Percolation and long-range correlations

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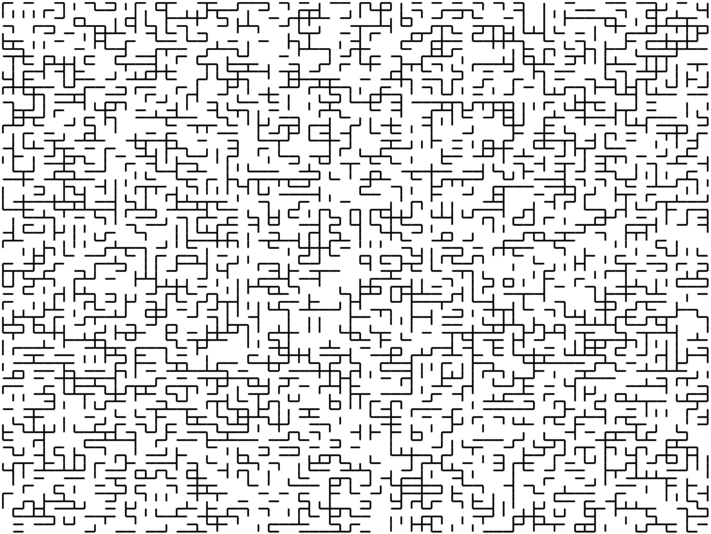
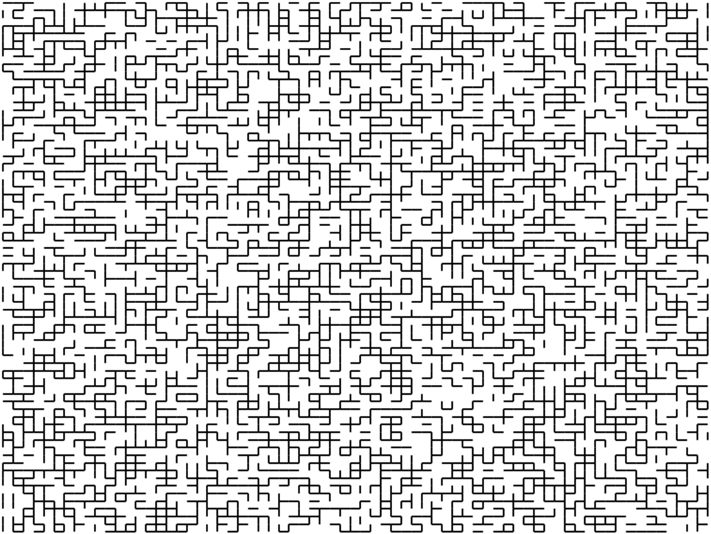
Bernoulli (bond) percolation

- Bernoulli percolation has first been investigated by chemists Flory and Stockmayer in the 1940s investigating the gelation of polymers, and then mathematically by Broadbent and Hammersley [BH57] in their research on gas masks;
- the model: each bond in $\mathbb{Z}^d$ is chosen to be “open” with probability $p \in (0,1)$, and “closed” otherwise (in an i.i.d. fashion);
- there exists $p_c \in (0,1)$ such that for $p \in (0,p_c)$ there exist only bounded connected component of open bonds, whereas for $p \in (p_c,1)$ there exists a (unique) unbounded connected component;

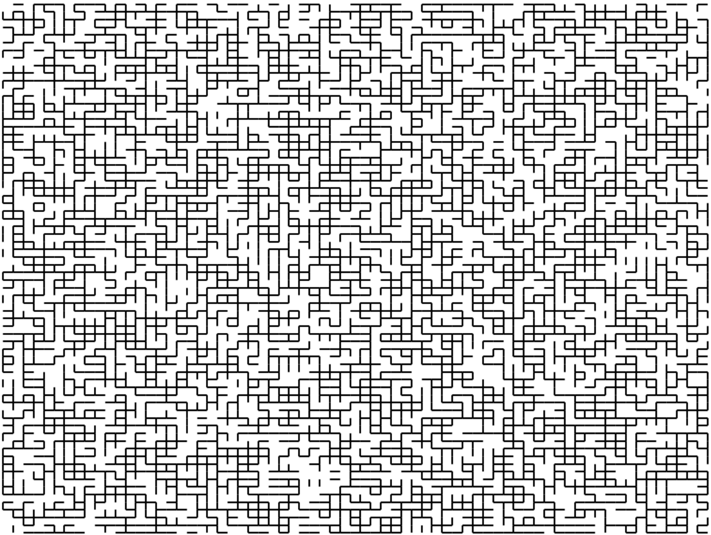
Bernoulli bond percolation ($p = 0.4$)



