Diffusion in Hamiltonian Systems

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based on work with O. Ajanki, A. Kupiainen and earlier work with J. Fröhlich.

Main goal

Prove that a Hamiltonian (or other reasonable deterministic) system exhibits diffusion for long times.
$\rightarrow$ Emergence of irreversibility from deterministic dynamics
common wisdom physically, but hard to make rigorous

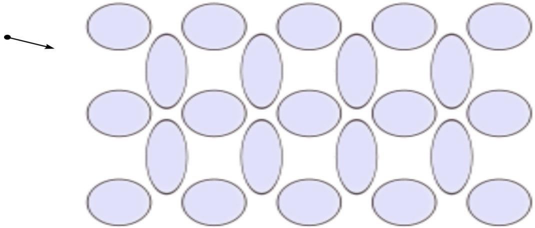

Figure: Billiards with finite horizon.Tracer particle bounces elastically off periodic objects. Diffusion proven by *Sinai and Bunimovich, 81,* . A similar setup (Coulombic potentials instead of scatterers) was done by *Knauf*

Plan

- Rayleigh gas: Diffusion and Markov scaling limit: Linear Boltzmann equation
- General strategy for proving diffusion assuming we can prove Markov scaling limit
- Main idea of the strategy: Random Walk in Random Environment (*a real honest theorem*)
- Mention of some result along these lines (*also honest but unstated*)
- Models based on waves: Markov scaling limits

Rayleigh gas: Tagged particle in ideal gas

- Ideal gas: N point particles with mass 1 in volume $\Lambda \subset \mathbb{R}^3$. Coordinates $q = (x, v) = (x_i, v_i)_{i=1,\dots,N} \in \Gamma_{\mathrm{E}}$.
- 'Tagged' Particle (not point-like, having radius 1) and mass 1, Coordinates $Q = (X, V)$.
- Billiard Dynamics: free flow

$$(\dot{x}(t), \dot{X}(t)) = (v(t), V(t)) \qquad (\dot{v}(t), \dot{V}(t)) = 0$$

up to the first collision time, when $\exists i : |x_i - X| = 1$. Then $(v_i, V) \to (v_i', V')$ by the rule

$$(v_i' - V')_{\parallel} := -(v_i - V)_{\parallel}, \qquad (v_i' - V')_{\perp} := (v_i - V)_{\perp}$$

where $a = a_{\parallel} + a_{\perp}$ such that $a_{\parallel} \parallel (q_i - Q)$ and $a_{\perp} \perp (q_i - Q)$.

$\to$ defines a dynamical system: $(q, Q)(0) \to (q, Q)(t)$ for a.a. initial conditions.

Rayleigh gas: particle in ideal gas

- Initial measure $\rho_0 = \rho_{\mathrm{S},0}(\mathrm{d}Q) \times \rho_{\mathrm{E}}(q)\mathrm{d}q$, with

$$\rho_{\mathrm{E}}(q) = \rho_{\mathrm{E},\beta,N,\Lambda}(q) = \prod_i \frac{1}{|\Lambda|} \mathrm{e}^{-\frac{\beta}{2} v_i^2}$$

  and $\rho_{\mathrm{S},0}$ localized around $(0,0)$. Hence, gas is in 'thermal state' (homogeneous Maxwellian).
- Dynamics defines flow on measures $\rho_0 \to \rho_t$, in particular the marginal $\rho_{\mathrm{S},0} \to \rho_{\mathrm{S},t}$.
- As Λ grows large, influence of the boundaries only after long time. $\to$ define distribution of $Q(t) = (X(t), V(t))$ in the limit

$$\Lambda \to \mathbb{R}^3, N \to \infty, \qquad \text{with } \epsilon = N/|\Lambda| \text{ fixed}$$

  (or, start in infinite volume, then law of original positions is Poisson Point process with intensity ϵ)

Rayleigh gas: Diffusion?

Does the particle diffuse?

$$\left\langle \frac{|X(t)|^2}{6t} \right\rangle \underset{t \nearrow \infty}{\longrightarrow} D, \qquad D > 0$$

.... CLT? invariance principle?

