Diffusion in Hamiltonian Systems

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based on work with O. Ajanki, A. Kupiainen and earlier work with J. Fröhlich.
Main goal

Prove that a Hamiltonian (or other reasonable deterministic) system exhibits diffusion for long times.

Emergence of irreversibility from deterministic dynamics common wisdom physically, but hard to make rigorous

Figure: Billiards with finite horizon. Tracer particle bounces elastically off periodic objects. Diffusion proven by \textit{Sinai and Bunimovich, 81, .} A similar setup (Coulombic potentials instead of scatterers) was done by \textit{Knauf}
Plan

- Rayleigh gas: Diffusion and Markov scaling limit: Linear Boltzmann equation
- General strategy for proving diffusion assuming we can prove Markov scaling limit
- Main idea of the strategy: Random Walk in Random Environment (*a real honest theorem*)
- Mention of some result along these lines (*also honest but unstated*)
- Models based on waves: Markov scaling limits
Rayleigh gas: Tagged particle in ideal gas

- Ideal gas: $N$ point particles with mass 1 in volume $\Lambda \subset \mathbb{R}^3$. Coordinates $q = (x, v) = (x_i, v_i)_{i=1,...,N} \in \Gamma_E$.
- ‘Tagged’ Particle (not point-like, having radius 1) and mass 1, Coordinates $Q = (X, V)$.
- Billiard Dynamics: free flow

\[
(\dot{x}(t), \dot{X}(t)) = (v(t), V(t)) \quad (\dot{v}(t), \dot{V}(t)) = 0
\]

up to the first collision time, when $\exists i : |x_i - X| = 1$. Then $(v_i, V) \to (v'_i, V')$ by the rule

\[
(v'_i - V')_\parallel := -(v_i - V)_\parallel, \quad (v'_i - V')_\perp := (v_i - V)_\perp
\]

where $a = a_\parallel + a_\perp$ such that $a_\parallel \parallel (q_i - Q)$ and $a_\perp \perp (q_i - Q)$.

→ defines a dynamical system: $(q, Q)(0) \to (q, Q)(t)$ for a.a. initial conditions.
Rayleigh gas: particle in ideal gas

- Initial measure \( \rho_0 = \rho_{S,0}(dQ) \times \rho_E(q) dq \), with

\[
\rho_E(q) = \rho_{E,\beta,N,\Lambda}(q) = \prod_i \frac{1}{|\Lambda|} e^{-\frac{\beta}{2} v_i^2}
\]

and \( \rho_{S,0} \) localized around \((0, 0)\). Hence, gas is in ’thermal state’ (homogeneous Maxwellian).

- Dynamics defines flow on measures \( \rho_0 \rightarrow \rho_t \), in particular the marginal \( \rho_{S,0} \rightarrow \rho_{S,t} \).

- As \( \Lambda \) grows large, influence of the boundaries only after long time. \( \rightarrow \) define distribution of \( Q(t) = (X(t), V(t)) \) in the limit

\[
\Lambda \rightarrow \mathbb{R}^3, N \rightarrow \infty, \quad \text{with } \epsilon = N/|\Lambda| \text{ fixed}
\]

(or, start in infinite volume, then law of original positions is Poisson Point process with intensity \( \epsilon \))
Does the particle diffuse?

\[ \left\langle \frac{|X(t)|^2}{6t} \right\rangle \rightarrow_{t \to \infty} D, \quad D > 0 \]

.... CLT? invariance principle?

If you assume each gas particle collides only once (not entirely accurate, also negative recollisions), then \( Q(t) \) is a Markov jump process (Linear Boltzmann Equation, see later). Diffusion follows trivially.

However, what about recollisions?
⇒ Markov property breaks down
Rayleigh gas: Scaling limit

- Make recollisions infinitely unlikely \(\Rightarrow\) easier problem. This is the idea of scaling limits. For example, let density of the gas particles be small and observe the system for long times.

\[
\text{density } \sim \epsilon, \quad \text{time } \sim \frac{1}{\epsilon}, \quad \epsilon \downarrow 0.
\]

In the limit, \(\epsilon \downarrow 0\), the probability of one collision, respectively a recollision is

\[
\left(\frac{1}{\epsilon}\right) \times \epsilon, \quad \left(\frac{1}{\epsilon}\right) \times \epsilon^2
\]

One expects that \((\epsilon X^\epsilon(\tau/\epsilon), V^\epsilon(\tau/\epsilon))\) converges to a Markov process (Linear Boltzmann equation) in \(\tau\) as \(\epsilon \downarrow 0\).

