From Hamiltonian particle systems to Kinetic equations

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WIAS, Berlin, February 2012

Rigorous derivation of the Boltzmann equation

Represent the expansion in terms of integrals of the initial datum.
Def a $n-$collision, $j-$particle tree, $G(j, n) = \{k_1 \dots k_n\}$ s.t.

$$k_1 \in I_j, k_2 \in I_{j+1}, \dots k_n \in I_{j+n}$$

where $I_s = \{1, 2, \dots s\}$.

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A natural graphical representation. For instance $G(2,5)$ given by $1, 2, 1, 3, 2$:

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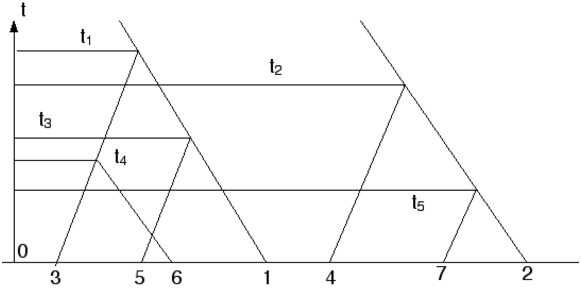


Rigorous derivation of the Boltzmann equation

Each branch (say $j+\ell$) represents a new particle (with incoming or outgoing velocities according to $\sigma_\ell = -1$ or $\sigma_\ell = 1$ respectively) created at time t_ℓ by a previous particle (branch) $k_\ell = 1, \dots j+\ell-1$.

The set of all such trees is denoted by $\mathcal{G}(j,n)$. In the following we shall write

$$\sum_{k_1 \dots k_n}{}' = \sum_{G(j,n) \in \mathcal{G}(j,n)} .$$

Note that the number of terms is $j(j+1)\dots(j+n-1)$.

Rigorous derivation of the Boltzmann equation

$\underline{\zeta}^\varepsilon(s) = (\underline{\xi}^\varepsilon(s), \underline{\eta}^\varepsilon(s))$ the positions and velocities of the particles created up to the time s. If $s \in (t_r, t_{r+1})$ we have $j + r$ particles whose positions and velocities are:

$$\underline{\xi}^\varepsilon(s) = (\xi_1^\varepsilon(s), \dots, \xi_{j+r}^\varepsilon(s))$$

and

$$\underline{\eta}^\varepsilon(s) = (\eta_1^\varepsilon(s), \dots, \eta_{j+r}^\varepsilon(s)).$$

The particle $j + r$ is created at time t_r by particle i in the position

$$\xi_{j+r}^\varepsilon(t_r) = \xi_i^\varepsilon(t_r) - \sigma_r \omega_r \varepsilon$$

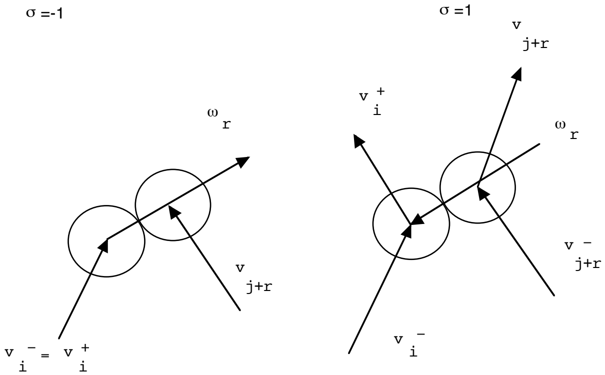
with velocity

$$\eta_{j+r}(t_r^-) = v_{j+r} + \omega_r \cdot (\eta_i^\varepsilon(t_r^+) - v_{j+r})\omega_r(\frac{\sigma_r + 1}{2})$$

If $\sigma_r = 1$ the pair $(\eta_i^\varepsilon(t_r^+), v_{j+r}))$ is post-collisional.

Rigorous derivation of the Boltzmann equation

Then also the velocity of the direct progenitor i changes according to the formula $\eta_i(t_r^-) = \eta_i(t_r^+) - \omega_r \cdot (\eta_i^\varepsilon(t_r) - v_{j+r})\omega_r$.



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Given the initial position and velocities of the new created particles, the (backward) flow is Φ^{-t}, the H-S flow, up to the next creation in which the procedure is repeated.

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Note that recollisions are unlikely, but they are the main responsible of the different behavior of the particle dynamics from the Boltzmann evolution.
The configuration $\underline{\zeta}^{\varepsilon}(s)$ depends on $G(j,n)$, on the creation times $(t_1, \dots t_n)$, on the sequence $\underline{\sigma_n}$ and on the velocities of the created particles $v_{j+1}, \dots, v_{j+n}$.