If you assume each gas particle collides only once (not entirely accurate, also negative recollisions), then $Q(t)$ is a Markov jump process (Linear Boltzmann Equation, see later). Diffusion follows trivially.

However, what about recollisions?
$\Rightarrow$ Markov property breaks down

Rayleigh gas: Scaling limit

- Make recollisions infinitely unlikely $\Rightarrow$ easier problem. This is the idea of scaling limits. For example, let density of the gas particles be small and observe the system for long times

  $$\text{density} \sim \epsilon, \qquad \text{time} \sim 1/\epsilon, \qquad \epsilon \searrow 0.$$

  In the limit, $\epsilon \searrow 0$, the probability of one collision, respectively a recollision is

  $$(1/\epsilon) \times \epsilon, \qquad (1/\epsilon) \times \epsilon^2$$

  One expects that $(\epsilon X^\epsilon(\tau/\epsilon), V^\epsilon(\tau/\epsilon))$ converges to a Markov process (Linear Boltzmann equation) in τ as $\epsilon \searrow 0$.
- This was done (rather, something similar) by Durr-Goldstein-Lebowitz in 1981. However, without scaling limit (i.e. ϵ fixed), no results available!

Linear Boltzmann equation

Recall distribution $\rho_{S,t}(X,V)$ and collision map for the tracer particle $(V,v) \to V'_n(V,v)$. Assume 1-particle density of the gas always given by the equilibrium $\mu(v) \sim \mathrm{e}^{-\beta v^2/2}$. Then

$$\partial_t \rho_{S,t}(X,V) = V \cdot \nabla_X \rho_{S,t}(X,V)$$
$$+ \int \mathrm{d}n \int \mathrm{d}v_0 \mathrm{d}V_0 \delta(V'_n(V_0,v_0) - V)|(V_0 - v_0)_{\|}|\mu(v_0)\rho_{S,t}(X,V_0)$$
$$- \int \mathrm{d}n \int \mathrm{d}v \mathrm{d}V' \delta(V'_n(V,v) - V')|(V - v)_{\|}|\mu(v)\rho_{S,t}(X,V)$$

Originates from nonlinear B.E. by replacing once $\rho_{S,t}$ (f_t in Pulvirenti's talk) by $\mu(v)$. $\Rightarrow$ condensed form

$$\partial_t \rho_{S,t}(X,V) = \int \mathrm{d}V' (r(V',V)\rho_{S,t}(X,V') - r(V,V')\rho_{S,t}(X,V)) + V \cdot \nabla_X \rho_{S,t}(X,V)$$

with $r(V,V')$ rate of jump $V \to V'$.

Nice, but Scaling limits are not my aim

The dynamical system gets adjusted as time grows.

$\Rightarrow$ Does not give information on the long-time limit of the fixed ϵ dynamical system. For example, *D* is different

Example

2D Anderson model is well-described by LBE for short times, but localized for large times: With probability $\exp^{-\lambda^{-2}}$, the particle is sent back to its starting place. **1D FPU-β chain**: Phonon boltzmann equation predicts wrong power law. In both examples, the scaling limit is lying!

Results (NOT exhaustive) on scaling limits

- *Yau, Erdös, '99, Yau, Erdös, Salmhofer, '05, Lukkarinen, Spohn, '08,* quantum or wave models
- *Toth, Holley, Dürr-Goldstein-Lebowitz, '81,* Rayleigh gas
- *Komorowski, Ryzhik, '04,* particle in random force field
- *Dolgopyat, Liverani, '10,* coupled Anosov systems.

Long-range nonmarkovian corrections

Correlation decay

Probability of a second collision with given gas particle decays polynomially in time: Assume that the test particle diffuses, $X^2 \sim t$. Then, a receding gas particle that wants to recollide after time t, should have a velocity smaller than $\sqrt{t}/t = 1/\sqrt{t}$, but

$$\int \mathrm{d}v \, \chi[|v| \leq t^{-1/2}] \mathrm{e}^{-\beta v^2/2} \sim t^{-d/2}$$

(this is not a correct estimate of the correlation decay, though)

Slow decorrelation is a generic feature of momentum converving Hamiltonian system, for interacting hard spheres like $t^{-d/2}$ (*Adler, Wainwright, '70*).