Bernoulli bond percolation ($p = 0.5$)



Bernoulli bond percolation ($p = 0.6$)



Bernoulli percolation on $\mathbb{Z}^d$ well-understood in off-critical regime

For $p \in (0, p_c)$:

- sharp phase transition / exponential decay of radius function [Men86] (cf. also [AB87]):

$$\psi_{\mathrm{Ber}}(p, n) := \mathbb{P}_p(0 \leftrightarrow \partial B(0, n)) \leq e^{-c_p n};$$

- $\rightsquigarrow$ finite expected cluster size $\chi(p) := \mathbb{E}_p[|\mathcal{C}_0|] < \infty$, with $\mathcal{C}_0$ the open cluster of the origin;

For $p \in (p_c, 1)$:

- uniqueness of infinite open cluster [AKN87] / [BK89];
- chemical distance [AP96];
- (stretched) exponential decay of radius / volume of finite open clusters [CCG$^+$89] / [ADS80] ;

For further background see Stauffer & Aharony [SA18], Grimmett [Gri99].

(Near-)critical percolation

For $p \approx p_c$, understanding has been obtained in two dimensions as well as in high dimensions:

- in $2d$ planar Bernoulli (bond) percolation, one has $p_c = \frac{1}{2}$ [Kes75] and there is no percolation at p_c [Har60];
- in planar settings of hexagonal / triangular lattice, critical exponents for Bernoulli percolation have been computed in [SW01] using conformal invariance and SLE;
  e.g., for *percolation function* $\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$, one has

$$\theta(p) = (p - 1/2)^{\frac{5}{36} + o(1)} \quad \text{as } p \downarrow p_c = 1/2,$$

  so critical exponent for θ is $\beta = 5/36$ in this setting;
- [HS90] used lace expansion to compute critical exponents in high dimensions (mean-field, cf. behavior on trees);

(Near-)critical percolation – physicists know more

For p close to p_c, *correlation length* $\xi = \xi(p) = |p - p_c|^{-\nu}$ describes the natural inherent length scale.

On smaller scales $L \ll \xi$, the system looks critical, while for $L \gg \xi$ its non-criticality becomes apparent. E.g., for $p \downarrow p_c$, there is $D < d$ such that

- for $r \ll \xi$ objects are expected to be fractal like

$$|\mathcal{C}_0 \cap B(r)| \approx r^D$$

- for $r \gg \xi$,

$$|\mathcal{C}_0 \cap B(r)| \approx \xi^D (L/\xi)^d$$

While the above is conjectured to be true for rather general percolation models, in $\mathbb{Z}^d$, $3 \leq d \leq 10$, however, far from determining critical exponents, in Bernoulli percolation it is not even proven that (as expected)

$$\theta(p_c) = 0.$$

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Gaussian free field

- G vertex set of a *transient* countably infinite graph with symmetric weights $\lambda_{x,y}$;
- SRW on G is the MC X with transition matrix

$$P(x,y) = \frac{\lambda_{x,y}}{\lambda_x},$$

where $\lambda_x = \sum_{z \sim x} \lambda_{x,z}$.