- This was done (rather, something similar) by Durr-Goldstein-Lebowitz in 1981. However, without scaling limit (i.e. \(\epsilon\) fixed), no results available!
Recall distribution $\rho_{S,t}(X, V)$ and collision map for the tracer particle $(V, v) \rightarrow V'_n(V, v)$. Assume 1-particle density of the gas always given by the equilibrium $\mu(v) \sim e^{-\beta v^2/2}$. Then

$$
\partial_t \rho_{S,t}(X, V) = V \cdot \nabla_X \rho_{S,t}(X, V)
+ \int d\nu_0 \int dV_0 \delta(V'_n(V_0, \nu_0) - V) \| (V_0 - \nu_0) \| \mu(\nu_0) \rho_{S,t}(X, V_0)
- \int d\nu \int dV' \delta(V'_n(V, \nu) - V') \| (V - \nu) \| \mu(\nu) \rho_{S,t}(X, V)
$$

Originates from nonlinear B.E. by replacing once $\rho_{S,t}$ ($f_t$ in Pulvirenti’s talk) by $\mu(v)$. ⇒ condensed form

$$
\partial_t \rho_{S,t}(X, V) = \int dV'(r(V', V) \rho_{S,t}(X, V') - r(V, V') \rho_{S,t}(X, V))
+ V \cdot \nabla_X \rho_{S,t}(X, V)
$$

with $r(V, V')$ rate of jump $V \rightarrow V'$. 
Nice, but Scaling limits are not my aim

The dynamical system gets adjusted as time grows.

⇒ Does not give information on the long-time limit of the fixed $\epsilon$ dynamical system. For example, $D$ is different

Example

2D Anderson model is well-described by LBE for short times, but localized for large times: With probability $\exp(-\lambda^{-2})$, the particle is sent back to its starting place. 1D FPU-$\beta$ chain: Phonon boltzmann equation predicts wrong power law. In both examples, the scaling limit is lying!

Results (NOT exhaustive) on scaling limits

- Yau, Erdös, ’99, Yau, Erdös, Salmhofer, ’05, Lukkarinen, Spohn, ’08, quantum or wave models
- Toth, Holley, Dürr-Goldstein-Lebowitz, ’81, Rayleigh gas
- Komorowski, Ryzhik, ’04, particle in random force field
- Dolgopyat, Liverani, ’10, coupled Anosov systems.
Long-range nonmarkovian corrections

Correlation decay

Probability of a second collision with given gas particle decays polynomially in time: Assume that the test particle diffuses, $X^2 \sim t$. Then, a receding gas particle that wants to recollide after time $t$, should have a velocity smaller than $\sqrt{t/t} = 1/\sqrt{t}$, but

$$\int d\mathbf{v} \chi[|\mathbf{v}| \leq t^{-1/2}] e^{-\beta v^2/2} \sim t^{-d/2}$$

(this is not a correct estimate of the correlation decay, though)

Slow decorrelation is a generic feature of momentum converging Hamiltonian system, for interacting hard spheres like $t^{-d/2}$ (Adler, Wainwright, ’70).
⇒ lies at the heart of anomalous diffusion in 1D and 2D systems. (Velocity-velocity autocorrelation not integrable)
We know that on time scales $t \approx \epsilon^{-1}$, the particle looks like a Markov jump process ($\sim$ random walk).

The corrections to this behaviour are manifestly non-Markovian and long-range in time.

This looks like the problem of proving an annealed central limit theorem for a random walk in a time-dependent random environment, with long-range memory.

More generally, this looks like doing perturbation theory around a stochastic system, rather than around the unperturbed Hamiltonian system. **Improvement, because perturbation preserves character**
Let $U_{\tau \in \mathbb{N}}$ be random transition kernels on $\mathbb{Z}^d$

