminus A_2, A_2 \setminus A_1, A_1 \cap A_2\big\}$$

Equivalence of ensembles

[G (2008)]

For all $\boldsymbol{\rho} \in (0,\infty)^2$, $h_{L,\mathbf{N}}\big(\mathbf{M}(\boldsymbol{\rho})\big) \to 0$ as $\mathbf{N}/L \to \boldsymbol{\rho}$.

If $\boldsymbol{\rho} \in A_i$ and $\nu^1_{\boldsymbol{\mu}^*}[\boldsymbol{\eta}^i_x = k] \sim k^{-b}$, $b > 2$ as $k \to \infty$, then

$$\frac{1}{L} \max_{x\in\Lambda_L} \eta^i_x \overset{\pi_{L,\mathbf{N}}}{\longrightarrow} (\rho_i - \rho_{c_i}) \ , \quad \text{as } \mathbf{N}/L \to \boldsymbol{\rho} \ .$$

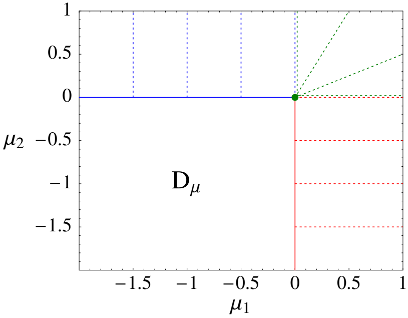
- thermodynamic formalism, convex analysis
- stationary product measures under curl-free condition on ZRP rates

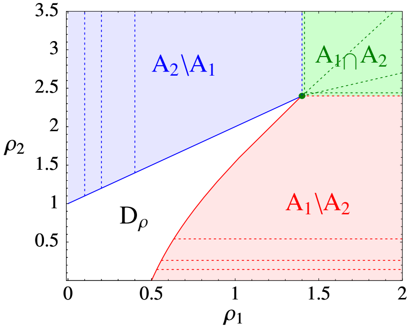
[Evans, Hanney (2003); G, Spohn (2003)]

Several conservation laws

$$w(k_1,k_2) \simeq k_1^{-b} \left(\frac{k_1+1}{k_1+2}\right)^{k_2} , \qquad \mu_1, \mu_2 \leq 0$$

[Evans, Hanney (2003)]



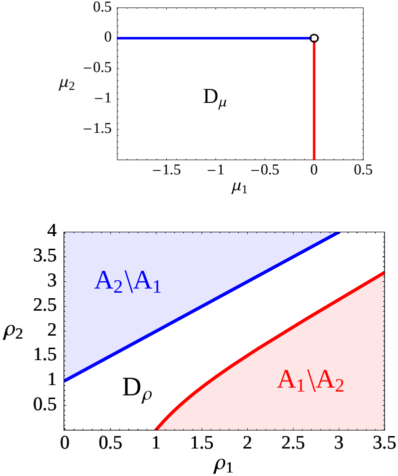


$$b = 4$$

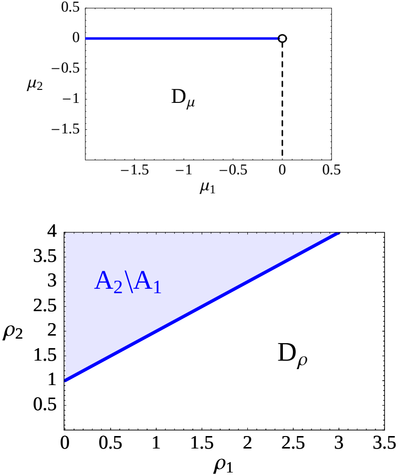
$$g_1 = \left(\frac{1+1/(k_1+1)}{1+1/k_1}\right)^{k_2} \left(1+\frac{b}{k_1}\right) \qquad g_2 = 1 + \frac{1}{k_1+1}$$

Several conservation laws

$2 \le b < 3$

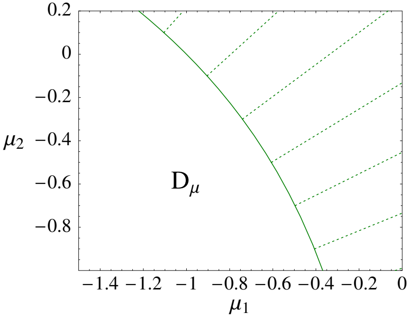


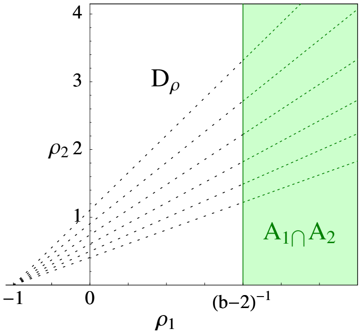
$b \le 2$



Several conservation laws

$$w(k_1, k_2) \simeq k_1^{-b}\ \frac{(k_1+1)^{k_2}}{k_2!}\ , \qquad \partial D_\mu = \{\boldsymbol{\mu} | e^{\mu_2} + \mu_1 = 0\}$$

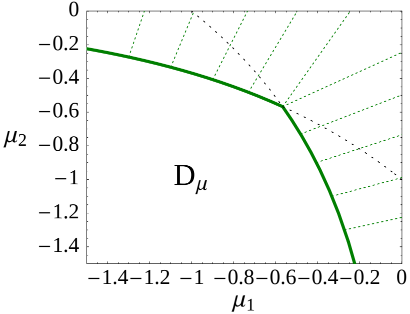


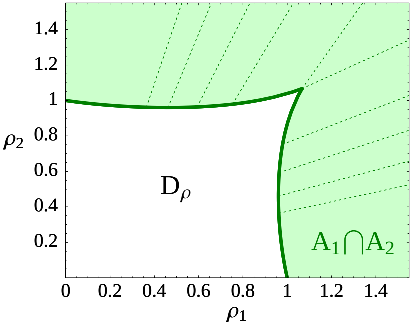


$$g_1 = \left(\frac{k_1}{1+k_1}\right)^{k_2} (1 + b/k_1) \qquad g_2 = \frac{k_2}{1+k_1}$$

Several conservation laws

$$w(k_1,k_2) = k_1^{-b}\ \frac{(k_1+1)^{k_2}}{k_2!} + k_2^{-b}\ \frac{(k_2+1)^{k_1}}{k_1!}$$





Part I - Stationary results

Several conservation laws

Constrained-driven condensation

continuous variables $m = (m_x : x \in \Lambda)$, $m_x \geq 0$

conserved quantities $\Sigma_L(m) = \sum_{x\in\Lambda} m_x$, $\Sigma_L^p(m) = \sum_{x\in\Lambda} m_x^p$, $p > 0$

$$\pi_{L,N,M}[dm] = \frac{\mathbb{1}\{\Sigma_L = N, \Sigma_L^p = M\}}{Z_{L,N,M}} \prod_{x\in\Lambda} w(m_x)$$

- $p = 2$, $w(m) \equiv 1$, $N = L$, $M = bL$ with $b > 2$ [Rumpf (2004); Chatterjee (2017)]

  single condensate with $\frac{M_L}{\sqrt{(b-2)L}} \stackrel{\pi_{L,N,M}}{\longrightarrow} 1$ as $L \to \infty$

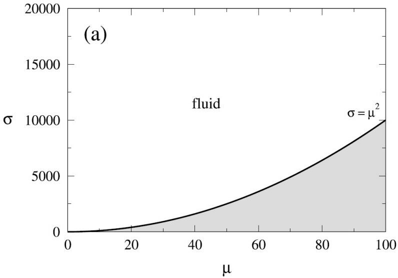
- $p > 1$, $w(m) \gg e^{-m^p}$, $N/L \to \mu$, $M/L \to \sigma \geq \mu^2$

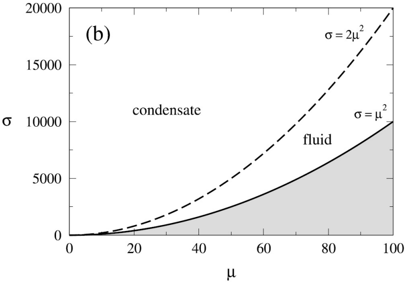
  single condensate of order $M_L \sim L^{1/p}$

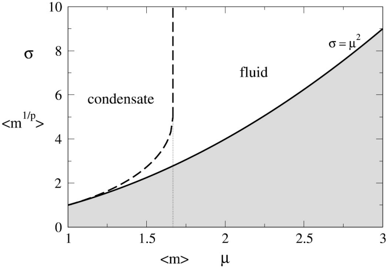
[Szavitz-Nossan, Evans, Majumdar (2014-16)]

Several conservation laws

$w(m) \ll e^{-m^p}$ $\quad p = 2 \quad$ $w(m) = re^{-rm}$







$w(m) \sim m^{-\gamma}$

Beyond product measures

Pair-factorized stationary measures on $\Lambda = \mathbb{Z}/L\mathbb{Z}$

$$\pi_{L,N}[d\eta] = \frac{\mathbb{1}\{\Sigma_L = N\}}{Z_{L,N}} \prod_{x\in\Lambda} w(\eta_x, \eta_{x+1}) \ , \quad w(m,n) = w(n,m)$$

possible dynamics $\quad c(\eta \to \eta^{x,x+1}) = \frac{w(\eta_x - 1, \eta_{x-1})}{w(\eta_x, \eta_{x-1})} \frac{w(\eta_x - 1, \eta_{x+1})}{w(\eta_x, \eta_{x+1})}$

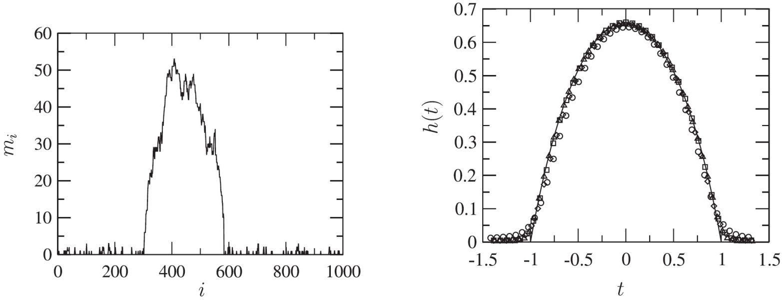
Example

- $w(\eta_x, \eta_{x+1}) = \exp\Big(- J|\eta_x - \eta_{x+1}| + \frac{U}{2}\big(\delta_{\eta_x,0} + \delta_{\eta_{x+1},0}\big)\Big)$

condensation for $J > U - \log(e^U - 1)$ with $M_L \sim \sqrt{L}$

- **condensation transition** with $\rho_b = \rho_c$ derived heuristically
- shape of the condensate varies with $w(m,n)$ from smooth to rectangular or single-site

Beyond product measures



[Evans, Hanney, Majumdar (2006); Waclaw, Sopik, Janke, Meyer-Ortmanns (2009)]

Beyond product measures

- **Chipping model** [Rajesh, Majumdar (2001)]

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} p(x,y)\Big(w\big[f(\eta^{x,y}) - f(\eta)\big] + \big[f\big(\eta + \eta_x(e_y - e_x)\big) - f(\eta)\big]\Big)$$

full **clusters move with rate** 1, loose a particle with **chipping rate** $w > 0$

Heuristic results for complete graph or regular lattice in any dimension $d \geq 1$:

Condensation transition with $\rho_c = \sqrt{w+1} - 1$ and a single condensate site

background density ρ_b depends on $\rho > \rho_c$.