Definition

The GFF is the centered Gaussian process (φ_x), $x \in G$, with

$$\mathrm{Cov}(\varphi_x, \varphi_y) = g(x,y) = \frac{1}{\lambda_y} \sum_{n \geq 0} P^n(x,y), \quad \forall x,y \in G.$$

Gaussian free field

- on finite subset of $\mathbb{Z}^d$ with edge set E, density with respect to product Lebesgue measure (modulo boundary conditions) is

$$\propto \prod_{(x,y)\in E} \exp\Big\{ -\frac{(\varphi_x - \varphi_y)^2}{2\sigma_{x,y}^2} \Big\}.$$

$\rightsquigarrow$ can be interpreted as d-dimensional analogue of Brownian motion;

- strong correlations

$$\mathrm{Cov}(\varphi_x, \varphi_y) = g(x,y) \sim c_d \|x-y\|_2^{2-d}$$

in $\mathbb{Z}^d$, as $\|x-y\|_2 \to \infty$.

Gaussian free field

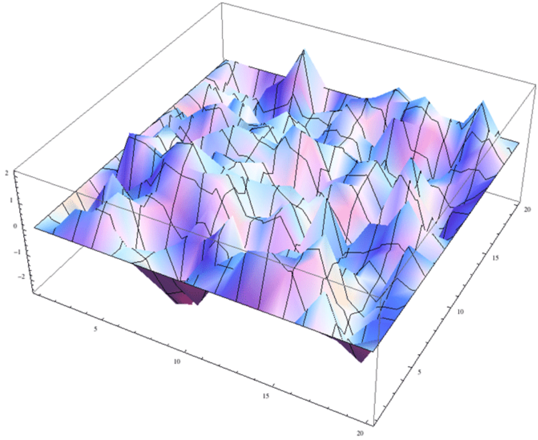


Figure: A realization of a (2d) Gaussian free field on a box with zero boundary condition

(By L. Coquille)

Percolation of GFF level sets

Introduce excursion sets

$$E^{\geq h}(G) := \{x \in G : \varphi_x \geq h\} \quad (= \varphi^{-1}([h, \infty)))$$

as percolation model with long-range correlations.

Critical parameter / level:

$$h_*(G) := \inf \{ h \in \mathbb{R} : \mathbb{P}(E^{\geq h}(G) \text{ has unbounded cluster}) = 0 \},$$

first introduced in [LS86] on $\mathbb{Z}^d$;

Previous (off-critical) results

- [BLM87]: $h_*(\mathbb{Z}^d) \geq 0$ for all $d \geq 3$, and $h_*(\mathbb{Z}^3) < \infty$;
- [RS13]:

$$h_*(\mathbb{Z}^d) < \infty \text{ for all } d \geq 3, \quad h_*(\mathbb{Z}^d) > 0 \quad \text{for } d \text{ large};$$

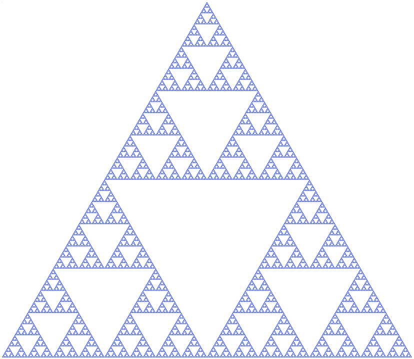
- [DPR18b]:

$$h_*(\mathbb{Z}^d) > 0 \quad \text{for all } d \geq 3;$$

- [DPR18a]: $\overline{h}(G) > 0$ for "regular G with dimension > 2";
  $\rightsquigarrow$ via isomorphism theorems also settles non-trivial phase transition ($u_*(G) > 0$) for vacant set percolation of Random Interlacements, confirming a conjecture of [Szn12];
- [DCGRS20]: Sharp phase transition for GFF level-set percolation in $\mathbb{Z}^d$, $d \geq 3$;

Previous results

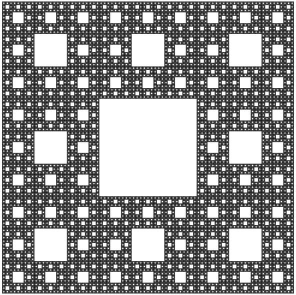
$S \times \mathbb{Z}$, with S the Sierpinski triangle;