$\Rightarrow$ lies at the heart of anomalous diffusion in 1D and 2D systems. (Velocity-velocity autocorrelation not integrable)

Strategy

- We know that on time scales $t \approx \epsilon^{-1}$, the particle looks like a Markov jump process ($\sim$ random walk).
- The corrections to this behaviour are manifestly non-Markovian and long-range in time.
- This looks like the problem of proving an annealed central limit theorem for a random walk in a time-dependent random environment, with long-range memory.
- More generally, this looks like doing perturbation theory around a stochastic system, rather than around the unperturbed Hamiltonian system. Improvement, because perturbation preserves character

RWRE I (random walk in random environment)

Let $U_{\tau\in\mathbb{N}}$ be *random* transition kernels on $\mathbb{Z}^d$

- Transition kernels

$$U_\tau(x,x')\geq 0, \qquad \sum_{x'} U_\tau(x,x')=1$$

- Law of U_τ invariant under space and time-translations.
- $\mathbb{E}(U_\tau)=T$ is transition kernel of simple random walk
- Hence, $U_\tau=T+B_\tau$ with B_τ 'dynamical disorder'.
- "Disorder correlations", for $\tau_1<\tau_2<\ldots<\tau_m$, and $\gamma>0$,

$$G(\tau_{1,\ldots,m}):=\sup_{x_1}\sum_{x'_1}\ldots\sup_{x_m}\sum_{x'_m}\mathrm{e}^{\gamma\sum_j|x'_j-x_j|} \left|\langle B_{\tau_1}(x_1,x'_1);\ldots;B_{\tau_m}(x_m,x'_m)\rangle^c\right|$$

RWRE II

Theorem (Ajanki, D.R., Kupiainen, *in preparation*)

Assume (for some γ, α and all m)

$$\sum_{1=\tau_1<\ldots<\tau_m} \prod_{j=2}^{m} (|\tau_j - \tau_{j-1}|^{\alpha}) G(\tau_{1,\ldots,m}) < \delta^m,$$

Then, if $\delta < \delta_0$ and $\alpha > 0$, there is annealed CLT

$$\sum_x \mathrm{e}^{-\mathrm{i}k\frac{x}{\sqrt{N}}} \left[\mathbb{E}(U_N \ldots U_1)\right](0, x) \underset{N \nearrow \infty}{\longrightarrow} \mathrm{e}^{-D^2 k}$$

- Similar framework for RWRE was pioneered in ′91 by Bricmont-Kupiainen. Here: much easier because integrable correlations. Quenched CLT requires also some spatial decay.
- Proof: renormalization group + cluster expansion.

Renormalization group for RWRE

Recall kernels (random except T)

$$U_\tau(x,x'), \qquad T(x,x'), \qquad B_\tau(x,x')$$

Renormalization step is

$$\mathcal{R}U_\tau(x,x') := L^d \left(U_{L^2\tau} U_{L^2\tau-1} \dots U_{L^2(\tau-1)+1} \right)(Lx, Lx')$$

for some $L >> 1$, but $\delta L << 1$.
We denote

$$U'_\tau = \mathcal{R}U_\tau, \qquad T' = \mathbb{E}(\mathcal{R}T_\tau), \qquad B'_\tau = U'_\tau - T'$$

Running coupling constant δ_n

$$\sum_{1=\tau_1<\dots<\tau_m} (\prod_j (\tau_{j+1}-\tau_j))^\alpha \mathbb{E}(B_{\tau_m} \otimes \dots \otimes B_{\tau_2} \otimes B_{\tau_1}) \sim \delta_n^m$$