- **mass migration models** [Fajfrová, Gobron, Saada (2016)]

- **Target process** [Luck, Godréche (2007)]

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} p(x,y)\mathbb{1}_{\eta_x>0}\Big(1 - \frac{b}{\eta_y + 1 + b}\Big)\big[f(\eta^{x,y}) - f(\eta)\big]$$

Rigorous condensation transition with $\rho_c = 1/(b-2)$ for symmetric $p(x,y)$.

For asymmetric dynamics in 1D only heuristic results.

Different scaling limits

- **Condensation on a fixed lattice**

[Ferrari, Landim, Sisko (2007); Rafferty, Chleboun, G (2018)]

$$w(n) \sim n^{-b},\ b > 1 \quad \Rightarrow \quad \frac{M_N(\eta)}{N} \xrightarrow{\pi_{L,N}} 1 \quad \text{as } N \to \infty\ .$$

Various results on dynamics (later)

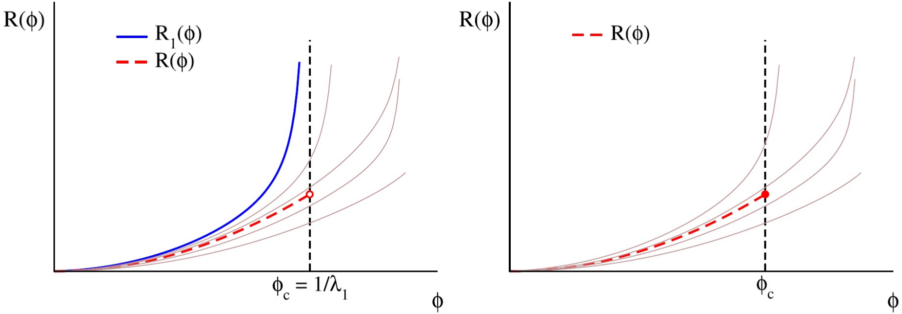
- **open boundary conditions** ZRP [Levine, Mukamel, Schütz (2005)]

pair-factorized measures [Nagel, Meyer-Ortmanns, Janke (2015)]

- **ZRP** with $N \gg L$ [Xu (2020)]

- fluctuating system size L [Godréche (2021)]

Inhomogeneous systems



harmonic function $\lambda_x = \sum_{y \in \Lambda} p(x,y)\lambda_y \quad \Rightarrow \quad w_x(n) = \lambda^n w(n)$

[Chleboun, G (2014)]

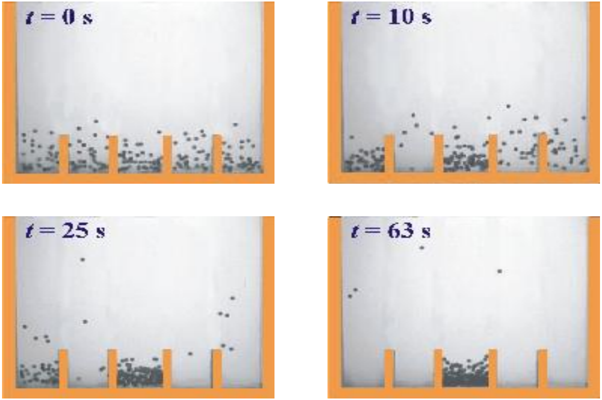
- **ZRP** with disordered rates [Evans (1996); Andjel et al. (2000); Ferrari Sisko (2007)]
- **ZRP** on networks [Waclaw et al. (2007/08). . .]
- **Inclusion process** [G, Redig, Vafayi (2011)]
- interplay between disorder and attraction

[G, Chleboun, Schütz (2008); Godréche, Luck (2012); Mailler, Mörters, Ueltschi (2016)]

Part I - Stationary results

Applications

Clustering in granular gases

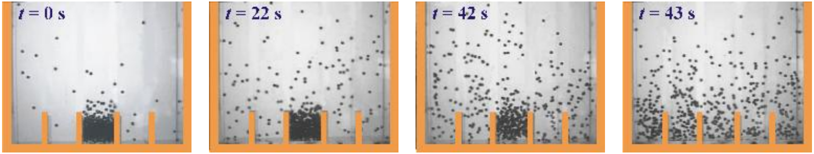


[van der Meer, van der Weele, Lohse, Mikkelsen, Versluis (2001-02)]

stilton.tnw.utwente.nl/people/rene/clustering.html

Applications

Clustering in granular gases



[van der Meer, van der Weele, Lohse, Mikkelsen, Versluis (2001-02)]

stilton.tnw.utwente.nl/people/rene/clustering.html

Model: ZRP with $g(k) = C\Big(\frac{k}{N}\Big)^2 e^{-\tilde{C}k/N}$ [Eggers (1999)]

$$g(k) = \frac{k}{N}\, e^{-1/(T_0+\Delta(1-k/N))}$$

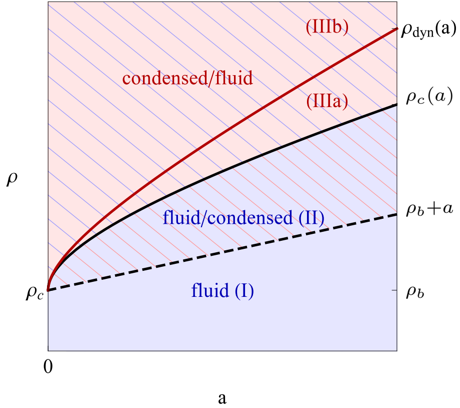
[Lipowski, Droz (2002), Coppex, Droz, Lipowski (2002)]

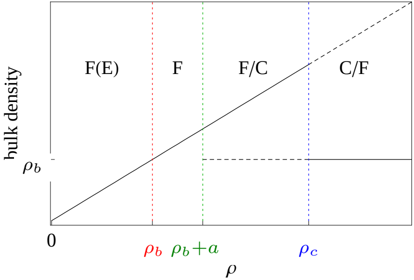
[Török (2005), van der Meer, Reimann, Lohse (2007)]

ZRP with size-dependent jump rates

Toy model [G., Schütz (2008); Chleboun, G (2015)]

$$g_L(n) := \begin{cases} c > 1 & \text{if } n \le aL, \\ 1 & \text{if } n > aL, \end{cases} \quad \text{for} \quad n \ge 1 \ , \quad g_L(0) = 0$$





ZRP with size-dependent jump rates

Stationary weights $w_L(n) = \begin{cases} c^{-n} & \text{for } n \leq aL \\ c^{-\lfloor aL \rfloor} & \text{for } n > aL \end{cases}$

Grand-canonical measures for $\mu < 0$

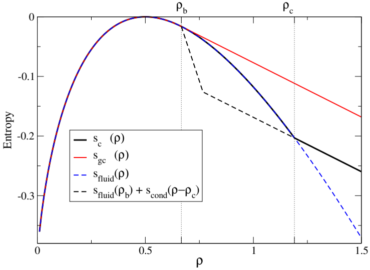
$$\nu^L_{\mu,L}[d\eta] = \frac{1}{z_L(\mu)^L} \prod_{x\in\Lambda} w_L(\eta_x) e^{\mu\eta_x} d\eta \quad \text{where} \quad z_L(\mu) = \sum_{n=0}^{\infty} w_L(n) e^{\mu n}$$

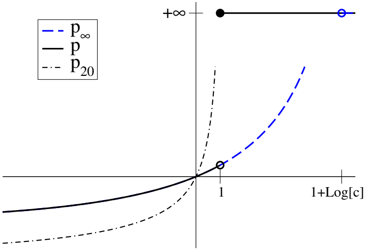
Canonical measures $\pi_{L,N}[d\eta] := \frac{\mathbb{1}_{E_{L,N}}}{Z_{L,N}} \prod_{x\in\Lambda} w_L(\eta_x) d\eta$

Pressure $p(\mu) = \lim_{L\to\infty} \log z_L(\mu) = \begin{cases} -\log(1 - e^\mu/c) & \text{if } \mu < 0, \\ \infty & \text{if } \mu \geq 0 \end{cases}$

Maximal gc density $R(\mu) = \partial_\mu p(\mu) = \frac{e^\mu}{c - e^\mu} \quad \Rightarrow \quad R(0) = 1/(c-1)$

Thermodynamic functions





Entropy $s_c(\rho) = \begin{cases} s_{\text{fluid}}(\rho) & \text{if } \rho \le \rho_c \quad \text{(fluid)} \\ s_{\text{fluid}}(\rho_b) + s_{\text{cond}}(\rho - \rho_b) & \text{if } \rho > \rho_c \quad \text{(condensed)} \end{cases}$

$$s_{\text{fluid}}(\rho) = \rho \log \rho - (1+\rho) \log(1+\rho) + \rho \log c \ ,$$

$$s_{\text{cond}}(m) = \lim_{\substack{L\to\infty \\ M/L\to m}} \frac{1}{L} \log w_L[\eta_1 = M] = -\begin{cases} m \log c & \text{if } m < a, \\ a \log c & \text{if } m \ge a \end{cases}$$

Grand-canonical entropy $s_{gc}(\rho) = p^*(\rho)$ (convex hull)

(Non-)Equivalence of ensembles

Theorem [G., Schütz (2008)]

In the thermodynamic limit $L, N \to \infty$, $N/L \to \rho$ we have

(I) $\pi_{L,N} \to \nu_{M(\rho)}$ and $\nu^L_{L,M(\rho)} \to \nu_{M(\rho)}$ for $\rho < \rho_b$,

(I/II) $\pi_{L,N} \to \nu_{M(\rho)}$ but $\nu^L_{L,M(\rho)} \to \nu_0$ for $\rho_b \leq \rho < \rho_c$,

(III) $\pi_{L,N} \to \nu_0$ for $\rho > \rho_c$,

where convergence holds in specific relative entropy.