(Picture by Beojan Stanislaus, CC BY-SA 3.0,
`https://commons.wikimedia.org/w/index.php?curid=8862246`)

Sierpinski carpet

- the d-dimensional Sierpinski carpet, $d \geq 3$;



(Picture by Josh Greig,

https://commons.wikimedia.org/wiki/File:Sierpinski_carpet.png)

A continuous model

Surprisingly, for an extension of the GFF, explicit computations are possible: $\rightsquigarrow$ "Cable system $\widetilde{\mathcal{G}}$" (goes back to [Var85] at least)

$\widetilde{\mathcal{G}}$ is obtained by adding line segments between neighboring vertices: for $x, y \in G$ neighboring vertices, on the line segment $I_{x,y}$ connecting x to y, conditionally on φ_x and φ_y, the GFF $(\widetilde{\varphi}_z)$, $z \in I_{x,y}$, behaves like a Brownian bridge $\rightsquigarrow$ "brings in analysis".

Then an edge $\{x, y\}$ is defined to be open iff Brownian bridge from φ_x to φ_y stays positive; have explicit formula

$$\mathbb{P}(\text{BB from } \varphi_x \text{ to } \varphi_y \text{ stays above } h \,|\, \varphi_x, \varphi_y) \\ = 1 - \exp\left\{2\lambda_{x,y}(\varphi_x \vee h)(\varphi_y \vee h)\right\}.$$

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Objects of interest

Want to obtain near-critical information on the following objects:

- Excursion sets $\widetilde{E}^{\geq h} := \{x \in \widetilde{\mathcal{G}} : \varphi_x \geq h\}$;
- cluster of "the origin" $\widetilde{\mathcal{K}}^h := \{x \in \widetilde{\mathcal{G}} : 0 \overset{\widetilde{E}^{\geq h}}{\leftrightarrow} x\}$;
- (non-)percolation function $\widetilde{\theta}(h) := \mathbb{P}(\widetilde{\mathcal{K}}^h \text{ is bounded})$;

$$\Big(\rightsquigarrow \text{ define critical parameter } \widetilde{h}_* := \inf\{h \in \mathbb{R} : \widetilde{\theta}(h) = 1\}\Big)$$

- truncated radius function
  $\widetilde{\psi}(h,n) := \mathbb{P}\big(0 \overset{\widetilde{E}^{\geq h}}{\leftrightarrow} \partial B(0,n), \widetilde{\mathcal{K}}^h \text{ is bounded}\big)$;
- truncated two-point function $\widetilde{\tau}_h^{\mathrm{tr}}(0,x) := \mathbb{P}(x \in \widetilde{\mathcal{K}}^h, \widetilde{\mathcal{K}}^h \text{ bounded})$;

Some previous work

- At level $h = 0$, the (truncated) two-point function $\tau^{\mathrm{tr}}_{h=0}(0, x)$ admits an exact formula, first observed in [Lup16]:

$$\widetilde{\tau}_0^{\mathrm{tr}}(0, x) = \frac{2}{\pi} \arcsin\left(\frac{g(0,x)}{\sqrt{g(0,0)g(x,x)}}\right)(\asymp d(0,x)^{2-d} \text{ in } \mathbb{Z}^d),$$

as $d(0, x) \to \infty$.

- For $\widetilde{\mathcal{G}} = \widetilde{\mathbb{Z}}^3$, [DW18] obtain bounds for truncated radius function $\psi(0, r)$:

$$cr^{-\frac{1}{2}} \leq \widetilde{\psi}(0, r) \leq C\left(\frac{r}{\log r}\right)^{-\frac{1}{2}}$$

Cluster capacity law

Crucial quantity in our investigations: For $K \subset G$, its *capacity* is

$$\operatorname{cap}(K) := \sum_{x \in \partial K} \lambda_x P_x(\widetilde{H}_K = \infty); \quad \textit{e.g.} \quad \operatorname{cap}(B(0,r)) \asymp r^{\nu}.$$

