Random walks on a Lévy-type random media

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Phase transitions in spatial particle systems
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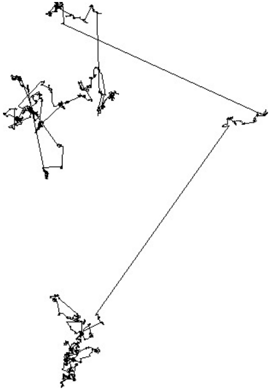
Super-diffusive motions



Super-diffusive trajectory

Main features

- long ballistic “flights“
- short disorder motion

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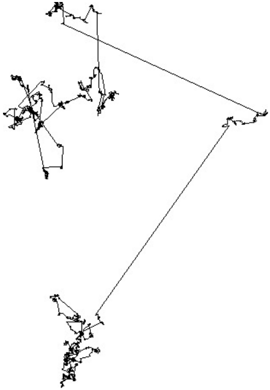

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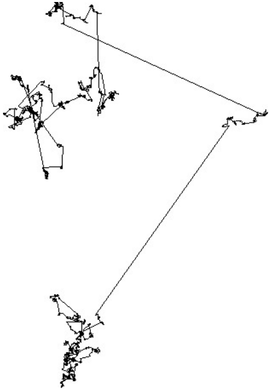

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- light particle in an optical lattice;
- molecular diffusion in porous media;
- predator hunting for food.

Models for super-diffusions

- $(\zeta_k)_{k\in\mathbb{N}}$ i.i.d. real r.v.'s in the domain of attraction of an α-stable r.v., with $\alpha \in (0,2]$:

$$\mathbb{P}(\zeta_k > x) \sim c x^{-\alpha}, \quad \text{for } x \to \infty$$

Note: $\alpha \in (0,2) \Longrightarrow \mathbb{E}(\zeta_k^2) = \infty$ and $\alpha \in (0,1) \Longrightarrow \mathbb{E}(|\zeta_k|) = \infty$

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- $\alpha \in (1,2]$: if $\mu = \mathbb{E}(\zeta_k)$ and $\bar{Z}_1$ centered α-stable r.v.

$$\frac{S_n - \mu n}{n^{1/\alpha}} \xrightarrow[n\to\infty]{d} \bar{Z}_1$$

Lévy flights and Lévy walks

Schlesinger, Klafter['85], [Zumofen, Klafter '93], Barkai, Dubkov ['17]

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Random walk on $\mathbb{R}^d$ with jumps length given by a sequence of i.i.d. α-stable- r.v., with $\alpha \in (0, 2)$. (but infinite second moment)

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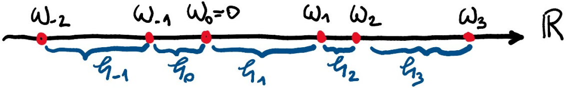
The super-diffusive behavior is intrinsic to the walker motion, and independent of the media. **Good behavior** but **naive models**.

Lévy random media

Define the environment $\omega = (\omega_k)_{k\in\mathbb{Z}}$ as the renewal P.P. on $\mathbb{R}$

$$\omega_0 = 0\,, \quad \omega_k - \omega_{k-1} = \zeta_k$$ Lévy random medium

with $(\zeta_k)_{k\in\mathbb{Z}}$ i.i.d. positive r.v.'s in the domain of attraction of an α-stable law, $\alpha \in (0,2]$



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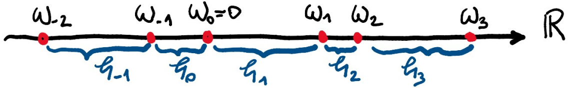

Remark: Since $\omega_k = \sum_{j=1}^{k} \zeta_j$, $\forall k > 0$ (and similarly for $k < 0$) setting

$\tilde{\omega}_x^{(n)} := \frac{\omega_{[nx]} - \mu[nx]}{n^{1/\alpha}}$, $x \in \mathbb{R}$, and $Z = \alpha$-stable Lévy process on $\mathbb{R}$,

$$(\tilde{\omega}_x^{(n)})_{x\in\mathbb{R}} \xrightarrow{w} Z = (Z_x)_{x\in\mathbb{R}} \quad \text{in } (D(\mathbb{R},\mathbb{R}), J_1)$$

Random walks on Lévy random media

Let $S = (S_n)_{n\in\mathbb{N}}$ be an underlying RW on $\mathbb{Z}$ with i.i.d. centered increments $(\xi_j)_{j\in\mathbb{N}}$ s.t. $\mathbb{E}(\xi_j^2) < \infty$.