We have a **condensation transition** with critical density ρ_c characterized by

$$s_{\text{fluid}}(\rho_c) = s_{\text{cond}}(\rho_c - \rho_b) + s_{\text{fluid}}(\rho_b) \ ,$$

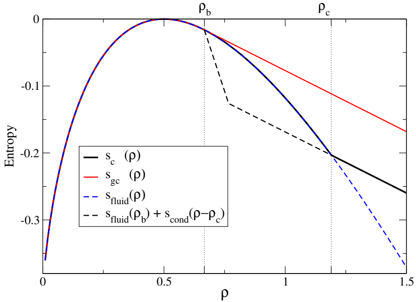
and **bulk density** $\rho_b = R(0) = \nu^1_0(\eta_x) = \frac{1}{c-1} < \rho_c$.

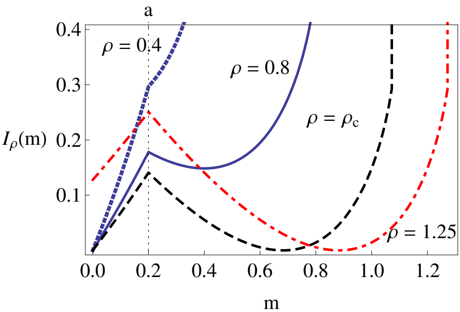
The limiting grand-canonical product measure is given by the marginals

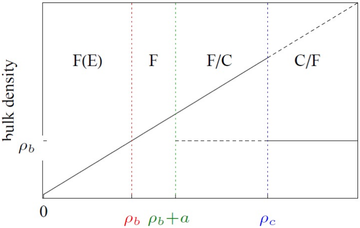
$$\nu^1_\mu[\eta_x = n] = \big(1 - e^\mu/c\big)\, c^{-n} e^{\mu n} \ , \quad \mu < \log c \ .$$

Discontinuous transition

Rate function $I_\rho(m) := \lim_{\substack{N/L\to\rho \\ M/L\to m}} \frac{1}{L} \log P_{L,N}\big[\max_x \eta_x = M\big]$



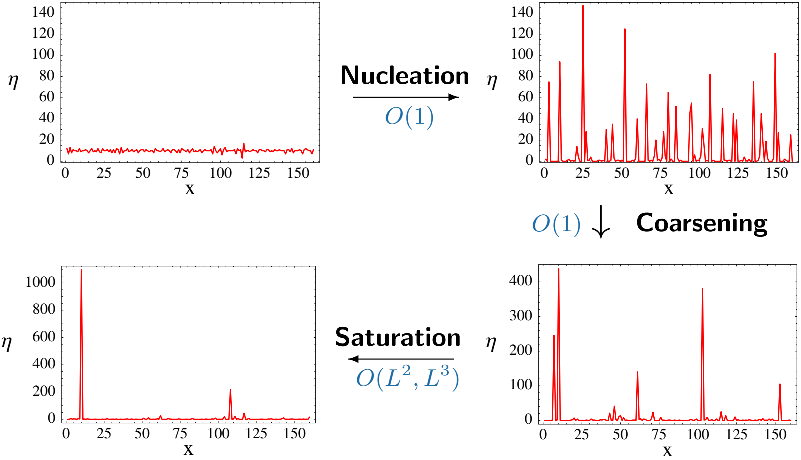




Part II - Dynamic results

Dynamics of condensation

ZRP with $g(k) = 1 + b/k$, $b = 4$, $\rho_c = 1/(b-2) = 0.5$, $\rho = 10$



hydrodynamics $O(L, L^2)$ [subcritical, Stamatakis (2015)]

stationary dynamics of condensate $O(L^{1+b})$

Rigorous results on condensation dynamics

Stationary dynamics for ZRP (metastability)

[Beltrán, Landim (2010,11,12,15)], [Landim (2014,2022), Seo (2019-2022)]
[Armendáriz, G., Loulakis (2017)], [Bovier, Neukirch (2014)]

Nucleation/Coarsening for ZRP

[Beltrán, Jara, Landim (2017)], [G., Jatuviriyapornchai (2016,19)], [Armendáriz, Beltrán, Cuesta, Jara (2023)]

Hydrodynamic limits for subcritical ZRP

[Stamatakis (2015)], [Stamatakis, Loulakis (2019)]

Inclusion process

[G., Redig, Vafayi (2013)], [Bianchi, Dommers, Giardiná (2017)], [Carinci, Giardina, Redig (2019)],
[Ayala, Carinci, Redig (2019)], [Kim, Seo (2021)], [Kim, Sau (2023)]

Metastability in ZRP

The setting

- **Generator** $\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} p(x,y)g(\eta_x)\big[f(\eta^{xy}) - f(\eta)\big]$
- **reversible, translation invariant** $p(x,y) = p(|y-x|)$ on $(\mathbb{Z}/L\mathbb{Z})^d$
- **jump rates** $g(0) = 0,\ g(1) = 1,\ g(k) = \big(\frac{k}{k-1}\big)^b\ ,\quad b > 2$
- **stationary weights** $w(0) = 1,\ w(n) = n^{-b}$, $n \geq 1$

  canonical measures $\pi_{L,N} = \dfrac{\mathbb{1}_{E_{L,N}}}{Z_{L,N}} \prod_{x\in\Lambda} w(\eta_x)$
- **condensation** $\dfrac{1}{L} M_L \overset{\pi_{L,N}}{\longrightarrow} \rho - \rho_c\ ,\quad N/L \to \rho > \rho_c \in (0,\infty)$
- **metastability** location of the condensate site

Metastability: dynamics of the condensate

Potential theoretic approach

[Bovier, Gayrard, Eckhoff, Klein (2001/02); [Bovier, den Hollander (2015)]...

Martingale approach [Beltrán Landim (2010-15); Beltrán, Seo (2019-22)

Trace process • metastable wells

$$\mathcal{E}^x := \{\eta_x \geq N - \rho_c L - \alpha_L,\, \eta_y \leq \beta_L,\, y \neq x\}\ , \quad \mathcal{E} = \cup_{x\in\Lambda}\mathcal{E}^x$$

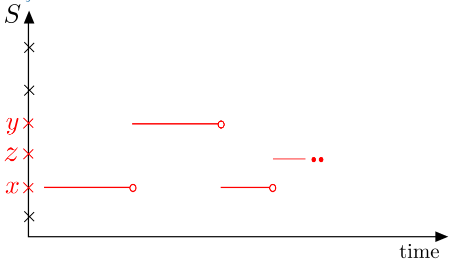
- $\eta^{\mathcal{E}}$ is a **Markov process** on $\mathcal{E}$ with rates

$$r^{\mathcal{E}}(\eta,\xi) = r(\eta,\xi) + \sum_{\zeta\in\Delta} r(\eta,\zeta)\mathbb{P}_\zeta[T_{\mathcal{E}} = T_\xi]$$

- **invariant measure**

$$\mu[\cdot] = \pi_{L,N}[\ \cdot \mid \mathcal{E}]$$

- $\pi_{L,N}[E_{L,N} \setminus \mathcal{E}] \ll 1/L$



Main result

Theorem. [Armendáriz, G, Loulakis (2017)]

The ZRP with $b > 21$, as $L, N \to \infty$, $N/L \to \rho > \rho_c$, $\Lambda = \mathbb{Z}/L\mathbb{Z}$
exhibits metastability w.r.t. the rescaled condensate location

$$Y_t^L := \frac{1}{L} \sum_{x \in \Lambda} \mathbb{1}_{\mathcal{E}^x}\big(\eta^{\mathcal{E}}(\theta_L t)\big) \in \mathbb{T} \quad \text{on the scale } \theta_L = L^{1+b} .$$

For all initial conditions $\eta_0^L \in \mathcal{E}^0$ we have weakly on pathspace

$$\big(Y_t^L : t \geq 0\big) \Rightarrow (Y_t : t \geq 0) \quad \text{with} \quad Y_0 = 0 ,$$

where $(Y_t : t \geq 0)$ is a **Lévy-type process** on $\mathbb{T}$ with generator

$$\mathcal{L}^{\mathbb{T}} f(u) = K_{b,\rho} \int_{\mathbb{T}\setminus\{0\}} \frac{1}{d(v,u)} \big(f(v) - f(u)\big)\, dv , \qquad (1)$$

where $d(v,u) = |v-u|\big(1 - |v-u|\big)$ is the distance in $\mathbb{T}$.

Proof

- $\big(Y_t^L : t \geq 0\big)$ is **tight** on $D\big([0,T],\mathbb{T}\big)$
- identify limit points $\big(Y_t : t \geq 0\big)$ as solutions of the **martingale problem**

$$f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}^{\mathbb{T}} f(Y_s)\, ds \quad \text{is a martingale .}$$

Introduce **auxiliary process** $\mathcal{L}^\Lambda$ on Λ with averaged rates

$$r^\Lambda(x,y) = \frac{1}{\mu[\mathcal{E}^x]} \sum_{\eta\in\mathcal{E}^x,\, \xi\in\mathcal{E}^y} \mu[\eta]\, r^{\mathcal{E}}(\eta,\xi) \ , \quad \text{and write}$$

$$\int_0^t \Big(\mathcal{L}^{\mathbb{T}} f(Y_s^L) - \theta_L \mathcal{L}^{\mathcal{E}}(f\circ Y^L)(\eta^{\mathcal{E}}(\theta_L s))\Big)\, ds =$$
$$\int_0^t \Big(\mathcal{L}^{\mathbb{T}} f(Y_s^L) - \theta_L \mathcal{L}^\Lambda f(Y_s^L)\Big) ds + \theta_L \int_0^t \Big(\mathcal{L}^\Lambda f(Y_s^L) - \mathcal{L}^{\mathcal{E}}(f\circ Y^L)(\eta^{\mathcal{E}}(\theta_L s))\Big) ds$$

1. central Lemma: **uniform bounds** on exit rates from wells
2. Prove **convergence of averaged dynamics** $\mathcal{L}^\Lambda$ to limit dynamics
3. Prove **equilibration within wells** / replacement by averaged dynamics

1 – Coupling to a branching system of BD processes

$m = 2^b$ largest arrival rate for ZRP

$x \in \Lambda$, couple $\big(\eta_x(t) : t \geq 0\big)$ with a growing system of BD chains $\zeta_x^{\mathbf{k}}$,

indexed by the m-regular tree $\mathcal{R}_m$

- At any time t, only m of the chains are coupled to $\eta_x(t)$, and the rest are evolving independently.
- Each chain ζ_x has birth rate 1 and death rate $g(\zeta_x)$.