Theorem [D-Prévost-Rodriguez] 2022

For all reasonably nice $\widetilde{\mathcal{G}}$, all $h \in \mathbb{R}$, and under $\mathbb{P}(\,\cdot\,, \emptyset \neq \widetilde{\mathcal{K}}^h$bounded), the random variable $\operatorname{cap}(\widetilde{\mathcal{K}}^h)$ has density given by

$$\varrho_h(t) = \frac{1}{2\pi t\sqrt{g(0,0)(t - g(0,0)^{-1})}} \exp\left\{-\frac{h^2 t}{2}\right\} \mathbb{1}_{t \geq g(0,0)^{-1}}.$$

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What happens close to / at the critical point $\widetilde{h}_* = 0$?

At level h close to (but different from) $h_* (= 0)$, the

$$\text{correlation length} \quad \xi = \xi(h) = |h|^{-\nu_c}$$

is expected to describe the natural inherent length scale of the system. $\nu_c \in (0, \infty)$ is the *critical exponent for the correlation length*. More generally, close to the critical point $\widetilde{h}_* = 0$, physicists expect observables to be described via 'power functions':

$$1 - \widetilde{\theta}(h) \approx |h|^{\beta} \text{ some } \beta \in (0, \infty);$$

$$\rightsquigarrow \text{can define critical exponent } \beta := \lim_{h \uparrow 0} \log(1 - \widetilde{\theta}(h)) / \log(|h|).$$

Similarly, for the truncated radius function on conjectures that

$$\rho := \lim_{n \to \infty} \log(\widetilde{\psi}(0, n)) / \log(n) \quad \text{exists.}$$

Critical exponents

Using (among other things) that unbounded, closed, connected sets have infinite capacity, we get the following.

Corollary [D-Prévost-Rodriguez] 2022

$$\widetilde{\theta}(h) = 2\Phi(h \wedge 0) \quad \text{for all } h \in \mathbb{R},$$

where $\Phi(t) = \mathbb{P}(\varphi_0 \leq t)$. In particular,

$$\widetilde{h}_* = 0 \quad \text{and} \quad \widetilde{\theta}(0) = 1.$$

Furthermore, $\widetilde{\theta} : \mathbb{R} \to [0, 1]$ is continuous, and

$$\lim_{h \uparrow 0} \frac{1 - \widetilde{\theta}(h)}{|h|} = \sqrt{\frac{2}{\pi g(0,0)}}; \quad \rightsquigarrow \beta = 1.$$

(recall that $\beta := \lim_{h \uparrow 0} \log(1 - \widetilde{\theta}(h)) / \log(|h|)$, if it exists)

See Prévost [Pré21] for graphs with $\widetilde{h}_* \neq 0$;

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Standing assumptions

- α-Ahlfors regular volume growth

$$cr^{\alpha} \leq \lambda(B(x,r)) \leq Cr^{\alpha} \quad \forall x \in G, r \geq 1;$$

- regular Green function decay

$$\begin{aligned} c \leq & g(x,x) \leq C, \\ cd(x,y)^{-\nu} \leq & g(x,y) \leq Cd(x,y)^{-\nu} \quad \forall x \neq y \in G; \end{aligned}$$

- technical assumptions: uniform ellipticity $\lambda_{x,y}/\lambda_x \geq c$ and existence of a certain infinite geodesic;

Critical exponents

Set $\xi(h) := |h|^{-2/\nu}$, which will play the role of the correlation length.

Theorem [D-Prévost-Rodriguez] 2023

For $\nu < 1$, $h \in \mathbb{R}$ and $r \geq 1$:

$$c_3 \widetilde{\psi}(0,r) \exp\big\{ -c_4 (r/\xi(h))^{\nu} \big\} \leq \widetilde{\psi}(h,r) \leq \widetilde{\psi}(0,r) \exp\big\{ -c_5 (r/\xi(h))^{\nu} \big\}.$$