- Transition kernels

$$U_{\tau}(x, x') \geq 0, \quad \sum_{x'} U_{\tau}(x, x') = 1$$

- Law of $U_{\tau}$ invariant under space and time-translations.
- $\mathbb{E}(U_{\tau}) = T$ is transition kernel of simple random walk
- Hence, $U_{\tau} = T + B_{\tau}$ with $B_{\tau}$ 'dynamical disorder'.
- "Disorder correlations", for $\tau_1 < \tau_2 < \ldots < \tau_m$, and $\gamma > 0$,

$$G(\tau_1, \ldots, m) := \sup_{x_1} \ldots \sup_{x'_1} \sum_{x_2} \ldots \sum_{x'_2} e^{\gamma \sum_j |x'_j - x_j|}$$

$$\left| \langle B_{\tau_1}(x_1, x'_1); \ldots; B_{\tau_m}(x_m, x'_m) \rangle^c \right|$$
Theorem (Ajanki, D.R., Kupiainen, *in preparation*)

Assume (for some $\gamma, \alpha$ and all $m$)

$$\sum_{1=\tau_1<...<\tau_m} \prod_{j=2}^{m} (|\tau_j - \tau_{j-1}|^\alpha) G(\tau_1,...,\tau_m) < \delta^m,$$

Then, if $\delta < \delta_0$ and $\alpha > 0$, there is annealed CLT

$$\sum_{x} e^{-ik \frac{x}{\sqrt{N}}} [\mathbb{E}(U_N \ldots U_1)] (0, x) \xrightarrow{N \to \infty} e^{-D^2k}$$

- Similar framework for RWRE was pioneered in ’91 by Bricmont-Kupiainen. Here: much easier because integrable correlations. Quenched CLT requires also some spatial decay.
- Proof: renormalization group + cluster expansion.
Renormalization group for RWRE

Recall kernels (random except $T$)

$$U_\tau(x, x'), \quad T(x, x'), \quad B_\tau(x, x')$$

Renormalization step is

$$\mathcal{R}U_\tau(x, x') := L^d \left( U_{L^2 \tau} U_{L^2 \tau - 1} \cdots U_{L^2 (\tau - 1) + 1} \right) (Lx, Lx')$$

for some $L >> 1$, but $\delta L << 1$.

We denote

$$U'_\tau = \mathcal{R}U_\tau, \quad T' = \mathbb{E}(\mathcal{R}T_\tau), \quad B'_\tau = U'_\tau - T'$$

Running coupling constant $\delta_n$

$$\sum_{1 = \tau_1 < \ldots < \tau_m} (\prod_{j}(\tau_{j+1} - \tau_j))^{\alpha} \mathbb{E}(B_{\tau_m} \otimes \ldots \otimes B_{\tau_2} \otimes B_{\tau_1}) \sim \delta_n^m$$
Flow $T \rightarrow T'$

\begin{align*}
T' &= L^d \mathbb{E}(T^{L^2}) + L^d \sum_{m=1}^{L^2} \mathbb{E}(T^m B_{L^2-m} T^{L^2-m-1}) \\
&\quad + L^d \sum_{m_1+m_2=L^2-2} \mathbb{E}(T^{m_1} B_{L^2-m_1} T^{m_2} B_{L^2-m_1-m_2-1} T^{L^2-m_1-m_2-2})
\end{align*}

By $\mathbb{E}(B_{\tau_1} \otimes B_{\tau_2}) \sim \delta_n^2$, second-order term is $L^{d+4} \delta_n^2$. \Rightarrow OK since $\delta_n$ contracts:

$$\delta_n \sim L^{-\kappa(n-1)} \delta_1, \quad \kappa > 0.$$  

Morally, $T(x, x') \sim e^{-\frac{(x'-x)^2}{2D}}$. Dropping irrelevant terms gives

\begin{align*}
T'(x, x') &\sim L^d(T^{L^2})(Lx, Lx') \sim L^d \frac{1}{\sqrt{L^2D^2}} e^{-\frac{(Lx'-Lx)^2}{2DL^2}} = T(x, x') \\
\end{align*}

Hence Fix point
Flow $B \rightarrow B'$

\[
\mathbb{E}(B'_{\tau_2} \otimes B'_{\tau_1}) = \sum_{m_1,m_2} \mathbb{E}(T^{m_2}B_{L^{2}\tau_2-m_2}T^{L^2-m_2-1} \otimes T^{m_1}B_{L^{2}\tau_1-m_1}T^{L^2-m_1-1})
\]

Naive estimate:

\[
\mathbb{E}(B'_{\tau_2} \otimes B'_{\tau_1}) \sim \sum_{m_1,m_2} \mathbb{E}(B_{L^{2}\tau_2-m_2} \otimes B_{L^{2}\tau_1-m_1}) =: \tau_2 \otimes \tau_1
\]