Flow $T \to T'$

$$T' = L^d \mathbb{E}(T^{L^2}) + L^d \sum_{m=1}^{L^2} \mathbb{E}(T^m B_{L^2-m} T^{L^2-m-1})$$

$$+ \quad L^d \sum_{m_1+m_2=L^2-2} \mathbb{E}(T^{m_1} B_{L^2-m_1} T^{m_2} B_{L^2-m_1-m_2-1} T^{L^2-m_1-m_2-2})$$

By $\mathbb{E}(B_{\tau_1} \otimes B_{\tau_2}) \sim \delta_n^2$, second-order term is $L^{d+4}\delta_n^2$. $\Rightarrow$ OK since δ_n contracts:

$$\delta_n \sim L^{-\kappa(n-1)}\delta_1, \qquad \kappa > 0.$$

Morally, $T(x,x') \sim \mathrm{e}^{-\frac{(x'-x)^2}{2D}}$. Dropping irrelevant terms gives

$$T'(x,x') \sim L^d (T^{L^2})(Lx, Lx') \sim L^d \frac{1}{\sqrt{L^2}^d} \mathrm{e}^{-\frac{(Lx'-Lx)^2}{2DL^2}} = T(x,x')$$

Hence Fix point

$$\mathbb{E}(B'_{\tau'_2} \otimes B'_{\tau'_1})$$

$$= \sum_{m_1, m_2} \mathbb{E}(T^{m_2} B_{L^2\tau'_2 - m_2} T^{L^2 - m_2 - 1} \otimes T^{m_1} B_{L^2\tau'_1 - m_1} T^{L^2 - m_1 - 1})$$

Naive estimate:

$$\mathbb{E}(B'_{\tau'_2} \otimes B'_{\tau'_1}) \sim \sum_{m_1, m_2} \mathbb{E}(B_{\underbrace{L^2\tau'_2 - m_2}_{=:\tau_2}} \otimes B_{\underbrace{L^2\tau'_1 - m_1}_{=:\tau_1}})$$

Since $\mathbb{E}(B_{\tau_2} \otimes B_{\tau_1}) \sim \delta^2(\tau_2 - \tau_1)^{-(1+\alpha)}$, we get

$$\mathbb{E}(B'_{\tau'_2} \otimes B'_{\tau'_1}) \sim \delta^2 \begin{cases} L^4(L^2(\tau'_2 - \tau'_1))^{-(1+\alpha)} & \tau'_2 - \tau'_1 > 1 \\ 1 & \tau'_2 - \tau'_1 = 1 \end{cases}$$

Hence, $\delta_n \sim \delta_{n-1}$, even for large $\alpha > 1 \Rightarrow$ Hopeless.

Ward Identity

Conservation of probability $\sum_{x'} U_\tau(x,x') = 1$ implies

$$\sum_{x'} B_\tau(x,x') = \sum_{x'} U_\tau(x,x') - \sum_{x'} \mathbb{E}(U_\tau(x,x')) = 1 - 1 = 0$$

Let us Fourier transform

$$T \to \hat{T} = \hat{T}(p), \qquad B_\tau \to \hat{B}_\tau = \hat{B}_\tau(p,p')$$

s.t. $\hat{B}_\tau(p,0) = 0$, and consider

$$(\hat{T}^m \hat{B}_\tau)(p,p') = \hat{T}^m(p')(\hat{B}_\tau(p,p') - \hat{B}_\tau(p,0))$$

Since $\hat{T}^m(p) \sim \mathrm{e}^{-mDp^2}$ (and using $\hat{B}(\cdot,\cdot)$ analytic):

$$(\hat{T}^m \hat{B}_\tau)(p,p') \sim \frac{1}{\sqrt{m}} \sup_{p''} |B_\tau(p,p'')|$$