Rigorous derivation of the Boltzmann equation

Next we set

$$\begin{aligned} \mathbf{t}_n &= (t_1, \dots, t_n), \\ \underline{\omega}_n &= (\omega_1, \dots, \omega_n) \\ \mathbf{V}_{j,n} &= (v_{j+1}, \dots, v_{j+n}) \\ \underline{\sigma}_n &= (\sigma_1, \dots, \sigma_n), \quad \sigma_j = \pm 1. \end{aligned}$$

Define

$$\begin{aligned} d\Lambda(\mathbf{t}_n, \underline{\omega}_n, \mathbf{V}_{j,n}) = &\chi(\{t_1 > t_2 \cdots > t_n\}) dt_1 \dots dt_n \\ &d\omega_1 \dots d\omega_n dv_{j+1} \dots dv_{j+n}. \end{aligned}$$

With these definitions we rewrite the Dyson expansion

Rigorous derivation of the Boltzmann equation

$$f_j^\varepsilon(Z_j;t) = \sum_{n=0}^{N-j} \alpha_n^\varepsilon(j) \sum_{G(j,n)\in\mathcal{G}(j,n)} \sum_{\underline{\sigma}_n} (-1)^{|\underline{\sigma}_n|}$$

$$\int d\Lambda(\mathbf{t}_n, \underline{\omega}_n, \mathbf{V}_{j,n}) \prod_{i=1}^{n} B(\omega_i; \eta_{k_i}^\varepsilon(t_i) - v_{j+i})\, f_{0,j+n}^\varepsilon(\underline{\zeta}^\varepsilon(0)),$$

where $B(\omega_i; \eta_{k_i}^\varepsilon(t_i) - v_{j+i}) = \omega_i \cdot (\eta_{k_i}^\varepsilon(t_i) - v_{j+i}))$ and k_i is the progenitor of the particle $j+i$.

$$\alpha_n^\varepsilon(j) = \varepsilon^{2n}(N-j)(N-j-1)\dots(N-j-n+1).$$

Rigorous derivation of the Boltzmann equation

We now treat the solution to the Boltzmann equation in the same manner. Let f solve B eq.n.

$$f_j(Z_j;t) = f(t)^{\otimes j}(Z_j) \tag{1}$$

Then $f_j(t)$ solves the Boltzmann hierarchy:

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}) f_j = C_{j+1} f_{j+1}, \tag{2}$$

Rigorous derivation of the Boltzmann equation

$$C_{j+1} = \sum_{k=1}^{j} C_{k,j+1}$$

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$$C^{+}_{k,j+1} g_{j+1}(x_1, \dots, x_j; v_1, \dots, v_j) = \int dv_{j+1} \int_{S^2_+} d\omega \omega \cdot (v_k - v_{j+1})$$

$$[g_{j+1}(x_1, \dots, x_j, x_k; v_1, \dots, v'_k \dots, v'_{j+1}),$$

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$$g_{j+1}(x_1,\dots,x_j,x_k;v_1,\dots,v_k,\dots,v_{j+1})].$$

The same as H-S hierarchy putting $\varepsilon = 0$.

Rigorous derivation of the Boltzmann equation

$$f_j(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n$$
$$S(t-t_1) C_{j+1} \dots S(t_{n-1}-t_n) C_{j+n} S(t_n) f_{0,n+j}.$$

where

$$f_{0,n+j} = (f_0)^{\otimes(n+j)}$$

and $S(t) g_j(X_j, V_j) = g_j(X_j - V_j t, V_j)$ is the free flow.

Rigorous derivation of the Boltzmann equation

We can do the same tree expansion as before readily arriving to the following expression

$$f_j(Z_j;t) = \sum_{n=0}^{\infty} \sum_{G(j,n)\in\mathcal{G}(j,n)} \sum_{\underline{\sigma}_n} (-1)^{|\underline{\sigma}_n|}$$

$$\int d\Lambda(\mathbf{t}_n, \underline{\omega}_n, \mathbf{V}_{j,n}) \prod_{i=1}^{n} B(\omega_i; \eta_{k_i}(t_i) - v_{j+i})\, f_{0,j+n}(\underline{\zeta}(0)),$$

where $B(\omega_i; \eta_{k_i}(t_i) - v_{j+i}) = \omega_i \cdot (\eta_{k_i}(t_i) - v_{j+i}))$ and k_i is the progenitor of the particle $j+i$.