For $\nu \geq 1$, $h \in \mathbb{R}$ and $r \geq 1$:

$$\widetilde{\psi}(h,r) \leq \widetilde{\psi}(0,r) \cdot \begin{cases} \exp\big\{ -c_5 \frac{(r/\xi(h))}{\log(r \vee 2)} \big\}, & \text{if } \nu = 1, \\ \exp\big\{ -c_5 r h^2 \big\}, & \text{if } \nu > 1. \end{cases}$$

There exists $c_6 \in (0,1)$ such that for $\nu = 1$ and all $|h| \leq c$,

$$\widetilde{\psi}(h,r) \geq c_3 \psi(0,r) \cdot \exp\Big\{ -c_4 \frac{(r/\xi(h))}{\log((r/\xi(h)) \vee 2)} \Big\}, \text{ if } \tfrac{r}{\xi(h)} \notin (1, (\log \xi(h))^{c_6}).$$

Critical exponents

Can derive similar estimates for the truncated two-point function $\tau_h^{\mathrm{tr}}(0,x)$

$\rightsquigarrow$ yields the following corollary, consistent with predictions of Weinrib & Halperin [WH83, Wei84] ("disorder relevance" (e.g. for $\mathbb{Z}^\alpha$ and $\alpha < 6$)).

Corollary [D-Prévost-Rodriguez] 2023

For $\nu \leq 1$,

$$\gamma \overset{\text{def.}}{=} -\lim_{h\to 0} \frac{\log(\mathbb{E}[|\mathcal{K}^h| 1\{|\mathcal{K}^h| < \infty\}])}{\log |h|}$$

exists and

$$\gamma = \frac{2\alpha}{\nu} - 2 \quad \Big(= \nu_c(2-\eta)\Big).$$

For $\nu < 1$ one has the stronger result

$$\mathbb{E}[|\mathcal{K}^h| 1\{|\mathcal{K}^h| < \infty\}] \asymp |h|^{-\frac{2\alpha}{\nu}+2} \text{ as } h \to 0.$$

Critical exponents [DPR23]

Exponent	α_c	β	γ	δ	Δ	ρ	ν_c	η
Value	$2-\frac{2d}{\nu}$	1	$\frac{2d}{\nu}-2$	$\frac{2d}{\nu}-1$	$\frac{2d}{\nu}-1$	$\frac{2}{\nu}$	$\frac{2}{\nu}$	$\nu-d+2$
Bernoulli $\mathbb{Z}^3$	≈ -0.63	≈ 0.41	≈ 1.7	≈ 5.3	≈ 2.2	≈ 2.1	≈ 0.87	≈ -0.06

Can determine red exponents, use scaling relations to conjecture further critical exponents:

$$2-\alpha_c=\gamma+2\beta=\beta(\delta+1),\quad \Delta=\delta\beta \quad \text{(scaling relations);}$$
$$d\rho=\delta+1,\quad d\nu_c=2-\alpha_c \quad \text{(hyperscaling relations);}$$

Cheat sheet:

β $\leftrightsquigarrow$ percolation probability
ρ $\leftrightsquigarrow$ radius function
ν_c $\leftrightsquigarrow$ correlation length
α_c $\leftrightsquigarrow$ clusters per vertex
γ $\leftrightsquigarrow$ truncated cluster size
δ $\leftrightsquigarrow$ cluster volume
Δ $\leftrightsquigarrow$ cluster moments
η $\leftrightsquigarrow$ truncated two-point function.

N.b.:

- valid for $\nu \in (0,1]$ except for β and η, which hold for all $\nu > 0$;
- as conjectured, critical exponents do not depend on the microscopic structure of the underlying graph $\rightsquigarrow$ universality;
- For diffusive RW, for $d \uparrow 6$ (or $\nu \uparrow 4$, equivalently), exponent converge to respective mean-field values for Bernoulli percolation ($\beta=\gamma=1$, $\Delta=\delta=2$, $\eta=0$);

Strategy for upper & lower bounds on radius function

Want to show: For $\nu \leq 1$, $h \in \mathbb{R}$ and $r \geq 1$:

$$c_3 \psi(0,r) \exp\Big\{ - c_4 (r/\xi(h))^{\nu} \Big\} \leq \psi(h,r) \leq \psi(0,r) \exp\Big\{ - c_5 (r/\xi(h))^{\nu} \Big\}.$$

$\nu \leq 1 \implies$ cluster radius can be understood in terms of cluster capacity.