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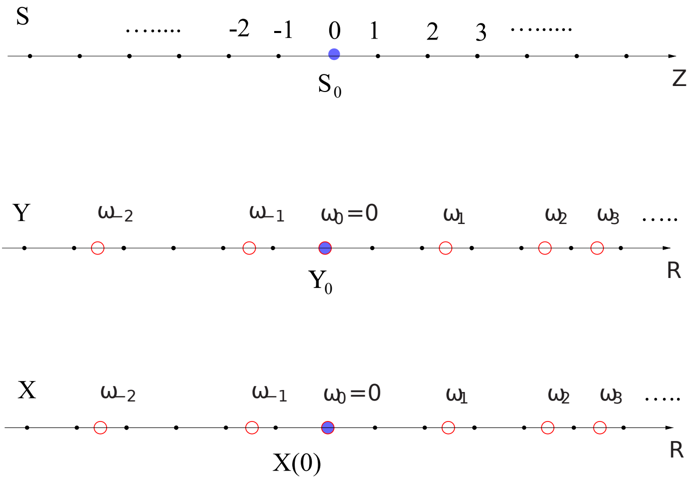
  - for $t \in [T_n, T_{n+1})$, set $X_t := Y_n + \mathrm{sgn}(\xi_{n+1})(t - T_n)$,