Since \(\mathbb{E}(B_{\tau_2} \otimes B_{\tau_1}) \sim \delta^2(\tau_2 - \tau_1)^{-(1+\alpha)}\), we get

\[
\mathbb{E}(B'_{\tau_2} \otimes B'_{\tau_1}) \sim \delta^2 \left\{ \begin{array}{cl}
L^4(L^2(\tau_2' - \tau_1'))^{-(1+\alpha)} & \tau_2' - \tau_1' > 1 \\
1 & \tau_2' - \tau_1' = 1
\end{array} \right.
\]

Hence, \(\delta_n \sim \delta_{n-1}\), even for large \(\alpha > 1 \Rightarrow \text{Hopeless.} \)
Ward Identity

Conservation of probability $\sum_{x'} U_\tau(x, x') = 1$ implies

$$\sum_{x'} B_\tau(x, x') = \sum_{x'} U_\tau(x, x') - \sum_{x'} E(U_\tau(x, x')) = 1 - 1 = 0$$

Let us Fourier transform

$$T \to \hat{T} = \hat{T}(p), \quad B_\tau \to \hat{B}_\tau = \hat{B}_\tau(p, p')$$

s.t. $\hat{B}_\tau(p, 0) = 0$, and consider

$$(\hat{T}^m \hat{B}_\tau)(p, p') = \hat{T}^m(p')(\hat{B}_\tau(p, p') - \hat{B}_\tau(p, 0))$$

Since $\hat{T}^m(p) \sim e^{-mDp^2}$ (and using $\hat{B}(\cdot, \cdot)$ analytic):

$$(\hat{T}^m \hat{B}_\tau)(p, p') \sim \frac{1}{\sqrt{m}} \sup_{p''} |B_\tau(p, p'')|$$
Flow $B \rightarrow B'$ with Ward identity

Using $T^m B_\tau \sim \frac{1}{\sqrt{m}} B_\tau$, we get

$$\mathbb{E}(B'_{\tau_2} \otimes B'_{\tau_1}) \sim \sum_{m_1, m_2} \frac{1}{\sqrt{m_1}} \frac{1}{\sqrt{m_2}} \mathbb{E}(B_{L^2 \tau_2 - m_2} \otimes B_{L^2 \tau_1 - m_1})$$

$$\sim \sum_{m_1, m_2} \frac{((L^2 \tau'_2 - m_2) - (L^2 \tau'_1 - m_1))^{-(1+\alpha)}}{\sqrt{m_1} \sqrt{m_2}}$$

hence, since $1 \leq m_1, m_2 \leq L^2$, by power counting

$$\delta_n^2 \sim L \times L \times L^{-2(1+\alpha)} \delta_{n-1}^2 \sim L^{-2\alpha} \delta_{n-1}^2$$

Contracting if $\alpha > 0$!
Higher order contributions are controlled by simple cluster expansions.

If the disorder is symmetric $U(x, x') = U(x', x)$, then there is another ‘Ward identity: $\hat{B}(\rho = 0, \rho') = 0$.

There are (probably) better approaches for RWRE, e.g. Kipnis-Varadhan martingale technique, but we need (for the sake of the Hamiltonian model) a robust scheme, not relying on positivity. Nevertheless, the stated result is not contained in the literature (Redig, Voellering 2011: CLT under stronger decay condition but disorder need not be small).
In Hamiltonian model, we have dynamics $V_t$ acting on full phase space $\Gamma_{SE}$ \Rightarrow lift to densities

$V_t : L^1(S \times E) \rightarrow L^1(S \times E), \quad (L^1(S \times E) \text{ short for } L^1(\Gamma_{SE}, dQd\mathbf{q}))$

Similarly, $V_{E,t}$ on $L^1(E)$.

- $U := V_{\epsilon^{-1}}$ (the time on which stochasticity is visible)
- Map $E : \mathcal{B}(L^1(S \times E)) \rightarrow \mathcal{B}(L^1(S))$, defined by

$$E(Z)\rho_S := \int_{\Gamma_E} Z(\rho_S \times \rho_E)$$

(Recall $\rho_E$ density of Gibbs measure).

- $T := E(U)$ is then the reduced particle evolution and $T \otimes V_{E,\epsilon^{-1}}$ a natural approximation for the full evolution.
- $B := U - T \otimes V_{E,\epsilon^{-1}}$. 

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What did we \textit{really} use in RWRE-analysis?