  Arrival events for $\eta_x(t)$ are used only for one of the coupled chains
- **Number of chains grows linearly with time**
- $\max_{\mathbf{k}} \zeta_x^{\mathbf{k}}(t) \geq \eta_x(t)$ for all times $t \geq 0$.
- control time spent outside $\mathcal{E}$ via mixing on extended wells
  $\to$ leads to choice of $\beta_L = (L^6 \log^2 L)^{1/(b-1)}$ (i.e. $b > 6$)

Uniform exit rate bound: $$\sup_{\eta \in \mathcal{E}^x} \sum_{\xi \notin \mathcal{E}^x} r^{\mathcal{E}}(\eta, \xi) \leq C \, \frac{1}{L^5 \log^2(L)}$$

1 – Coupling to a branching system of BD processes

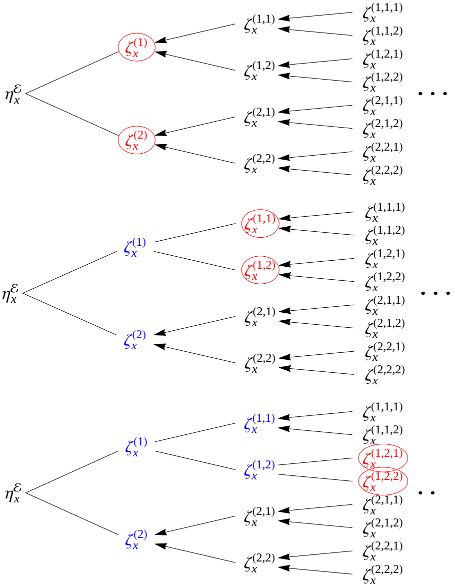
Example for $m = 2$

arrows $\rightarrow$: identical copies

coupled chains : red encircled

independent chains : in blue

- generation $n = 1$ only two coupled
- particle arrives at x (middle)
  chains in 1st gen. turn independent
  2 descendants get coupled
- second particle arrives, etc.



2 – Mean rates as capacities

$$\mu[\mathcal{E}^{A_1}]r^\Lambda(A_1,A_2) = \mu[\mathcal{E}^{A_1}]\,\frac{1}{|A_1|}\sum_{\substack{x\in A_1\\ y\in A_2}} r^\Lambda(x,y) \qquad A_1,\ A_2\subset\Lambda$$

$$= \frac{1}{2}\Big(\mathrm{cap}\big(\mathcal{E}^{A_1},\mathcal{E}\setminus\mathcal{E}^{A_1}\big)+\mathrm{cap}\big(\mathcal{E}^{A_2},\mathcal{E}\setminus\mathcal{E}^{A_2}\big)-\mathrm{cap}\big(\mathcal{E}^{A_1\cup A_2},\mathcal{E}\setminus\mathcal{E}^{A_1\cup A_2}\big)\Big)$$

[Bovier, den Hollander, *Metastability - a potential theoretic approach*]

with $\mathrm{cap}\big(\mathcal{E}^{A_1},\mathcal{E}\setminus\mathcal{E}^{A_1}\big) = \inf\Big(\mathcal{D}(F)\Big|F:E_{L,N}\to\mathbb{R},\ F|_{\mathcal{E}^A}\equiv 1,\ F|_{\mathcal{E}\setminus\mathcal{E}^A}\equiv 0\Big)$

Dirichlet form $\mathcal{D}(F) = \frac{1}{2}\sum_{\eta\in E_{L,N}}\sum_{x\in\Lambda}\sum_{z=-1,1}\pi_{L,N}[\eta]g(\eta_x)\big[F(\eta^{x,x+r})-F(\eta)\big]$

$$\theta_L\,\mathrm{cap}\big(\mathcal{E}^{A_1},\mathcal{E}\setminus\mathcal{E}^{A_1}\big)\le K(b,\rho)\,(1+\bar{\epsilon}_L)\sum_{\substack{x\in A\\ y\notin A}}\mathrm{cap}_\Lambda(x,y)$$

$$\theta_L\,\mathrm{cap}\big(\mathcal{E}^{A_1},\mathcal{E}\setminus\mathcal{E}^{A_1}\big)\ge K(b,\rho)\,(1-\underline{\epsilon}_L)\sum_{\substack{x\in A\\ y\notin A}}\mathrm{cap}_\Lambda(x,y)$$

where $\mathrm{cap}_\Lambda(x,y) = \frac{1}{|x-y|\,(L-|x-y|)}$ capacities of symmetric rw on Λ.

2 – Regularization

- Total exit rate from a well is $\sim \log L$
- Upper and lower bounds for rates $r^\Lambda(x,y)$ do not match

[Bovier, Neukirch (2014)]

- **Coarse graining** in Λ and **Lipschitz test functions** to regularize

$$\theta_L \mathcal{L}^\Lambda f(x) = \sum_{m=1}^{\bar{L}} r^\Lambda(V_0, V_m) \left(f\left(\frac{x+\ell m}{L}\right) - f\left(\frac{x}{L}\right)\right) + o(1)$$

with $|V_i| = \ell \propto \alpha_L \log^3 L \to \infty$, $\bar{L} = L/\ell$.

$\to$ leads to choice of $\alpha_L = L^{1/2+5/(2b)}$, $(b > 6)$

- matching bounds from capacity representation for $r^\Lambda(V_0, V_m)$

$$\sup_{\eta\in\mathcal{E}} \mathbb{E}_\eta \Big| \int_0^t \Big(\mathcal{L}^{\mathbb{T}} f(Y_s^L) - \theta_L \mathcal{L}^\Lambda f(Y_s^L)\Big) ds \Big| \to 0$$

3 – Equilibration within a well

Restricted process to a well by ignoring jumps outside, $\mu^x = \mu[\,\cdot\,|\mathcal{E}^x]$

- bound on **relaxation time** t_{rel}, **mixing time** $t_{\mathsf{mix}}(\epsilon)$

$$t_{\mathsf{rel}} \leq CL^4 \quad \text{and} \quad t_{\mathsf{mix}}(\epsilon) \leq t_{\mathsf{rel}} \log\left(\frac{1}{\epsilon\mu_{\mathsf{min}}}\right) \leq CL^5 \log\left(1/\epsilon\right)$$

- **ergodic L^2 bound** for functions with $\mu^x(h) = 0$, $x \in \Lambda$

$$\mathbb{E}_\mu \Big| \int_0^t h(\eta_u^{\mathcal{E}})\, du \Big|^2 \leq 24t\, t_{\mathsf{rel}} \sum_{x\in\Lambda} \mu\big[\mathcal{E}^x\big]\, \mu^x\big(h^2\big), \tag{2}$$

[J. Beltrán and C. Landim *Martingale approach to metastability*]

- Apply (2) + 1. + bounds on $\sum_{y\neq x} r^\Lambda(x,y)$ from 2. to $h = r^{\mathcal{E}} - r^\Lambda$ to get

$$\sup_{\eta\in\mathcal{E}} \mathbb{E}_\eta \Big| \theta_L \int_0^t \Big(\mathcal{L}^\Lambda f(Y_s^L) - \mathcal{L}^{\mathcal{E}}(f\circ Y^L)(\eta^{\mathcal{E}}(\theta_L s))\Big) ds \Big| \to 0$$

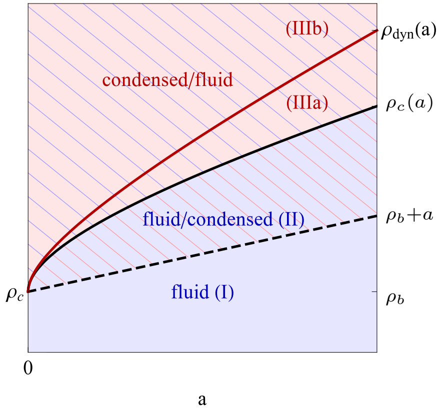
Difficulties. exponential size vs. polynomial depth

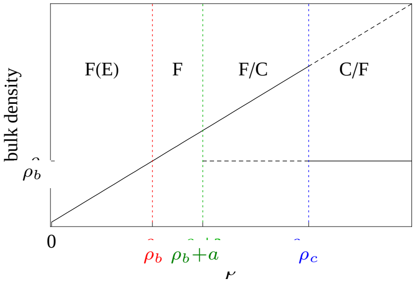
Metastability for toy ZRP

Toy model

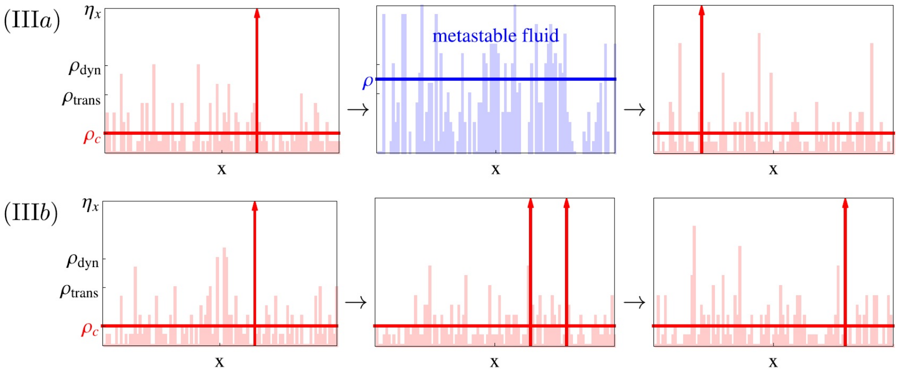
[G., Schütz (2008); Chleboun, G (2015)]

$$g_L(n) := \begin{cases} c > 1 & \text{if } n \leq aL, \\ 1 & \text{if } n > aL, \end{cases} \quad \text{for} \quad n \geq 1 \ , \quad g_L(0) = 0$$



