Workshop on Numerical Methods and Analysis in CFD

Weierstrass Institute for Applied Analysis and Stochastics

Berlin, July 5–8, 2022

www.wias-berlin.de/workshops/NUM_CFD22

Organizers:

Carsten Carstensen	(Humboldt-Universität zu Berlin)
Volker John	(WIAS Berlin, Freie Universität Berlin)
Stefan Sauter	(University of Zurich)

This workshop ist supported by:

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1 General Information

Dear Paricipants,

welcome to the "Workshop on Numerical Methods and Analysis in CFD" at the Weierstrass Institute for Applied Analysis and Stochastic in Berlin.

This workshop is devoted to modern numerical methods for solving equations from fluid dynamics, such as Stokes and Navier-Stokes equations, convection-diffusion-reaction or reaction-diffusion equations, and coupled systems. Topics include as well the numerical analysis of such methods as their use in simulating problems from applications.

The workshop is sponsored by the Joint Seed Funding Call 2021 of the Humboldt-Universität zu Berlin and the the University of Zurich.

This booklet contains the workshop program, the abstracts for talks, information about the workshop dinner and places of lunch near the Insitute. Lectures are given in the Erhard-Schmidt lecture room on the ground floor.

You can connect to the internet via the **eduroam** network, where the login credentials are provided by your home institute. If your institution does not participate in the eduroam service, you can log in to with **user account: WIAS67x**, **user name: cfdjuly** and **password: cF+OkLv8**. Please be aware that this account is used by all workshops participants. Therefore do not leave any confidential data in its home directory. All remaining files will be deleted after the workshop.

Assistance in the case of questions will be giben by anybody of the WIAS staff participating in the workshop (**w**earing blue badges).

For the compliance with the rules related to Covid on Tuesday and Thursday masks and self-tests will be given. are provided by WIAS.

Carsten (Carstensen), Volker (John), Stefan (Sauter)

2 Program

Tuesday, 05.07.2022	
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15:00	**Giuseppe Vacca (Bari)** Virtual elements for a fluid-structure interaction problem
15:30	**Dirk Peschka (Berlin)** Discretization of compressible Stokes flow using Hamiltonian and Onsager structures
16:00 - 16:30	Coffee Break
16:30	**Mourad Oulghelou (Angers)** Numerical analysis of a residual-based stabilization-motivated POD-ROM for incompressible flows
17:00	**Franco Dassi (Milano)** Vorticity-stabilized virtual elements for the Oseen equation
17:30	**Xu Li (Jinan)** An EMA-conserving, pressure-robust and Re-semi-robust reconstruction method for incompressible Navier–Stokes simulations

Wednesday, 06.07.2022	
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09:45	**Natalia Kopteva (Limerick)** A posteriori error control in the maximum norm for convection-dominated convection-diffusion equations
10:15 - 10:45	COFFEE BREAK
10:45	**Derk Frerichs-Mihov (Berlin)** On reducing spurious oscillations in discontinuous Galerkin methods for convection-diffusion equations
11:15	**Philip Freese (Augsburg)** Super-localized orthogonal decomposition for convection-dominated diffusion problems
11:45	**Abhinav Jha (Aachen)** Adaptive grids for algebraic stabilizations of convection-diffusion-reaction equations
12:15 - 14:15	LUNCH BREAK
14:15	**Markus Bause (Hamburg)** On space-time finite element methods for the Navier–Stokes equations
14:45	**Jan Philipp Thiele (Hannover)** A PU localization for space-time goal oriented adaptive refinement for incompressible flow problems
15:15	**Christian Wieners (Karlsruhe)** Space-time discontinuous Galerkin methods for weak solutions of hyperbolic linear symmetric Friedrichs systems
15:45 - 16:15	COFFEE BREAK
16:15	**Camilla Belponer (Berlin)** Multiscale and homogenized modeling of vascular tissues
16:45	**Pablo Alexei Gazca Orozco (Prague)** Nonlinear iterative approximation of steady flow of chemically reacting fluids
17:15	**Paul Stocker (Göttingen)** Numerical treatment of the vectorial equations of solar oscillations
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Thursday, 07.07.2022	
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09:45	**Gunar Matthies (Dresden)** Pressure-robust discretisations for the generalised Stokes problem on non-affine meshes
10:15 - 10:45	COFFEE BREAK
10:45	**Christian Merdon (Berlin)** Inf-sup stabilized Scott–Vogelius pairs on general simplicial grids by Raviart–Thomas enrichment
11:15	**Chunjae Park (Seoul)** Local computation of pressure in high order finite element methods for Stokes equations
11:45	**Stefan Sauter (Zurich)** On the inf-sup stabillity of Crouzeix–Raviart Stokes elements in 3D
12:15 - 14:15	LUNCH BREAK
14:15	**Sonia Gomes (Campinas, SP)** A multiscale hybrid-mixed model for Stokes–Brinkman–Darcy flows with divergence-compatible velocity-pressure elements
14:45	**Bosco García-Archilla (Sevilla)** Robust error bounds for the Navier–Stokes equations using implicit-explicit second order BDF method with variable steps
15:15	**Utku Kaya (Magdeburg)** Local pressure-corrections for incompressible flows
15:45 - 16:15	COFFEE BREAK
16:15	**Felipe Galarce (Berlin)** Inverse problems on non-parametric domains. Flow reconstruction from medical data using non linear dimensionality reduction
16:45	**Torsten Keßler (Saarbrücken)** Entropy-based methods for rarefied gas flows
17:15	**Tianwei Yu (Duebendorf)** Asymptotic-preserving plasma model in 3D