- $\mathbb{E}(U) = T$ is manifestly diffusive $\Rightarrow$ Still OK
- $\int_{\Gamma_S} B \rho_{SE} = 0$ for any $\rho_{SE}$, $\Rightarrow$ Not true but still
  $\int_{\Gamma_{SE}} B \rho_{SE} = 0$ for any $\rho_{SE}$. Therefore, we can use the Ward Identity only once in each correlation function. $\Rightarrow$ Will need stronger condition on $\alpha$.
- $\mathbb{E}(B_{\tau_2} \otimes B_{\tau_1}) \sim \delta^2(\tau_2 - \tau_1)^{-1+\alpha}$. $\Rightarrow$ True, but needs definition.
### Analogy with RWRE

<table>
<thead>
<tr>
<th>1 timestep</th>
<th>time-interval $\epsilon^{-1}[\tau - 1, \tau]$ (stochasticity visible)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_\tau$</td>
<td>unitary dynamics in time $\epsilon^{-1}[\tau - 1, \tau]$</td>
</tr>
<tr>
<td>$T$</td>
<td>Markov approx. (emerges on timescales $\epsilon^{-1}$)</td>
</tr>
<tr>
<td>$B_\tau$</td>
<td>$U_\tau - T \otimes V_{E,\epsilon^{-1}}$ effect of recollisions</td>
</tr>
<tr>
<td>$\mathbb{E}$</td>
<td>integrate over the environment wrt. Gibbs state</td>
</tr>
<tr>
<td>$\delta$</td>
<td>function of $\epsilon$, measures smallness of $B_\tau$</td>
</tr>
<tr>
<td>$\alpha &gt; 0$</td>
<td>Need $\alpha &gt; 1/2$ because Ward on just one $B$</td>
</tr>
</tbody>
</table>

Controlling all cumulants (as required in our approach) seems out of reach for models like Rayleigh gas. **But we can do it for a certain quantum model**
Waves instead of particles: smoother and ‘more Gaussianity’.

Quantum mechanics allows to put everything on lattice ⇒ no high-velocity problems.

Consider a tagged particle with internal structure (‘spin’ or ‘molecule’) but still Hamiltonian.

Coupling small and particle mass large.

Gas consists of optical phonons (dispersion relation matters)

Under these conditions we prove diffusion in $3D$ (D.R., Kupiainen) building on (D.R. Frohlich, 2010) in $4D$. The diffusion constant is close to, but not equal to that of the relevant Markovian approximation.

**Message of hope**: In the quantum case: there is only analysis: no probability, no trajectories, no coupling arguments: Should be much better for classical wave models
Remark

- **Realistic Hamiltonian models for diffusion**. 3D case included, but **CHALLENGE** to get rid of large mass and spin (seems even harder than Rayleigh)

- **Only soft mathematics required**: Markov scaling limit and perturbation of stochastic systems. Thanks to the introduction of a new time-scale (energies of internal degree of freedom ’molecule’).

- **Phenomenology that can be derived**: Diffusion, decoherence, thermalization, transport, fluctuation-dissipation

- Much simpler Kipnis-Varadhan approach for symmetric disorder $U(x, x') = U(x', x)$. The above theorem barely exploits this. However, our Hamiltonian model is reversible, shortcut possible?
Another model: Particle coupled to wave equation

- Wave equation for $\varphi(z, t), \pi(z, t)$

$$
\dot{\varphi} = \pi, \quad \dot{\pi} = \Delta \varphi
$$

derived from the Hamiltonian

$$
H_E = H_E(\varphi, \pi) = \frac{1}{2} \int \mathrm{d}z \left( |\nabla \varphi(z)|^2 + |\pi(z)|^2 \right)
$$

- Particle $(x, p)$ (set again $m = 1$)

$$
\dot{x} = v := p, \quad \dot{v} = 0
$$

Hence Hamiltonian

$$
H_S(x, p) = \frac{p^2}{2}
$$
Free wave equation

Solution

\[
\begin{pmatrix}
\dot{\varphi}(t) \\
\dot{\pi}(t)
\end{pmatrix} = e^{t\mathcal{L}} \begin{pmatrix}
\varphi(0) \\
\pi(0)
\end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix}
0 & 1 \\
\Delta & 0
\end{pmatrix}
\]