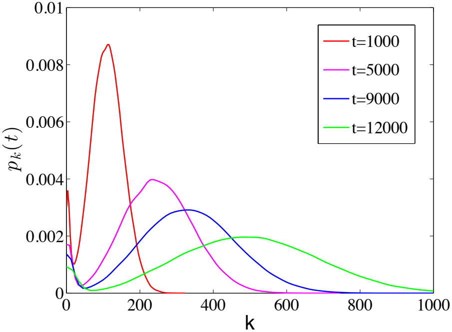


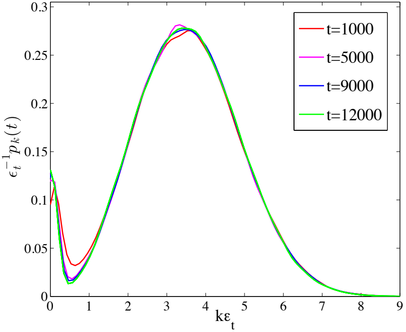
Metastability for toy ZRP



Part II - Dynamic results

Dynamics of condensation





$$f_k(t) = \mathbb{P}[\eta_x(t) = k] \quad \text{and} \quad p_k(t) = k\, f_k(t)/\rho$$

Mean-field equation

SPS with generator $\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda} p(x,y)c(\eta_x,\eta_y)\big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta})\big)$

Empirical measures $F_k^L(\eta) = \frac{1}{L}\sum_{x\in\Lambda}\delta_{\eta_x,k} \in [0,1]$

Assume

- **complete graph** $p(x,y) = 1/(L-1)$
- **jump rates** $c(k,l) \leq C_1 k(C_2 + l)$
- **initial conditions** $\eta(0)$ such that $F_k^L(\eta(0)) \to f(0)$ on $\mathbb{N}_0$

  $$m_0(0) = 1\ ,\quad m_1(0) = \sum_k k f_k(0) = \rho < \infty\ ,\quad m_2(0) < \infty$$

  and $\alpha_1,\ \alpha_2 > 0$ such that for all $L \geq 1$

  $$\eta(0) \in \Omega_\alpha := \Big\{\eta : \frac{1}{L}\sum_{x\in\Lambda}\eta_x < \alpha_1,\ \frac{1}{L}\sum_{x\in\Lambda}\eta_x^2 < \alpha_2\Big\}$$

  $\to$ for example $\eta_x(0) \sim f(0)$ i.i.d. bounded

Mean-field equation

Theorem – LLN for empirical process [G, Jatuviriyapornchai (2019)]

Under above assumptions, for all $k \in \mathbb{N}_0$ the empirical processes $\big(F_k^L(\eta(t)) : t \geq 0\big)$ converge weakly on path space to $\big(f_k(t) : t \geq 0\big)$ as $L \to \infty$, which are given as the **unique solution** of the **mean-field (rate) equation**

$$\frac{d}{dt} f_k(t) = \sum_{l \geq 0} \Big(c(k+1,l) f_l(t) f_{k+1}(t) + c(l,k-1) f_l(t) f_{k-1}(t) \Big) - \sum_{l \geq 0} \big(c(k,l) + c(l,k) \big) f_l(t) f_k(t) \quad \text{for all } k \geq 0 \ , \qquad \text{(MFE)}$$

with initial condition $f(0)$ given above.

- In particular we show uniqueness of the solution to (MFE) for given $f(0)$.
- Implies convergence of expectations,

$$f_k^L(t) := \mathbb{E}^L\big[F_k^L(\eta(t))\big] = \frac{1}{L} \sum_{x \in \Lambda} \mathbb{P}^L\big[\eta_x(t) = k\big] \to f_k(t) \ .$$

Propagation of chaos

Assume in addition symmetry of initial conditions, i.e.

the law of $\{\eta_x(0) : x \in \Lambda\}$ is permutation invariant for each $L \geq 1$.

Corollary – Propagation of chaos (see e.g. [dai Pra (2017)])

Under the conditions of the Theorem and above, for any **finite-dimensional marginal** with distinct $x_1, \ldots, x_m \in \Lambda$, $m \geq 1$, we have for any $T > 0$

$$\big(\eta_{x_i}(t) : t \in [0,T]\big) \text{ converge to independent birth-death chains },$$

with (non-linear) master equation (MFE) and **generator**

$$\mathcal{L}_{f(t)} h(k) = \alpha_k(t)[h(k+1) - h(k)] + \beta_k(t)[h(k-1) - h(k)] ,$$

with rates $\alpha_k(t) = \sum_{l \geq 0} c(l,k) f_l(t)$ and $\beta_k(t) = \sum_{l \geq 0} c(k,l) f_l(t)$.

[Gärtner (1988) WASEP; Rezakhanlou (1994) SSEP and ZRP, (1996) multi-type model ...]

Proof of main result

1. existence of limits $t \mapsto f(t)$ via tightness
2. limits are solutions of (MFE)
3. uniqueness of solutions of (MFE)

Moments. $m_n^L(t) := \mathbb{E}^L\Big[\sum_{k\geq 0} k^n F_k^L(\eta(t))\Big] = \sum_{k\geq 0} k^n f_k^L(t)$

$$m_0^L(t) \equiv 1 \quad \text{and} \quad m_1^L(t) \equiv m_1^L(0) \stackrel{L\to\infty}{\longrightarrow} \rho \ .$$

Lemma. $C > 0$ such that $m_2^L(t) \leq (\alpha_2 + Ct)e^{Ct}$ for all $t \geq 0$, $L \geq 1$,

$$\text{using} \qquad \frac{d}{dt}\mathbb{E}^L\big[F_k^L(\eta(t))\big] = \mathbb{E}^L\big[\mathcal{L}F_k^L(\eta(t))\big] \quad \text{and Gronwall .}$$

Proof of main result

1. Tightness. For each bounded $h : \mathbb{N}_0 \to \mathbb{R}$ the law of

$$t \mapsto H(\eta(t)) := \sum_{k\geq 0} h_k F_k^L(\eta(t))$$

on path space $D_{[0,\infty)}(\mathbb{R})$ is tight as $L \to \infty$.

Using a version of **Aldous' criterion**, $\big|\sum_k h_k F_k^L(\eta)\big| \leq \|h\|_\infty$ and Markov's inequality we need to establish

$$\limsup_{L\to\infty} \sup_{t<\delta} \sup_{\zeta\in\Omega_\alpha} \mathbb{E}_\zeta^L\big[|H(\eta(t)) - H(\zeta)|\big] \to 0 \quad \text{as } \delta \to 0^+ \ .$$

Itô's formula $\quad M_h(t) := H(\eta(t)) - H(\eta(0)) - \int_0^t \mathcal{L}H(\eta(s))ds$

is a **martingale** with pred. QV $\quad \langle M_h\rangle(t) = \int_0^t [\mathcal{L}H^2 - 2H\mathcal{L}H](\eta(s))ds \ .$

$$\mathcal{L}H(\eta) = \sum_{k\geq 0} h_k \Big[F_{k-1}^L(\eta) \sum_{l\geq 1} c(l, k-1) F_l^L(\eta) + F_{k+1}^L(\eta) \sum_{l\geq 0} c(k+1, l) F_l^L(\eta)$$

$$- F_k^L(\eta) \sum_{l\geq 0} \big(c(k,l) + c(l,k)\big) F_l^L(\eta)\Big](1 + 1/L) + \Delta_L(\eta)$$

Proof of main result

$$\mathbb{E}^L\Big[\big|H(\eta(t))-H(\eta(0))\big|\Big] \le \underbrace{\int_0^t \mathbb{E}^L|\mathcal{L}H(\eta(s))|ds}_{(1)} + \underbrace{\mathbb{E}^L\big[\langle M_h\rangle(t)\big]}_{(2)}.$$

with estimates

$$(1) \le t\|h\|_\infty\Big(4C_1\alpha_1\left(\alpha_1+C_2\right)+\frac{C}{L}(1+t)e^{Ct}\Big)$$
$$(2) \le t\|h\|_\infty^2\frac{1}{L}\Big(4C_1\alpha_1\left(\alpha_1+C_2\right)+\frac{C}{L}(1+t)e^{Ct}\Big)$$

Both vanish as $t\le\delta\to 0$ uniformly in L which implies **tightness**.

2. Estimate for (2) implies $\mathbb{E}^L[\langle M_h\rangle(t)]\to 0$ as $L\to\infty$ for all $t\ge 0$,
so the martingale vanishes and each limit **solves a weak version of (MFE)**

$$\sum_{k\ge 0} h_k\big(f_k(t)-f_k(0)\big) = \int_0^t \sum_{k\ge 0} h_k\big(\mathcal{L}^\dagger_{f(s)}f(s)\big)_k\,ds\ .$$

Proof of main result

3. Uniqueness of solutions of (MFE)

- **moments** $m_n(t) = \sum_{k\geq 0} k^n f_k(t)$

$$m_0(t) \equiv m_0(0) \quad \text{and} \quad m_1(t) \equiv m_1(0) = \rho \quad \text{are conserved.}$$

Gronwall estimate $m_2^L(t) \leq (\alpha_2 + Ct)e^{Ct}$ for all $t \geq 0$.

- Consider $f(t), \hat{f}(t)$ with $f(0) = \hat{f}(0) \in \mathcal{P}(\mathbb{N}_0)$ and establish Gronwall for

$$\theta(t) := \sum_{k\geq 0} (k+1)\big|\Delta_k(t)\big| \quad \text{where} \quad \Delta_k(t) := f_k(t) - \hat{f}_k(t) \ .$$

[Esenturk (2017), Schlichting (2018)], following classical proof [Ball, Penrose (1986)]

Future work.

- quantitativve propagation of chaos with **uniform-in-time** error bounds
- **large deviations** for F_k^L
- instantaneous gelation for EDG-model with $\gamma > 2$

Properties of solutions to MFE

For all $k \geq 0$ with $\beta_0(t) \equiv 0$ and $f_{-1}(t) \equiv 0$ we have

$$\frac{d}{dt} f_k(t) = \alpha_{k-1}(t) f_{k-1}(t) + \beta_{k+1}(t) f_{k+1}(t) - \big(\alpha_k(t) + \beta_k(t)\big) f_k(t) \ ,$$

where $\alpha_k(t) = \sum_{l \geq 0} c(l,k) f_l(t)$ and $\beta_k(t) = \sum_{l \geq 0} c(k,l) f_l(t)$.