Flow $B \to B'$ with Ward identity

Using $T^m B_\tau \sim \frac{1}{\sqrt{m}} B_\tau$, we get

$$\begin{aligned}\mathbb{E}(B'_{\tau'_2} \otimes B'_{\tau'_1}) &\sim \sum_{m_1,m_2} \frac{1}{\sqrt{m_1}} \frac{1}{\sqrt{m_2}} \mathbb{E}(B_{L^2\tau_2 - m_2} \otimes B_{L^2\tau_1 - m_1}) \\ &\sim \sum_{m_1,m_2} \frac{((L^2\tau'_2 - m_2) - (L^2\tau'_1 - m_1))^{-(1+\alpha)}}{\sqrt{m_1}\sqrt{m_2}}\end{aligned}$$

hence, since $1 \leq m_1, m_2 \leq L^2$, by power counting

$$\delta_n^2 \sim L \times L \times L^{-2(1+\alpha)} \delta_{n-1}^2 \sim L^{-2\alpha} \delta_{n-1}^2$$

Contracting if $\alpha > 0$!

RG conclusion

- Higher order contributions are controlled by simple cluster expansions
- If the disorder is symmetric $U(x,x') = U(x',x)$, then there is another 'Ward identity: $\hat{B}(p=0,p')=0$.
- There are (probably) better approaches for RWRE, e.g. Kipnis-Varadhan martingale technique, but we need (for the sake of the Hamiltonian model) a **robust** scheme, not relying on positivity. Nevertheless, the stated result is not contained in the literature (Redig, Voellering 2011: CLT under stronger decay condition but disorder need not be small)

Analogy RWRE-Hamiltonian model

In Hamiltonian model, we have dynamics V_t acting on full phase space Γ_{SE} $\Rightarrow$ lift to densities

$$V_t : L^1(\mathrm{S}{\times}\mathrm{E}) \to L^1(\mathrm{S}{\times}\mathrm{E}), \qquad (L^1(\mathrm{S}{\times}\mathrm{E})\,\text{short for}\, L^1(\Gamma_{\mathrm{SE}}, \mathrm{d}Q\mathrm{d}q))$$

Similarly, $V_{\mathrm{E},t}$ on $L^1(\mathrm{E})$.

- $U := V_{\epsilon^{-1}}$ (the time on which stochasticity is visible)
- Map $\mathbb{E} : \mathcal{B}(L^1(\mathrm{S}\times\mathrm{E})) \to \mathcal{B}(L^1(\mathrm{S}))$, defined by

$$\mathbb{E}(Z)\rho_{\mathrm{S}} := \int_{\Gamma_{\mathrm{E}}} Z(\rho_{\mathrm{S}} \times \rho_{\mathrm{E}})$$

  (Recall ρ_{E} density of Gibbs measure).
- $T := \mathbb{E}(U)$ is then the reduced particle evolution and $T \otimes V_{\mathrm{E},\epsilon^{-1}}$ a natural approximation for the full evolution.
- $B := U - T \times V_{\mathrm{E},\epsilon^{-1}}$.

Analogy RWRE-Hamiltonian model

What did we *really* use in RWRE-analysis?

- $\mathbb{E}(U) = T$ is manifestly diffusive $\Rightarrow$ Still OK
- $\int_{\Gamma_{\mathrm{S}}} B\rho_{\mathrm{SE}} = 0$ for any ρ_{SE}, $\Rightarrow$ Not true but still $\int_{\Gamma_{\mathrm{SE}}} B\rho_{\mathrm{SE}} = 0$ for any ρ_{SE}. Therefore, we can use the Ward Identity only once in each correlation function. $\Rightarrow$ Will need stronger condition on α.
- $\mathbb{E}(B_{\tau_2} \otimes B_{\tau_1}) \sim \delta^2(\tau_2 - \tau_1)^{-(1+\alpha)}$. $\Rightarrow$ True, but needs definition.