Lévy-Lorentz gas (Barkai, Fleurov,Klafter['00]) corresponds to S simple and symmetric.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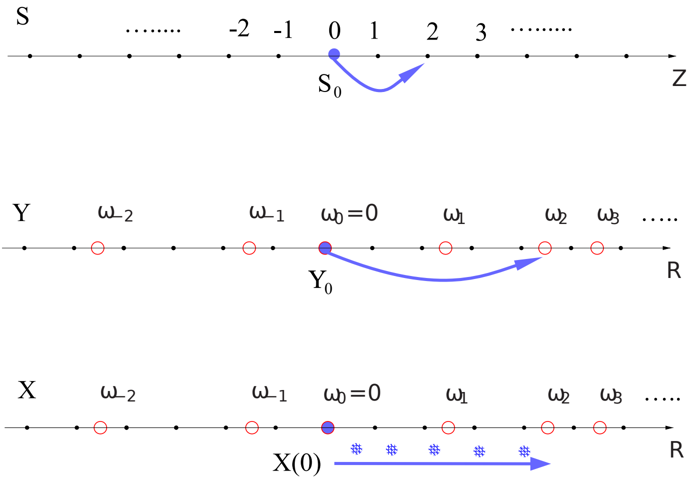
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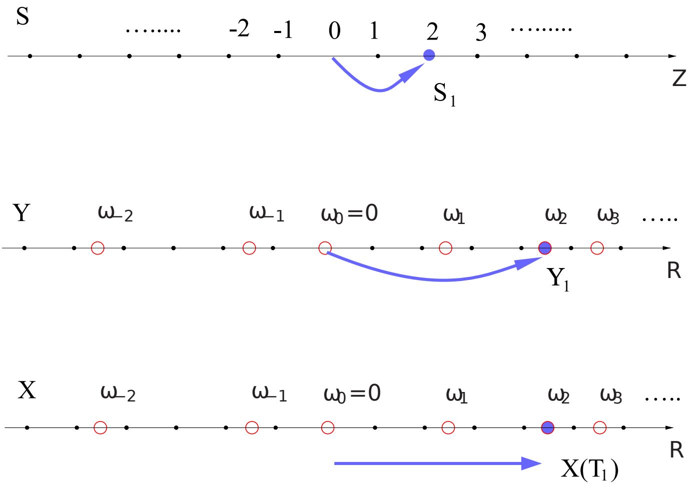
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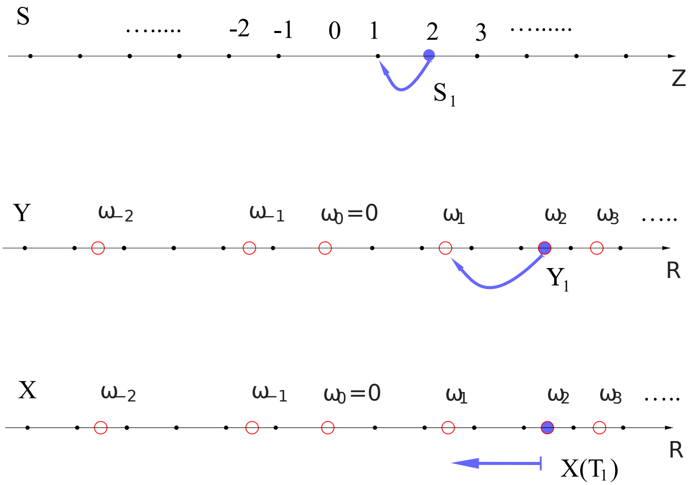
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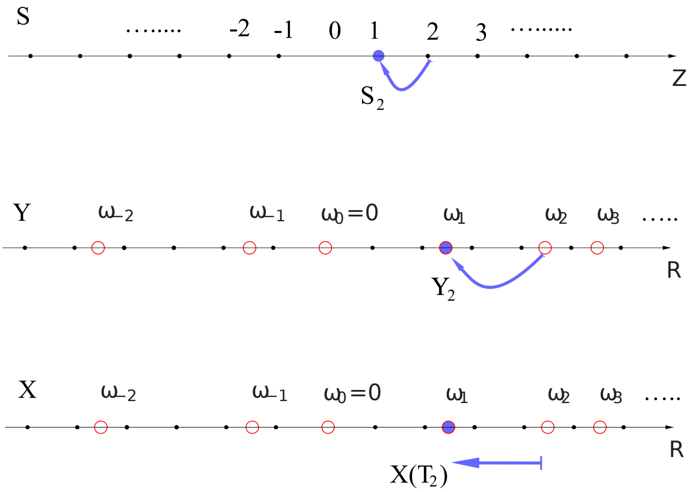
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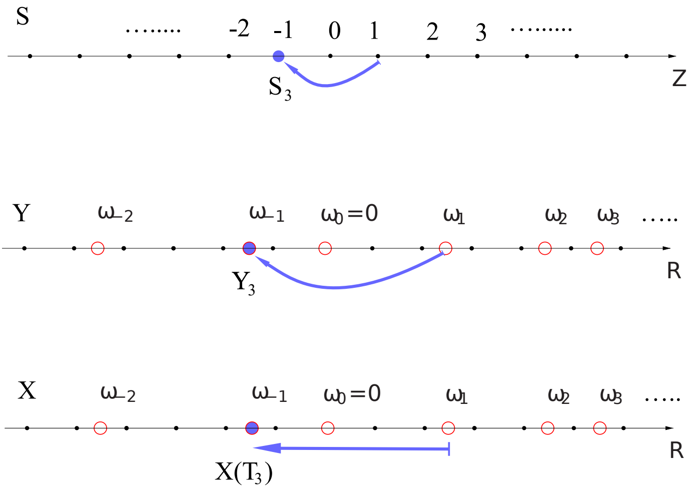
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Goal: Analyze the super-diffusive behavior of $(Y_n)_{n\in\mathbb{N}}$ and $(X_t)_{t\geq 0}$:

- Case $\alpha \in (1, 2]$: Integrable media
- Case $\alpha \in (0, 1]$: Non-integrable media

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We will consider:

- the quenched law of X and Y, denoted P_ω, for any fixed medium ω.
- the annealed law of X and Y, denoted $\mathbb{P}$, obtained averaging P_ω over the environments.

Integrable media: $\alpha \in (1, 2]$

Scaling limits

- **Annealed and quenched CLT:** (B., Cristadoro, Lenci, Ligabó ['16])
  Set $\mu = \mathbb{E}(\zeta)$, $m = \mathbb{E}(|\xi|)$, and $\sigma^2 = Var(\xi)$ and. For P-a.e. ω
  - $\frac{Y_n}{\sqrt{n}} \xrightarrow{d} N(0, \mu^2\sigma^2) \quad$ w.r.t. P_ω
  - $\lim_{t\to\infty} \frac{X_t}{\sqrt{t}} \overset{d}{=} N(0, \frac{\mu\sigma^2}{m}) \quad$ w.r.t. P_ω

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- **f-CLT for Lévy-Lorentz gas**: (Magdziarz, Szczotka ['18])
  Set $Y_t^{(n)} := \frac{Y_{[nt]}}{\sqrt{n}}$, for $t \geq 0$. Then, w.r.t to $\mathbb{P}$,

$$(Y_t^{(n)})_{t\geq 0} \xrightarrow[n\to\infty]{w} \mu B \quad \text{in } (D(\mathbb{R}^+, \mathbb{R}), J_1)$$

where B is the standard Brownian motion on $\mathbb{R}$.