- **moments** $m_n(t) = \sum_{k \geq 0} k^n f_k(t)$

  $m_0(t) \equiv m_0(0) = 1$ and $m_1(t) \equiv m_1(0) = \rho$ are conserved.

- **stationary solutions:** invariant product measures of SPS

  exist if and only if $$\frac{c(k,l)}{c(l+1,k-1)} = \frac{c(k,0)}{c(1,k-1)} \frac{c(1,l)}{c(l+1,0)}$$

  [Fajvrova, Gobron, Saada (2017)]

  $$f_k^\rho = \frac{1}{z(\phi)} \phi^k \underbrace{\prod_{i=1}^{k} \frac{c(1,i-1)}{c(i,0)}}_{w(k)} \quad \text{with} \quad z(\phi) = \sum_k w(k) \phi^k \ ,$$

  pick **fugacity** $\phi \in [0, \phi_c]$ to fix the **density** $\sum_k k f_k^\rho = \rho \in [0, \rho_c]$

Properties of solutions to MFE

- **detailed balance** $c(k,l-1)w(k)w(l-1)=c(l,k-1)w(l)w(k-1)\ , \quad k,l\geq 1$

- so the **relative entropy** $\mathcal{H}(f|f^\rho)=\sum_{k\geq 0} f_k \log\frac{f_k}{f_k^\rho}$

  is a (non-negative) **Lyapunov function** with

  $$\frac{d}{dt}\mathcal{H}(f(t)|f^\rho)=-\frac{1}{2}\sum_{k,l\geq 1} c(k,l-1)w(k)w(l)\Psi\Big(\frac{f_k f_{l-1}}{w(k)w(l-1)},\frac{f_{k-1}f_l}{w(l)w(k-1)}\Big)\leq 0$$

  where $\Psi(a,b)=(a-b)\big(\log a-\log b\big)$.

- can be used to establish **ergodicity**, i.e. for $\rho=m_1(0)$

  $$f(t)\to\begin{cases} f^\rho\ , & \rho\leq\rho_c \quad \text{(strong)}\\ f^{\rho_c}\ , & \rho\geq\rho_c \quad \text{(weak with bdd. test functions)}\end{cases}$$

  [Schlichting (2018)]

- **gradient flow** structure $\frac{d}{dt}f(t)=-\mathcal{K}[f]D\mathcal{H}(f|f^\rho)$

Scaling analysis for ZRP

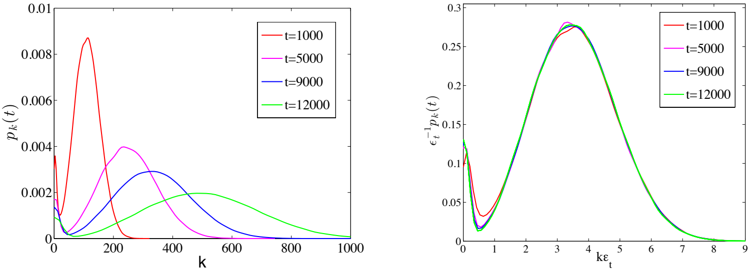
Scaling ansatz for phase separated solution with $m_1(t) = \rho > \rho_c$

$$f_k(t) = f_k^{\text{bulk}}(t) + \epsilon_t^2 h(k\epsilon_t) \quad \text{as } t \to \infty$$

with scale $\epsilon_t \to 0$ and scaling function $h(u)$, $u > 0$, and $h(u) \to 0$ as $u \to \infty$

We have $f^{\text{bulk}}(t) \to f^{\rho_c}$ and $\displaystyle\sum_{k>0} k\epsilon_t^2 h(k\epsilon_t) \to \int_{u>0} uh(u)\, du = \rho - \rho_c$.

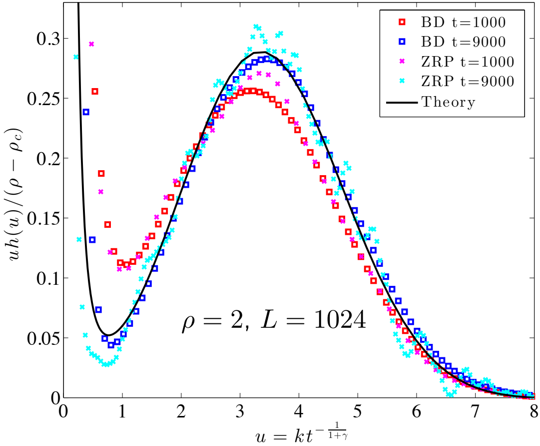
ZRP with rates $g(k) = 1 + b/k$, $b = 4$, $\rho_c = 1/2$, $\rho = 10$



Scaling analysis for ZRP

$$\epsilon_t = t^{-1/2} \ , \quad h''(u) + \Big(\frac{u}{2} - A + \frac{b}{u}\Big)h'(u) + \Big(1 - \frac{b}{u^2}\Big)h(u) = 0$$

[Godréche (2003); J., G. (2016); Godréche, Drouffe (2016)]



Particle approximations

Simulate m copies $(Y_t^i : t \geq 0)$ of the **single-site BD chain** with generator

$$\mathcal{L}^m G(\mathbf{k}) := \sum_{i=1}^{m} \Big(\bar{\alpha}_i(\mathbf{k}) \big[G(\mathbf{k}+\mathbf{e}_i) - G(\mathbf{k}) \big] + \bar{\beta}_i(\mathbf{k}) \big[G(\mathbf{k}-\mathbf{e}_i) - G(\mathbf{k}) \big] \Big)$$

with **empirical rates** $\bar{\alpha}_i(\mathbf{k}) = \frac{1}{m} \sum_{j=1}^{m} c(k_j, k_i)$ and $\bar{\beta}_i(\mathbf{k})$ analogously

Empirical measure $\bar{f}_k^m(\mathbf{Y}_t) = \dfrac{1}{m} \displaystyle\sum_{i=1}^{m} \delta_{Y_t^i, k} \to f_k(t)$

with quantitative error bounds [Miclo, del Moral (2004); Rousset (2006)]

Problems

$\to$ $G(\mathbf{k}) := \sum_{i=1}^{m} k_i$ is a martingale with QV linear in time;
absorbing state $\mathbf{k} = \mathbf{0}$ affects sampling at times of order m^2

$\to$ decreasing volume fraction of condensed phase (poor statistics)

Size-biased particle approximations

Size-biased dynamics $(X_t : t \geq 0)$ on state space $\mathbb{N}_+$

$p_k(t) = k\, f_k(t)/\rho$, $k \geq 1$, use (MFE) to get

$$\frac{d}{dt}p_k(t) = \alpha_{k-1}(t)p_{k-1}(t) + \beta_{k+1}(t)\frac{k}{k+1}p_{k+1}(t) + \frac{\alpha_{k-1}(t)}{k-1}p_{k-1}(t) - \Big(\alpha_k(t) + \beta_k(t)\frac{k-1}{k}\Big)p_k(t) - \frac{\beta_k(t)}{k}p_k(t)$$

$$\frac{d}{dt}p_1(t) = \beta_2(t)\frac{1}{2}p_2(t) - \alpha_1(t)p_1(t) + \alpha_0(t)\frac{f_0(t)}{\rho} - \beta_1(t)p_1(t)$$

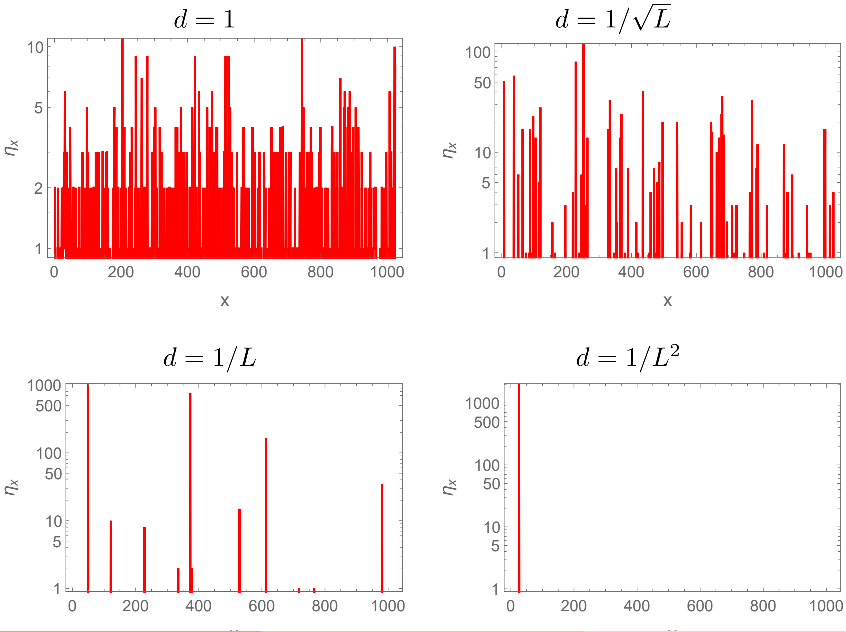
where $f_0(t) = 1 - \rho\sum_{k\geq 1} p_k(t)/k$.