Analogy with RWRE

1 timestep	time-interval $\epsilon^{-1}[\tau-1,\tau]$ (stochasticity visible)
U_τ	unitary dynamics in time $\epsilon^{-1}[\tau-1,\tau]$
T	Markov approx. (emerges on timescales ϵ^{-1})
B_τ	$U_\tau - T \otimes V_{\mathrm{E},\epsilon^{-1}}$ effect of recollisions
$\mathbb{E}$	integrate over the environment wrt. Gibbs state
δ	function of ϵ, measures smallness of B_τ
$\alpha > 0$	Need $\alpha > 1/2$ because Ward on just one B

Controlling all cumulants (as required in our approach) seems out of reach for models like Rayleigh gas. But we can do it for a certain quantum model

A fortunate quantum model

- Waves instead of particles: smoother and ‘more Gaussianity’.
- Quantum mechanics allows to put everything on lattice $\Rightarrow$ no high-velocity problems.
- Consider a tagged particle with internal structure (’spin’ or ’molecule’) but still Hamiltonian.
- Coupling small and particle mass large.
- Gas consists of optical phonons (dispersion relation matters)

Under these conditions we prove diffusion in $3D$ (D.R., Kupiainen) building on (D.R. Frohlich, 2010) in $4D$. The diffusion constant is close to, but not equal to that of the relevant Markovian approximation.

- Message of hope: In the quantum case: there is only analyis: no probability, no trajectories, no coupling arguments: Should be much better for classical wave models

Remark

- Realistic Hamiltonian models for diffusion. 3D case included, but CHALLENGE to get rid of large mass and spin (seems even harder than Rayleigh)
- Only soft mathematics required: Markov scaling limit and perturbation of stochastic systems. Thanks to the introduction of a new time-scale (energies of internal degree of freedom 'molecule').
- Phenomenology that can be derived: Diffusion, decoherence, thermalization, transport, fluctuation-dissipation
- Much simpler Kipnis-Varadhan approach for symmetric disorder $U(x, x') = U(x', x)$. The above theorem barely exploits this. However, our Hamiltonian model is reversible, shortcut possible?

Another model: Particle coupled to wave equation

- Wave equation for $\varphi(z,t), \pi(z,t)$

$$\dot{\varphi} = \pi, \qquad \dot{\pi} = \Delta\varphi$$

derived from the Hamiltonian

$$H_{\mathrm{E}} = H_{\mathrm{E}}(\varphi, \pi) = 1/2 \int \mathrm{d}z \left(|\nabla\phi(z)|^2 + |\pi(z)|^2 \right)$$

- Particle (x, p) (set again $m = 1$)

$$\dot{x} = v := p, \qquad \dot{v} = 0$$

Hence Hamiltonian

$$H_{\mathrm{S}}(x,p) = p^2/2$$

.

Free wave equation

- Solution

$$\begin{pmatrix} \dot\varphi(t) \\ \dot\pi(t) \end{pmatrix} = \mathrm{e}^{t\mathcal{L}} \begin{pmatrix} \varphi(0) \\ \pi(0) \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

Fourier trf: $\varphi(z), \pi(z) \to \hat\varphi(k), \hat\pi(k)$ and introduce

$$a(k) = \mathrm{i}|k|\hat\varphi(k) + \hat\pi(k), \qquad a^*(k) = -\mathrm{i}|k|\hat\varphi(k) + \hat\pi(k)$$

Then

$$a(k,t) = a(k,0)\mathrm{e}^{-\mathrm{i}|k|t}, \qquad a^*(k,t) = a^*(k,0)\mathrm{e}^{\mathrm{i}|k|t}$$

- Initial measure is the Gibbs measure, which is Gaussian:

$$\rho_{\mathrm{E}} \sim \frac{1}{Z(\beta)} \mathrm{e}^{-\beta H_{\mathrm{E}}(\varphi,\pi)}$$

So the initial condition will a.s. not be finite-energy.