To show the upper bound, use differential inequalities to infer upper bounds of the form

$$\psi(h,r) \leq \psi(0,r) e^{-ch^2 f_\nu(r)},$$

with $f_\nu(r) = r^\nu$ for $\nu < 1$ (logarithmic corrections for $\nu = 1$) and recalling $\xi(h) = |h|^{-2/\nu}$.

Tool to obtain differential inequalities: Cameron Martin theorem allows to compare capacities of $\mathcal{K}^h$ at different levels h; then use strong Markov property to derive the general formula comparing well-behaved functionals of GFF at different shifts.

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Strategy for lower bounds on radius function ($\nu \leq 1$)

Main tools:

- Change of measure / entropy formula: allows for comparing original GFF with a GFF shifted on a compact set;
- isomorphism theorems: coupling two GFFs $(\widetilde{\varphi}_x)$, $x \in \widetilde{\mathcal{G}}$, $(\widetilde{\psi}_x)_{x \in \widetilde{\mathcal{G}}}$, and interlacement local times $(\widetilde{\ell}_{x,u})_{x \in \widetilde{\mathcal{G}}}$, at level $u > 0$,

$$\widetilde{\varphi}_x + \sqrt{2u} = \widetilde{\psi}_x 1_{x \notin \widetilde{\mathcal{C}}_u^\infty} + \sqrt{\widetilde{\psi}_x^2 + 2\widetilde{\ell}_{x,u}} 1_{x \in \widetilde{\mathcal{C}}_u^\infty},$$

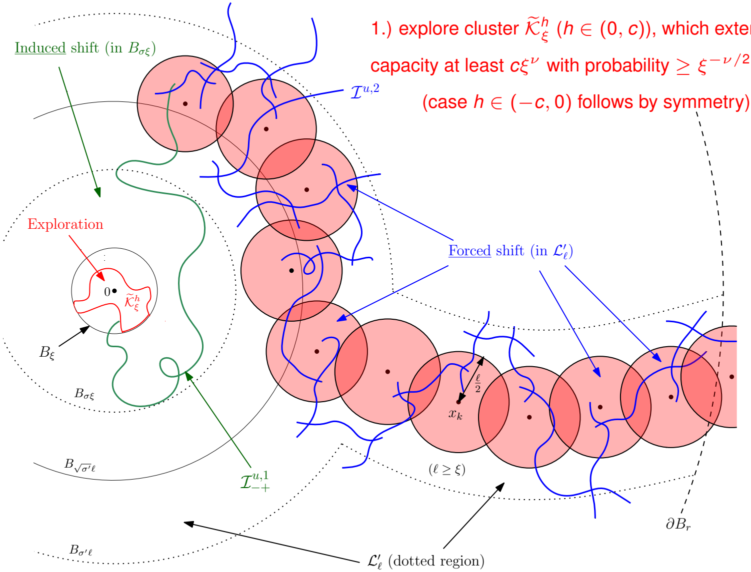
  with $\widetilde{\mathcal{C}}_u^\infty := \{x \in \widetilde{\mathcal{G}} : \widetilde{\ell}_{x,u} > 0\}$, and $(\widetilde{\psi}_x)_{x \in \widetilde{\mathcal{G}}}$ is independent from $(\widetilde{\ell}_{x,u})_{x \in \widetilde{\mathcal{G}}}$;

  $\rightsquigarrow$ connections in $E^{\geq h} = \{x \in \widetilde{\mathcal{G}} : \widetilde{\varphi}_x \geq h\}$, $h < 0$, can be made using random interlacements $\mathcal{I}^u = \{x \in \widetilde{\mathcal{G}} : \widetilde{\ell}_{x,u} > 0\}$;
- critical local uniqueness for Random Interlacements: with asymptotically non-vanishing probability and for $u \approx R^{-\nu}$, there is a unique giant connected component of $\widetilde{\mathcal{I}}^u$ in ball $B(0, R)$;