$\Rightarrow$ BD chain with long-range jumps $k \to l$ with rate $\frac{c(k,l-1)}{k} f_{l-1}(t)$

- **no** additional conservation law
- **no** absorbing state for m copies $(X_t^i : t \geq 0)$
- fixed volume fraction of condensed phase
- $m_2(t) = \mathbb{E}[\eta_x^2(t)]$ is well approximated by $\frac{1}{m}\sum_{i=1}^m X_t^i$

Part II - Dynamic results

Condensation in the inclusion process



Stationary measures

canonical measures are Dirichlet multinomials

$$\pi_{L,N}[d\eta] = \frac{\mathbb{1}_{X_{L,N}}(\eta)}{Z_{L,N}} \prod_{x\in\Lambda} w(\eta_x)\, d\eta$$

where $w(n) = \frac{\Gamma(n+d)}{n!\Gamma(d)} \simeq d\, n^{d-1}$ and $Z_{L,N} = \frac{\Gamma(N+dL)}{N!\Gamma(dL)}$

order statistics $\hat{\eta} := \big(\eta_{(L)}, \ldots, \eta_{(1)}\big)$ for $\eta \in X_{L,N}$

size-biased sample $\tilde{\eta} := \big(\eta_{\sigma(1)}, \ldots, \eta_{\sigma(L)}\big)$

$$\sigma(1) = x \in \Lambda \text{ w.prob. } \frac{\eta_x}{N}\ , \quad \sigma(2) = y \in \Lambda \setminus \sigma(1) \text{ w.prob. } \frac{\eta_y}{N - \eta_{\sigma(1)}}\ \ldots$$

partitions $\frac{1}{N}\tilde{\eta} \in \Delta := \big\{(q_1, q_2, \ldots) : q_k \geq 0 : \sum_k q_k = 1\big\}$

$$\frac{1}{N}\hat{\eta} \in \nabla := \big\{(q_1, q_2, \ldots) : q_1 \geq q_2 \geq \ldots \geq 0 : \sum_k q_k = 1\big\}$$

Condensation in IP

Asymptotic behaviour

[Jatuviriyapornchai, Chleboun, G (2020)]

IP $(\pi_{L,N})_{L,N}$ exhibits a CT with $\rho_c = 0$ as $N/L \to \rho, d = d_L \to 0$ and

- $(\eta_1, \ldots, \eta_k) \xrightarrow{D} (0, \ldots, 0)$ for all fixed $k \geq 1$
- $dL \to \infty$: $d\big(\tilde{\eta}_1, \ldots, \tilde{\eta}_k\big) \xrightarrow{D} i.i.d.\ \mathrm{Exp}(1/\rho)$ for all fixed $k \geq 1$
- $dL \to \theta \in [0, \infty)$: $\frac{1}{N}\tilde{\eta} \xrightarrow{D} \mathrm{GEM}(\theta)$ or $\frac{1}{N}\hat{\eta} \xrightarrow{D} \mathrm{PD}(\theta)$
- $dL \log L \to 0$: complete condensation with $N - \eta_{(L)} \xrightarrow{D} 0$

Let $U_1, U_2, \ldots \sim$ Beta$(1, \theta)$ iidrvs on $[0, 1]$ with PDF $\theta(1-x)^{\theta-1}$.

A random partition $V = (V_k : k \in \mathbb{N}) \in \Delta$ is $\mathrm{GEM}(\theta)$ distributed if

$$V_1 = U_1 \ , \quad V_2 = (1-U_1)U_2 \ , \quad V_3 = (1-U_1)(1-U_2)U_3 \ , \ \ldots$$

Then the order statistics $\nabla \ni \hat{V} \sim \mathrm{PD}(\theta)$ have Poisson-Dirichlet distribution.

[Kingman (1975), Griffiths (1980), Engen (1978), McCloskey (1965)]

Generalized IP-like models

Consider a system with $\pi_{L,N}[d\eta] = \frac{\mathbb{1}_{X_{L,N}}(\eta)}{Z_{L,N}} \prod_{x\in\Lambda} w_L(\eta_x)\, d\eta$ and

$$(A1) \qquad \|w_L - w\|_\infty \to 0 \quad \text{where wlog} \quad \sum_{n=0}^{\infty} w(n) = 1\ , \quad w(0) > 0\ ,$$

and $\sup_n \big|w(n-1) \wedge w(n)\big| > 0$ or $w(0) = 1$, as well as

$$(A2) \qquad \lim_{J\to\infty} \lim_{L\to\infty} \sup_{n>J} \big|n w_L(n) L - \theta\big| = 0\ .$$

Theorem

[Chleboun, Gabriel, G (2022)]

As $L, N \to \infty$, $N/L \to \rho$ the system exhibits a **condensation transition** with

$$\rho_c = \sum_{n=0}^{\infty} n w(n) \in [0, \infty) \quad \text{and } \textbf{background density } \rho_c \text{ for } \rho > \rho_c\ .$$

The condensed mass fraction $\alpha = \alpha(\rho) = (\rho - \rho_c)/\rho$ is distributed as

$$\pi_{L,N}\Big[\frac{1}{N}\hat{\eta} \in \cdot\ \Big] \xrightarrow{D} \mathrm{PD}_{[0,\alpha]}(\theta) \quad \text{as } L, N \to \infty,\ N/L \to \rho \geq \rho_c\ .$$

Generalized IP-like models

Proof. $\mathrm{PD}(\theta)$ is the unique reversible distribution of **split-merge dynamics** on ∇

$$\mathcal{G}_\theta f(q) = \sum_{i\neq j} q_i q_j \Big[f(\widehat{M}_{ij} q) - f(q) \Big] + \theta \sum_i q_i^2 \Big[\int_0^1 f(\widehat{S}_i^u q) du - f(q) \Big]$$

proven for $\theta \in [0,1]$ [Zerner et al. (2004), Schramm (2005)]

Consider a discrete approximation [Ioffe, Tóth (2020)]

$$\mathcal{G}_\theta^{N,\epsilon} f(q) = \tfrac{N}{N-1} \sum_{i\neq j} q_i q_j \mathbb{1}_{q_i, q_j \geq \epsilon} \Big[f(\widehat{M}_{ij} q) - f(q) \Big] + \tfrac{\theta}{N-1} \sum_i q_i \mathbb{1}_{q_i \geq 2\epsilon} \Big[\sum_{k=\epsilon N}^{N(q_i - \epsilon)} f(\widehat{S}_i^{k/(Nq_i)} q) du - f(q) \Big]$$

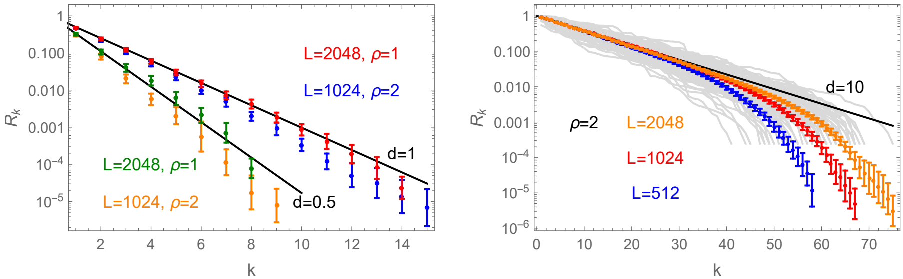
and show that $\mathcal{G}_\theta^{N,\epsilon} \to \mathcal{G}_\theta$ as $N \to \infty$, $\epsilon \to 0$ on $C_b(\overline{\nabla})$ and

$$\Big| \pi_{L,N} \Big(f\big(\tfrac{1}{N}\hat{\eta}\big) \mathcal{G}_\theta^{N,\epsilon} g\big(\tfrac{1}{N}\hat{\eta}\big) \Big) - \pi_{L,N} \Big(g\big(\tfrac{1}{N}\hat{\eta}\big) \mathcal{G}_\theta^{N,\epsilon} f\big(\tfrac{1}{N}\hat{\eta}\big) \Big) \Big| \to 0$$

as $L, N \to \infty$, $N/L \to \rho \geq 0$.

GEM/PD regime for IP

$$dL \to \theta \in (0, \infty) \quad : \quad \frac{1}{N}\tilde{\eta} \xrightarrow{D} \mathrm{GEM}(\theta)$$

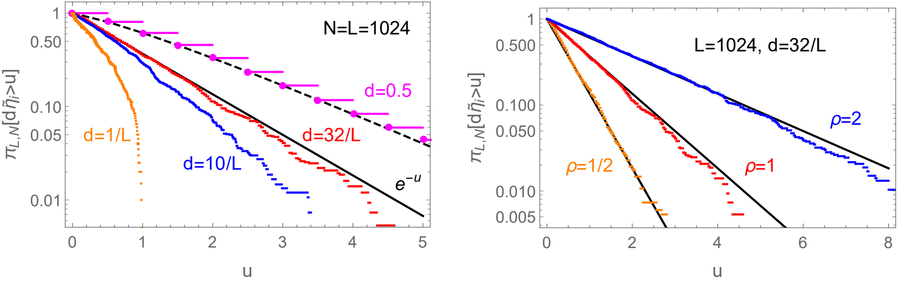


observe $R_k(\eta) = 1 - \frac{1}{N}\sum_{i=1}^{k} \tilde{\eta}_i$, then

$$\langle R_k \rangle_{L,N} \to \left(\frac{\theta}{1+\theta}\right)^k \quad \text{as } L, N \to \infty,\ N/L \to \rho,\ dL \to \theta\ .$$

Intermediate regime for IP

$$dL \to \infty \quad : \quad d\big(\tilde{\eta}_1, \ldots, \tilde{\eta}_k\big) \xrightarrow{D} i.i.d.\ \mathrm{Exp}(1/\rho)$$



IP on the complete graph

Generator $\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda}(\eta_x\eta_y + d_L\eta_x)\big(f(\eta^{xy}) - f(\eta)\big)$, $\eta \in E_{L,N}$

Empirical measure $\mu^\eta := \frac{1}{N}\sum_{x\in\Lambda}\eta_x\delta_{x/L} \in \mathcal{M}_1([0,1])$

with $\mu^\eta(h) = \frac{1}{N}\sum_{x\in\Lambda}\eta_x h(x/L)$, $h \in C([0,1])$

$$\mathcal{L}\mu^\eta(h) = \underbrace{\frac{1}{N}\sum_{x,y\in\Lambda}^{N}\eta_x\eta_y\Big[h\big(\tfrac{y}{L}\big) - h\big(\tfrac{x}{L}\big)\Big]}_{=0} + \underbrace{\frac{1}{N}\sum_{x\in\Lambda}\eta_x\frac{d_L L}{L}\sum_{y\in\Lambda}\Big[h\big(\tfrac{y}{L}\big) - h\big(\tfrac{x}{L}\big)\Big]}_{\to\mu^\sigma(\mathfrak{A}h)}$$

with **mutation operator** $\mathfrak{A}h(v) = \theta\int_0^1 \big[h(u) - h(v)\big]du$

carré du champ $\Gamma\mu^\sigma(h) = \mathfrak{L}\mu^\sigma(h)^2 - 2\mu^\sigma(h)\mathfrak{L}\mu^\sigma(h) \simeq 4\big(\mu^\sigma(h^2) - \mu^\sigma(h)^2\big)$

Fleming-Viot process on type space

$\big(\mu^{\sigma(t)} : t \geq 0\big) \xrightarrow{D} (\mu_t : t \geq 0)$ **Fleming-Viot process** which is a

measure-valued diffusion $d\mu_t(h) = \mu_t(\mathfrak{A}h)dt + dM_t(h)$

with martingale $M_t(h)$ with QV $\int_0^t \Gamma\mu_s(h)ds = 4\int_0^t \big(\mu_s(h^2) - \mu_s(h)^2\big)ds$,

and $\mathfrak{A}h(v) = \theta\int_0^1 \big[h(u) - h(v)\big]du$

[Ethier, Kurtz (1993)]

is equivalent to **Poisson-Dirichlet diffusion** $\big(q(t) : t \geq 0\big)$ on ∇ with generator

$$\mathcal{L}_{PD}f(q) = \sum_{i,j=1}^{\infty} q_i q_j (\partial_{q_i} - \partial_{q_j})^2 f(q) - \theta \sum_{i=1}^{\infty} q_i \partial_{q_i} f(q)$$

defined on a **core** $1, \phi_2, \phi_3 \ldots$ with $\phi_m(q) = \sum_i q_i^m$ [Ethier, Kurtz (1981)]

The partition $q(t) \in \nabla$ corresponds to the ordered **atoms** of μ_t.