Integrable media: $\alpha \in (1, 2]$

Moments

- **Quenched moments of** Y: (B., Cristadoro, Lenci, Ligabó ['16])
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$$\mathbb{E}(X_t^2) \sim \begin{cases} t^{\frac{2+2\alpha-\alpha^2}{1+\alpha}} & \text{if } \alpha \in (0,1) \quad \text{superdiffusive behavior} \\ t^{\frac{5}{2}-\alpha} & \text{if } \alpha \in [1,\frac{3}{2}] \quad \text{superdiffusive behavior} \\ t & \text{if } \alpha \in (\frac{3}{2},2) \quad \text{diffusive behavior} \end{cases}$$

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- **Moments of the Lévy-Lorentz gas**: (Zamparo ['22])
  As an effect of averaging over the environment,

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Non-integrable media: $\alpha \in (0,1)$ - Process Y

Theorem 1 (B., Lenci, Pène '20).

For $n \in \mathbb{N}$, let $\tilde{Y}^{(n)} = (\tilde{Y}^{(n)}(t))_{t \geq 0}$ such that

$$\tilde{Y}^{(n)}(t) := \frac{Y_{[nt]}}{n^{1/2\alpha}}, \quad \text{for all } t \geq 0\,.$$

Under $\mathbb{P}$ and taking $n \to \infty$, the finite-dim. distributions of $\tilde{Y}^{(n)}$ converge to those of $Z \circ B$.

Remark:

- The process Y displays superdiffusive behavior with scaling exponent $1/2\alpha$.
- The result can not be extended to a functional limit theorem w.r.t to the Skorokhod topology as $Z \circ B$ has discontinuities without one-sided limits.

Proof (ideas):
From definitions, it turns out that

- $\tilde{\omega}^{(n)} = \left(\frac{\omega_{[nx]}}{n^{\frac{1}{\alpha}}} \right)_{x \in \mathbb{R}} \xrightarrow{w} Z$ $\alpha -$ stable process
- $\tilde{S}^{(n)} = \left(\frac{S_{[nt]}}{n^{\frac{1}{2}}} \right)_{t \geq 0} \xrightarrow{w} B$ invariance principle

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For a suitable scaled process $(\tilde{T}^{(n)}(t))_{t \geq 0}$ [to be given!], we get

$$\tilde{X}^{(n)}(t) := \frac{X_{[nt]}}{n^{1/(\alpha+1)}} \simeq \tilde{\omega}^{(\sqrt{k_n})} \circ \tilde{S}^{(k_n)} \circ (\tilde{T}^{(n)}(t))^{-1}$$

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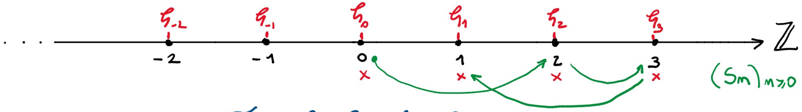
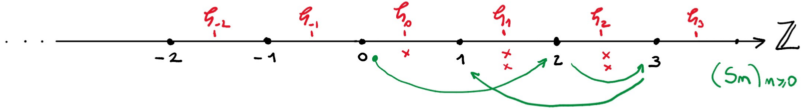
By [Kesten, Spitzer '79], the RWRS on (vertices of) $\mathbb{Z}$ is

$$\mathcal{T}_n := \sum_{j=0}^{n} \zeta_{S_j} = \sum_{k\in\mathbb{Z}} N_n(k)\zeta_k \ , n \in \mathbb{N}$$

where $N_n(k) = \sharp\{j \in \{0,\dots,n\} \,:\, S_j = k\}$ are **local times** of S.



RWRS : $T_3 = \tau_0 + \tau_2 + \tau_3 + \tau_1$



RWRS on bonds : $T_3 = (\tau_0 + \tau_1) + \tau_2 + (\tau_2 + \tau_3)$

Convergence of RWRS: Kesten-Spitzer process

Theorem 2 (Kesten, Spitzer '79).