Here the backward flow $\underline{\zeta}(s)$ is constructed as before with the difference that the new particles are created exactly in the same place of their progenitor. Obviously we do not have recollisions (particles are points). In other words everything goes as before just putting formally $\varepsilon = 0$.

Rigorous derivation of the Boltzmann equation

Hypotheses on the initial data

1) f_0 is a continuous, bounded probability density, s.t. , for some $\beta > 0$

$$f_0(x, v) \leq Ce^{-\beta v^2}$$

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We cannot assume the same initial datum for the H-S hierarchy.
Correlations at time zero due to the h-s non-overlapping condition.
We require

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4) $\lim_{\varepsilon \to 0} f^{\varepsilon}_{0,j} = f_0^{\otimes j}$

uniformly on compact sets outside the manifold

$$\{Z_j | x_i = x_s, \quad \text{for some} \quad i \neq s\}.$$

Rigorous derivation of the Boltzmann equation

Lanford '75

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Lanford '75

Theorem

There exists $t_0 > 0$ *s.t., for* $t \leq t_0$ *and for all* $j = 1, 2, \ldots$

$$\lim_{\varepsilon \to 0} f_j^{\varepsilon}(t) = f_j(t) \quad a.e.$$

Moreover

$$f_j(t) = f(t)^{\otimes j} \quad a.e.,$$

where $f(t)$ *solves the Boltzmann equation.*

Rigorous derivation of the Boltzmann equation

Compare the two series expansion

$$f_j(Z_j;t) = \sum_{n=0}^{\infty} \sum_{G(j,n)\in\mathcal{G}(j,n)} \sum_{\underline{\sigma}_n} (-1)^{|\underline{\sigma}_n|}$$
$$\int d\Lambda(\mathbf{t}_n, \underline{\omega}_n, \mathbf{V}_{j,n}) \prod_{i=1}^{n} B(\omega_i; \eta_{k_i}(t_i) - v_{j+i})\, f_{0,j+n}(\underline{\zeta}(0)),$$

and

$$f_j^\varepsilon(Z_j;t) = \sum_{n=0}^{N-j} \alpha_n^\varepsilon(j) \sum_{G(j,n)\in\mathcal{G}(j,n)} \sum_{\underline{\sigma}_n} (-1)^{|\underline{\sigma}_n|}$$
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$$\alpha_n^\varepsilon(j) = \varepsilon^{2m}(N-j)(N-j-1)\dots(N-j-n+1).$$

Outline the differences.

Rigorous derivation of the Boltzmann equation

$$\alpha_n^\varepsilon(j) \to 1.$$

For a.a. Z_j, $0 \leq s \leq t$

$$\underline{\zeta}^\varepsilon(s) \to \underline{\zeta}(s)$$

We have convergence term by term.

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Rigorous derivation of the Boltzmann equation

Remind

$$f_j^\varepsilon(t) = \sum_{n=0}^{N-j} \sum_{\substack{\sigma_1,\dots,\sigma_n \\ \sigma_i=\pm 1}} \sum_{k_1,\dots,k_n}{}' (-1)^{|\underline{\sigma}_n|} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n$$

$$\alpha_n^\varepsilon(j) U^\varepsilon(t-t_1) C_{k_1,j+1}^{\varepsilon,\sigma_1} \dots U^\varepsilon(t_{n-1}-t_n) C_{k_n,j+n}^{\varepsilon,\sigma_n} U^\varepsilon(t_n) f_{0,n+j}^\varepsilon.$$

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where

$$C_{j+1}^\varepsilon f_{j+1}^N(x_1,v_1,\dots x_j,v_j) = -\sum_{k=1}^{j} \int dn \int dv_{j+1} n\cdot(v_k - v_{j+1})$$

$$f_{j+1}^N(x_1,v_1,\dots x_k,v_k,\dots,x_k+\varepsilon n, v_{j+1})$$

Rigorous derivation of the Boltzmann equation

Suppose that $|v_i| \leq C$. To simplify (large velocities are not a problem).

$$|n \cdot (v_k - v_{j+1})| \leq C$$

Then

$$\|C^{\varepsilon}_{j+1} f^N_{j+1}\|_{L^\infty} \leq Cj \|f^N_{j+1}\|_{L^\infty}$$

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The generic term is uniformly bounded by

$$\frac{t^n}{n!} C^n j(j+1)\dots(j+n-1) \leq (2z)^j (tC)^n$$

if $\|f_{0,j}\|_{L^\infty} \leq z^j$. The series is converging for t small.

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Same estimate for the Boltzmann hierarchy.