[Griffiths, Ruggerio, Spanò, Zhou (2021)]

Measure-valued process $d_L L \to \theta$

Generator $\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda}(\eta_x\eta_y + d_L\eta_x)\big(f(\eta^{xy}) - f(\eta)\big)$, $\eta \in E_{L,N}$

Empirical measure on mass space $\mu^\eta := \sum_{x\in\Lambda} \frac{\eta_x}{N}\delta_{\frac{\eta_x}{N}} \in \mathcal{M}_1([0,1])$

$$\mu^\eta(h) = \sum_{x\in\Lambda} \tfrac{\eta_x}{N} h\big(\tfrac{\eta_x}{N}\big) = \sum_{x\in\Lambda} \tilde{h}\big(\tfrac{\eta_x}{N}\big) , \quad h \in C([0,1])$$

state space $E := \mu^{(\cdot)}(\overline{\nabla}) \subsetneq \mathcal{M}_1([0,1])$

Lemma

The closure of $(\mathcal{G}, \mathcal{D}_\mathcal{G})$ with

$$\mathcal{G}\prod_{k=1}^n \mu(h_k) = \sum_{k,l=1}^n \big(\mu(\tilde{h}_k'\tilde{h}_l') - \mu(\tilde{h}_k')\mu(\tilde{h}_l')\big)\prod_{m\neq k,l}\mu(h_m) + \sum_{k=1}^n \mu(\mathcal{A}h_k)\prod_{m\neq k}\mu(h_m)$$

domain $\mathcal{D}_\mathcal{G} =$ sub-algebra of $C(E)$ generated by $\mu \mapsto \mu(h)$, $h \in C^3([0,1])$

generates a Feller process on E, where $\tilde{h}'(z) = \big(zh(z)\big)' = h(z) + zh'(z)$ and

$$\mathcal{A}h(z) = z(1-z)h''(z) + \big(2 - (2+\theta)z\big)h'(z) + \theta\big(h(0) - h(z)\big) .$$

Measure-valued process $d_L L \to \theta$

Theorem [Chleboun, Gabriel, G (2023)]

Let $\mu^{\eta(0)} \xrightarrow{D} \mu_0 \in E$. Then for all $\rho > 0$ as $N/L \to \rho$, $d_L L \to \theta \geq 0$

$$\big(\mu^{\eta(t)} : t \geq 0\big) \xrightarrow{D} (\mu_t : t \geq 0) \qquad \text{on } D\big([0,\infty), E\big)$$

where $(\mu_t : t \geq 0)$ is a **measure-valued process** with generator $\mathcal{G}$.

$$\mathcal{A}h(z) = z(1-z)h''(z) + \big(2 - (2+\theta)z\big)h'(z) + \theta\big(h(0) - h(z)\big)$$

- **measure-valued diffusion** $\quad d\mu_t(h) = \mu_t(\mathcal{A}h)dt + d\mathcal{M}_t(h)$
- mass is conserved $\big(h(z) \equiv 1\big)$, $\quad \delta_0$ describes **mass below macro. scale**
  $h(z) = z$ describes second moment of the mass partition
- Let $(Z_t : t \geq 0)$ be the process on $[0,1]$ with generator $\mathcal{A}$, then we have
  the **duality** $\quad \mathbb{E}_{\mu_0}\big[\mu_t(h)\big] = \mathbf{E}_{\mu_0}\big[h(Z_t)\big]$ for all $t \geq 0$.
- Equivalence to PD diffusion $\big(q(t) : t \geq 0\big)$: $\quad (\mu_t : t \geq 0) \sim \big(\mu^{q(t)} : t \geq 0\big)$

Measure-valued process $d_L L \to \infty$

Empirical measure on mass scale ρ/d_L: $\bar{\mu}^\eta := \sum_{x\in\Lambda} \frac{\eta_x}{N} \delta_{d_L L \frac{\eta_x}{N}} \in \mathcal{M}_1([0,\infty))$

$$\frac{1}{dL}\mathcal{L} \prod_{k=1}^{n} \bar{\mu}^\eta(h_k) = \sum_{k=1}^{n} \bar{\mu}^\eta(\bar{\mathcal{A}}h_k) \prod_{m\neq k} \bar{\mu}^\eta(h_m) + o(1) \ ,$$

where $\bar{\mathcal{A}}h(z) = zh''(z) + (2-z)h'(z) + (h(0)-h(z))$, $z \in [0,\infty)$

Theorem

[Chleboun, Gabriel, G (2023)]

Let $\mu^{\eta(0)} \xrightarrow{D} \mu_0 \in \mathcal{M}([0,\infty])$. Then for all $\rho > 0$ as $N/L \to \rho$, $d_L L \to \infty$

$$\big(\bar{\mu}^{\eta(t/(d_L L))} : t \geq 0\big) \xrightarrow{D} (\bar{\mu}_t : t \geq 0) \qquad \text{on } D\big([0,\infty), \mathcal{M}([0,\infty])\big)$$

where $(\mu_t : t \geq 0)$ is a **measure-valued process** with generator

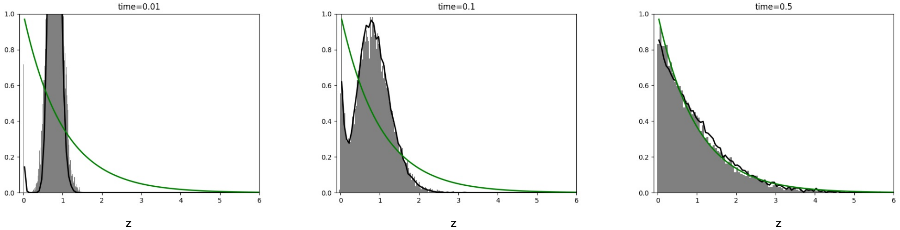
$$\bar{\mathcal{G}} \prod_{k=1}^{n} \bar{\mu}(h_k) = \sum_{k=1}^{n} \bar{\mu}(\bar{\mathcal{A}}h_k) \prod_{m\neq k} \bar{\mu}(h_m) \ , \quad h_k \in C_c^3([0,\infty]) \cap \text{constants}$$

Measure-valued process $d_L L \to \infty$

- **deterministic process** $d\bar{\mu}_t(h) = \bar{\mu}_t(\bar{\mathcal{A}}h)dt$ and $\bar{\mu}_t(h) = \mathbf{E}_{\mu_0}\big[h(\bar{Z}_t)\big]$
- **Duality** $\bar{\mu}_t = \mathrm{Law}(\bar{Z}_t)$ with $(\bar{Z}_t : t \geq 0)$ on $[0, \infty)$ with generator $\bar{\mathcal{A}}$
- $\bar{\mu}_t[dz] = f(t,z)dz$, **FPE** $\partial_t f(t,z) = z\partial_z^2 f(t,z) + z\partial_z f(t,z)$, $f(t, 0+) = 1$

  stationary distribution $\bar{\mu}_t \to \mathrm{Exp}(1)$

  [Avrachenkov et al. (2013); De Marco (2011)]
- **mass at** ∞: $\bar{\mu}_t[0,\infty) = 1 - (1 - \bar{\mu}_0[0,\infty))e^{-t}$



jump diffusion $\bar{\mathcal{A}}$ (histogram) , inclusion process $\mathcal{L}$ (black)

$N = L = 1024,\ d_L = L^{-1/2} = \frac{1}{32}$

Summary

Future work extend this approach to ZRP and EDG

Collaborators

- Paul Chleboun (Warwick Statistics)
- Simon Gabriel (Warwick Mathematics $\rightarrow$ Münster)
- Mim – Watthanan Jatuviriyapornchai (Mahidol University Bangkok)
- Frank Redig (TU Delft)
- Gunter Schütz (FZ Jülich/Lissabon)
- Herbert Spohn (TU München)

Thank you!

Rigorous results on condensation dynamics

Stationary dynamics for ZRP (metastability)

- L fixed, $N \to \infty$, $p(x, y)$ reversible [Beltrán, Landim (2010,11,12,15)]

$$Y^N\big(\eta(tN^{1+b})\big) \to Y_t \quad \text{RW on (subset of) } \Lambda \ , \quad \text{rates } \propto \mathrm{cap}_\Lambda(x, y)$$

  L fixed, $N \to \infty$, $p(x, y)$ asymmetric [Landim (2014), Seo (2018)]

- $L, N \to \infty$, $N/L \to \rho > \rho_c$, $p(x, y)$ symmetric on rescaled torus $\subset \mathbb{T}$

$$Y^L\big(\eta(tL^{1+b})\big) \to Y_t \quad \text{Lévy-type on } \mathbb{T}$$

[Armendáriz, G., Loulakis (2017)]
[Bovier, Neukirch (2014)]

Nucleation/Coarsening for ZRP

- L fixed, $N \to \infty$, $p(x, y)$ irreducible [Beltrán, Jara, Landim (2017)]
[Armendáriz, Beltrán, Cuesta, Jara (2023)]

$$\boldsymbol{\eta}(tN^2)/N \to \mathbf{X}_t \quad \text{absorbed diffusion on } \Delta_L$$

Inclusion process

- L fixed, $N \to \infty$, $d = d_N \ll 1/\log N$, time scale t/d_N

  Coarsening for $p(x, y)$ symmetric, $Nd_N \to \infty$ [G., Redig, Vafayi (2013)]

  Stat. dynamics with multiple scales [Bianchi, Dommers, Giardiná (2017); Kim Seo (2021)]