Particle-wave coupling

Coupling by the interaction Hamiltonian

$$H_I = \int \mathrm{d}z \varphi(z)\rho(z-x)$$

with ρ the *form factor*; it describes indeed the form of the particle (e.g. the charge distribution it carries)

$$\dot{\pi}(z) = \Delta\varphi(z) - \rho(z-x), \qquad \dot{\varphi} = \pi$$

$$\dot{p} = \int \mathrm{d}z \varphi(z)\nabla\rho(z-x) = \int \mathrm{d}k \hat{\varphi}(k) k \hat{\rho}(k) \mathrm{e}^{\mathrm{i}kx}, \qquad \dot{x} = v$$

Particle coupled to wave equation

Let $f(t)$ be the force acting on the particle at time t

$$
\begin{aligned}
f(t) &= \int \mathrm{d}k \frac{k\hat{\rho}(k)}{\mathrm{i}|k|}(a(k,t) - a^*(k,t))\mathrm{e}^{\mathrm{i}kx(t)} \\
&= \int \mathrm{d}k \frac{k\hat{\rho}(k)}{\mathrm{i}|k|}(a(k,0)\mathrm{e}^{-\mathrm{i}|k|t} - a^*(k,0)\mathrm{e}^{\mathrm{i}|k|t})\mathrm{e}^{\mathrm{i}k(x(0)+tv)} + \mathcal{O}(\rho^2)
\end{aligned}
$$

Integrate this approximation of the force over a time $T \gg 1$;

$$F_T = \int_0^T \mathrm{d}t\, f(t)$$

Then $\langle F_T \rangle = 0$ and

$$\langle F_T F_T \rangle = \frac{T}{\beta} \int \mathrm{d}k \delta(|k| - |v \cdot k|) \frac{k \cdot k |\hat{\rho}(k)|^2}{|k|^2} + o(T)$$

using smoothness of $\hat{\rho}$, hence decay of the correlation function $\langle f(s')f(s) \rangle$ in $s' - s$.

Particle coupled to wave equation: Scaling

Let $\rho \to \lambda\rho$ with λ a small coupling strength. We saw then that, for some $\sigma = \mathcal{O}(1)$

$$\langle F_T F_T \rangle \sim \lambda^2 T \sigma^2,$$

Hence particle needs time $T \sim \lambda^{-2}$ to feel influence of the field. Natural to define new coordinates (τ, χ) such that $t = \lambda^{-2}\tau, x = \lambda^{-2}\chi$. Then

$$dv(\tau) = \sigma \mathrm{d}B_\tau + \mathcal{O}(1) + \mathcal{O}(\lambda), \qquad d\chi(\tau) = v(\tau)\mathrm{d}\tau + \mathcal{O}(\lambda)$$

The $\mathcal{O}(1)$ term is of course what is missing to make the resulting equation detailed balance

‘Vague’ Conjecture on Markov scaling limit

In the scaling limit $t = \lambda^{-2}\tau, x = \lambda^{-2}\chi, \lambda \to 0$, the process of the particle converges (weakly) to the solution of the Fokker-Planck equation

$$\mathrm{d}v = \sigma(v)\mathrm{d}B_\tau - \gamma(v)\mathrm{d}\tau, \qquad \mathrm{d}\chi(\tau) = v(\tau)\mathrm{d}\tau$$

with

$$\sigma^2 = \frac{1}{\beta}\int \mathrm{d}k\,\delta(|k| - |v\cdot k|)\frac{k\cdot k|\hat{\rho}(k)|^2}{|k|^2}$$

and

$$\gamma = -\nabla\sigma^2 + \beta\sigma^2 v$$

Remarks

- At $\beta = \infty$, σ vanishes but not γ. There can still be friction.
- Note that for $|v| \leq 1$, $\sigma = 0$. Indeed, electrons moving faster than the speed of light are slowed down (Cerenkov radiation), but
- Below the speed of light, no friction force. Instead, a quasi-particle forms: Electron dressed with photon cloud (lots of work on Q mdoel). Classicaly, this corresponds to a soliton (Komech, Spohn, 98)
- Instead of $|k|$: choose dispersion $\omega(k)$ so that $\omega(k) - kv = 0$ always solution. Then, expect diffusion at $\beta < \infty$ and friction at $\beta = \infty$.
- Fokker-Planck derived in different regime by Eckmann, PIllet, Rey-Bellet 99. No weak coupling but fine-tuning of form factor to make the system essentially explicitly solvable.