Let $\alpha \in (0,1)$. *Under* $\mathbb{P}$, *and taking* $n \to \infty$, *it holds*

$$\tilde{\mathcal{T}}^{(n)} := \left(\frac{\mathcal{T}_{[ns]}}{n^{\frac{1+\alpha}{2\alpha}}} \right)_{s \geq 0} \xrightarrow{w} \Delta \quad \text{in } (D(\mathbb{R}^+, \mathbb{R}), J_1),$$

where $\Delta(t) = \int_{-\infty}^{+\infty} L_t(x) dZ(x)$ *Kesten-Spitzer process*,

$L_t = (L_t(x))_{x \in \mathbb{R}}$ *is the* ***local time*** *of the Brownian motion B and Z is an* α*-stable process on* $\mathbb{R}$.

Non-integrable media: $\alpha \in (0,1)$ - Process X

Assumption on the underlying RW: $\mathbb{E}(|\xi_1|^{2/\alpha+\varepsilon}) < \infty$.

Proposition 1 (B., Lenci, Pène '20).

Let $\alpha \in (0,1)$. *Under* $\mathbb{P}$, *and taking* $n \to \infty$, *it holds*

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Theorem 3 (B., Lenci, Pène '20).

Under $\mathbb{P}$*, and taking* $n \to \infty$*, the finite-dimensional distributions of* $\tilde{X}^{(n)} := \left(\frac{X_{[nt]}}{n^{1/(1+\alpha)}} \right)_{t \geq 0}$ *converge to those of* $Z \circ B \circ \Delta^{-1}$.

Remark: The process X displays superdiffusive behavior with scaling exponent $1/(\alpha+1)$.

Random Walks on Lévy random media (II)

Starting idea: The results of [Kesten, Spitzer '79] on RWRS apply to the following more general setting:

- Scenery on $\mathbb{Z}$: $(\zeta_k)_{k\in\mathbb{Z}}$ i.i.d. $\sim \alpha$-stable r.v.'s corresponding to $\omega = (\omega_k)_{k\in\mathbb{Z}}$ with $\omega_k - \omega_{k-1} = \zeta_k$
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Goal: Scaling limit of Y - RW on the Lévy medium

Remark: The study of X requires a control on collision times T_n, that is missing under these weaker assumptions on the moments of the underlying RW. (open problem)

RWRM: Results for $\alpha \in (0,1)$

Recall that $Y_n = \omega_{S_n}$, $n \in \mathbb{N}$, with ω the Lévy medium.

Theorem 4 (B., Bet, Lenci, Magnanini, Stivanello '21).

*Let $\alpha \in (0,1)$ (**medium with infinite mean**)*

- *If $\beta \in (0,1)$ or $\beta \in (1,2]$ with $\mathbb{E}(\xi_k) = 0$, then, under $\mathbb{P}$, the finite-dimensional distributions of $\left(\frac{Y_{[ns]}}{n^{1/\alpha\beta}}\right)_{s\geq 0}$ converge to those of $Z^\alpha \circ Z^\beta$.*
- *If $\beta \in (1,2]$ with $\mathbb{E}(\xi_k) = \mu \neq 0$, then, under $\mathbb{P}$,*

$$\left(\frac{Y_{[ns]}}{n^{1/\alpha}}\right)_{s\geq 0} \xrightarrow[n\to\infty]{w} sgn(\mu)|\mu|^{1/\alpha} Z^\alpha \qquad in\ (D(\mathbb{R}^+,\mathbb{R}), J_2)$$

RWRM: Results for $\alpha \in (1, 2]$

Theorem 5 (B., Bet, Lenci, Magnanini, Stivanello '21).

*Let $\alpha \in (1, 2]$ with $\nu := \mathbb{E}(\zeta_k)$ (**medium with finite mean**) :*

- *If $\beta \in (0, 1)$ or $\beta \in (1, 2]$ with $\mathbb{E}(\xi_k) = 0$, then, under $\mathbb{P}$,*

$$\left(\frac{Y_{[ns]}}{n^{1/\beta}} \right)_{s \geq 0} \xrightarrow[n\to\infty]{w} \nu Z^{\beta} \qquad \text{in } (D(\mathbb{R}^+, \mathbb{R}), J_1)$$

- *If $\beta \in (1, 2]$ with $\mathbb{E}(\xi_k) = \mu \neq 0$, then, under $\mathbb{P}$,*

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$\longrightarrow$ *Fluctuations around the mean: scaling and functional limit theorem for* $\bar{Y}^{(n)}(t) := Y_{[nt]} - \nu\mu[nt]$.