Strategy for lower bound on the radius function ($\nu \leq 1$)

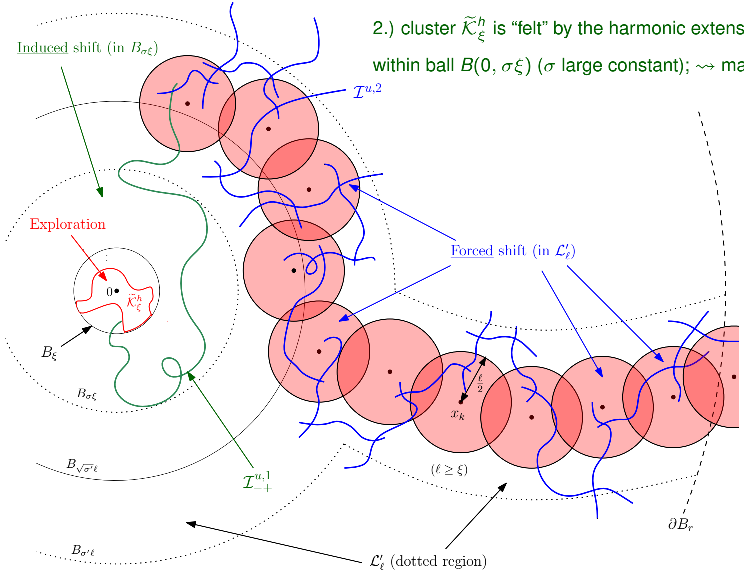
1.) explore cluster $\widetilde{\mathcal{K}}^h_\xi$ ($h \in (0, c)$), which extends to capacity at least $c\xi^\nu$ with probability $\geq \xi^{-\nu/2}$;

(case $h \in (-c, 0)$ follows by symmetry)



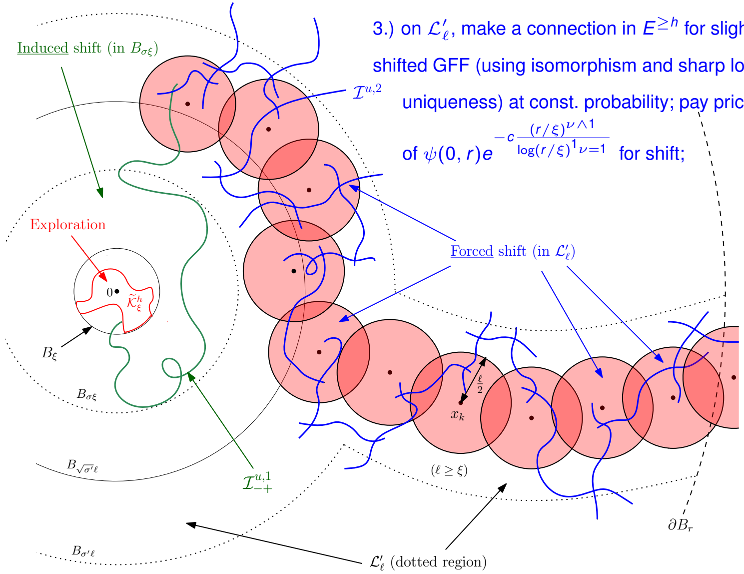
Strategy for lower bound on the radius function ($\nu \leq 1$)

2.) cluster $\widetilde{\mathcal{K}}^h_\xi$ is “felt” by the harmonic extension within ball $B(0, \sigma\xi)$ (σ large constant); $\rightsquigarrow$ make



Strategy for lower bound on the radius function ($\nu \leq 1$)

3.) on $\mathcal{L}'_\ell$, make a connection in $E^{\geq h}$ for slightly shifted GFF (using isomorphism and sharp local uniqueness) at const. probability; pay price of $\psi(0, r)e^{-c\frac{(r/\xi)^{\nu\wedge 1}}{\log(r/\xi)^{1_{\nu=1}}}}$ for shift;



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