Fourier trf: \(\varphi(z), \pi(z) \rightarrow \hat{\varphi}(k), \hat{\pi}(k)\) and introduce

\[
a(k) = i|k|\hat{\varphi}(k) + \hat{\pi}(k), \quad a^*(k) = -i|k|\hat{\varphi}(k) + \hat{\pi}(k)
\]

Then

\[
a(k, t) = a(k, 0)e^{-i|k|t}, \quad a^*(k, t) = a^*(k, 0)e^{i|k|t}
\]

Initial measure is the Gibbs measure, which is Gaussian:

\[
\rho_E \sim \frac{1}{Z(\beta)}e^{-\beta H_E(\varphi, \pi)}
\]

So the initial condition will a.s. not be finite-energy.
Particle-wave coupling

Coupling by the interaction Hamiltonian

\[ H_I = \int dz \varphi(z) \rho(z - x) \]

with \( \rho \) the form factor; it describes indeed the form of the particle (e.g. the charge distribution it carries)

\[ \dot{\pi}(z) = \Delta \varphi(z) - \rho(z - x), \quad \dot{\varphi} = \pi \]

\[ \dot{p} = \int dz \varphi(z) \nabla \rho(z - x) = \int dk \hat{\varphi}(k) k \hat{\rho}(k) e^{ikx}, \quad \dot{x} = v \]
Let $f(t)$ be the force acting on the particle at time $t$

\[
f(t) = \int dk \frac{k\hat{\rho}(k)}{i|k|}(a(k, t) - a^*(k, t))e^{ikx(t)}
\]

\[
= \int dk \frac{k\hat{\rho}(k)}{i|k|}(a(k, 0)e^{-i|k|t} - a^*(k, 0)e^{i|k|t})e^{ik(x(0)+tv)} + O(\rho^2)
\]

Integrate this approximation of the force over a time $T \gg 1$;

\[
F_T = \int_0^T dt \ f(t)
\]

Then $\langle F_T \rangle = 0$ and

\[
\langle F_T F_T \rangle = \frac{T}{\beta} \int dk \delta(|k| - |v \cdot k|) \frac{k \cdot k|\hat{\rho}(k)|^2}{|k|^2} + o(T)
\]

using smoothness of $\hat{\rho}$, hence decay of the correlation function $\langle f(s')f(s) \rangle$ in $s' - s$. 

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Let $\rho \to \lambda \rho$ with $\lambda$ a small coupling strength. We saw then that, for some $\sigma = O(1)$

$$\langle F_T F_T \rangle \sim \lambda^2 T \sigma^2,$$

Hence particle needs time $T \sim \lambda^{-2}$ to feel influence of the field. Natural to define new coordinates $(\tau, \chi)$ such that $t = \lambda^{-2} \tau, x = \lambda^{-2} \chi$. Then

$$d\nu(\tau) = \sigma dB_\tau + O(1) + O(\lambda), \quad d\chi(\tau) = v(\tau) d\tau + O(\lambda)$$

The $O(1)$ term is of course what is missing to make the resulting equation detailed balance.
In the scaling limit $t = \lambda^{-2} \tau$, $x = \lambda^{-2} \chi$, $\lambda \to 0$, the process of the particle converges (weakly) to the solution of the Fokker-Planck equation

$$d v = \sigma (v) dB_{\tau} - \gamma (v) d \tau, \quad d \chi (\tau) = v(\tau) d \tau$$

with

$$\sigma^2 = \frac{1}{\beta} \int dk \delta (|k| - |v \cdot k|) \frac{k \cdot k |\hat{\rho} (k)|^2}{|k|^2}$$

and

$$\gamma = - \nabla \sigma^2 + \beta \sigma^2 v$$
At $\beta = \infty$, $\sigma$ vanishes but not $\gamma$. There can still be friction.

Note that for $|v| \leq 1$, $\sigma = 0$. Indeed, electrons moving faster than the speed of light are slowed down (Cerenkov radiation), but

Below the speed of light, no friction force. Instead, a quasi-particle forms: Electron dressed with photon cloud (lots of work on Q mdoel). Classically, this corresponds to a soliton (Komech, Spohn, 98)

Instead of $|k|$: choose dispersion $\omega(k)$ so that $\omega(k) - kv = 0$ always solution. Then, expect diffusion at $\beta < \infty$ and friction at $\beta = \infty$.

Fokker-Planck derived in different regime by Eckmann, Pillet, Rey-Bellet 99. No weak coupling but fine-tuning of form factor to make the system essentially explicitly solvable.