First passage time and leapover

In the same general setting, consider:

- **Ladder times of Y**: $(\tau_n)_{n\in\mathbb{N}_0}$
  $\tau_0 = 0$, $\tau_n \equiv \tau_n(Y) := \min\{k > \tau_{n-1} \; : \; Y_k > Y_{\tau_{n-1}}\}$, $n \in \mathbb{N}$
  with τ_1 first passage time on $\mathbb{R}^+$.
- **Ladder heights of Y**: $(Y_{\tau_n})_{n\in\mathbb{N}_0}$
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Remark:

- By construction, the ladder times of Y coincide with those of the underlying RW, S: $\tau_n(Y) = \tau_n(S)$, $n \in \mathbb{N}$.
- The **dependence among the increments** of Y makes the analysis of its ladder heights non-trivial.

Classical results: Ladder times and heights of S

Let S be a RW with i.i.d. symmetric increments on $\mathbb{R}$.
Then $(\tau_n)_{n\in\mathbb{N}}$ and $(S_{\tau_n})_{n\in\mathbb{N}}$ have i.i.d. increments with law

- $$\mathbb{P}(\tau_1 > x) \sim c x^{-1/2}, \quad \text{as } x \to +\infty$$

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- If the increments of S are in the domain of a β-stable law,

$$\mathbb{P}(S_{\tau_1} > x) \sim cx^{-\beta/2}\,, \quad \text{as } x \to +\infty$$

[Sinai '57]; [Rogozin '64, '71] - without centering; [Greenwood, Omey, Teugels '82],
[Greenwood, Doney '93] - for the joint law of (τ_n, S_{τ_n}).

Results: Ladder heights of Y

Recall: $\begin{cases} \omega & \text{i.i.d. } \alpha-\text{stable increments} \\ S & \text{i.i.d. } \beta-\text{stable increments} \end{cases}$, $Y_n = \omega_{S_n}$

Theorem 6 (B., Cristadoro, Pozzoli '22).

If S has symmetric increments, and for all $n \in \mathbb{N}$,

- *If $\alpha \in (0,1)$: $\mathbb{P}(Y_{\tau_n} > x) \sim c_1 n x^{-\alpha\beta/2}$, as $x \to \infty$*
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with c_1, c_2 explicit constants.

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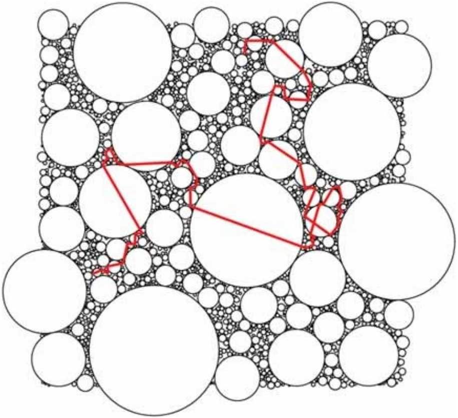
Proof (ideas):

- *Express Y as a suitable RWRS on bonds;*
- *Derive a simplified formula characteristic function of Y_{τ_n};*
- *Apply a generalized Spitzer-Baxter identity.*

Open Problem: Process in 2D



Lévy glass: image taken from [Barthelemy, Bertolotti1, Wiersma; Nature '08]

Conclusions

- Lévy Lorentz gas as a RW in random media, with collision times $=$ RWRS $\longrightarrow$ convergence to Kesten Spitzer process

If $\alpha \in (1,2)$: (integrable media) quenched CLT for discrete and continuous time processes [BCLL'16]. Quenched diffusive behavior.

If $\alpha \in (0,1)$: (non-integrable media) annealed functional LT for discrete and continuous time processes [BLP'20]. Annealed superdiffusive behavior.

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  - Functional limit theorems [BBLMS'21].
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- Construction of Lévy media on $\mathbb{R}^2$: Open problems:
  - Transience or recurrence when $\alpha \in (1,2)$.
  - Scaling and limit theorems when $\alpha \in (1,2)$.

Thank you for your attention!