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09:45	**Volker Kempf (Neubiberg)** Anisotropic Brezzi-Douglas-Marini interpolation with applications to incompressible flows
10:15 - 10:45	COFFEE BREAK
10:45	**Thomas Richter (Magdeburg)** Fluid-rigid body interactions with large motion and contact
11:15	**Ondrej Partl (Berlin)** Reconstruction of flow domain boundaries from velocity data via multi-step optimization of distributed resistance
11:45	**Falko Ruppenthal (Dortmund)** Bathymetry reconstruction via optimal control in well-balanced finite element methods for the shallow water equations

3 Abstracts

On space-time finite element methods for the Navier–Stokes equations

Bause, Markus
Helmut Schmidt University/University of the Federal Armed Forces, Germany

The numerical simulation of incompressible viscous flow continues to remain a challenging task, in particular, if three space dimensions are involved. Using space-time finite element methods (STFEM) allows the natural construction of higher order methods; cf. [5, 1, 2, 3]. They offer the potential to achieve accurate results on computationally feasible grids with a minimum of numerical costs. Geometric multigrid methods (GMG) are known as the most efficient iterative methods for the solution of large linear systems arising from the discretization of partial differential equations. To exploit their potential, they need to be adapted to STFEM; cf. [5, 2]. Time-dependent domains put a further facet of complexity on the numerical simulation of the flow problems; cf. [3] .

Here, families of STFEM with discrete solutions of increasing regularity in time are studied for the Navier–Stokes equations. Firstly, the potential of a Newton–GMRES solver with GMG preconditioning is illustrated for discontinuous Galerkin time-stepping schemes and the 3D DFG flow benchmark. Then, the approximation of the pressure trajectory in families of continuous in time Galerkin schemes is reviewed carefully. The lack of an initial pressure value yields a source of trouble for the definition of an equal-order in time pressure trajectory. A post-processing of the pressure trajectory by collocation techniques is introduced and optimal order error estimates are proved for the pressure approximation. Finally, extensions of SFTEM to flow problems on evolving domains and/or Galerkin time discretizations of higher regularity are discussed.

This is a joint work with M. Anselmann (Helmut Schmidt University), G. Matthies (University of Dresden) and F. Schieweck (University of Magedeburg).

References:

[1] M. Anselmann, M. Bause, *Higher order Galerkin-collocation time discretization with Nitsche's method for the Navier–Stokes equations*, Math. Comput. Simul., **189** (2021), pp. 141–162.
[2] M. Anselmann, M. Bause, *A geometric multigrid method for space-time finite element discretizations of the Navier–Stokes equations and its application to 3D flow simulation*, ACM Trans. Math. Softw., **submitted** (2021), pp. 1–27; arXiv:2107.10561.
[3] M. Anselmann, M. Bause, *CutFEM and ghost stabilization techniques for higher order space-time discretizations of the Navier–Stokes equations*, Int. J. Numer. Methods Fluids, **accepted** (2022),
doi: 10.1002/fld.5074, pp. 1–31; arXiv:2103.16249.
[4] M. Anselmann, M. Bause, G. Mathies, S. Schieweck, *Optimal order pressure approximation for the Stokes problem by a variational method in time with post-processing*, **in progress**, (2022).
[5] S. Hussain, F. Schieweck, S. Turek, *Efficient Newton-multigrid solution techniques for higher order space-time Galerkin discretizations of incompressible flow*, Appl. Numer. Math., **83** (2014), pp. 51–71.

Multiscale and homogenized modeling of vascular tissues

Belponer, Camilla
Weierstraß-Institut, Germany

In the context of medical imaging technique, Magnetic Resonance Elastography (MRE) can be used to create an elastogram, i.e., to obtain information about elastic tissue parameters. In clinical applications, MRE can then be used to characterize healthy and pathologic tissues and for the non-invasive diagnosis and monitoring of tissue diseases such as cancer and fibrosis. The goal of this work is to develop, analyze, and validate a multiscale and multiphysics description of a vascularized tissue. The considered model combines a reduced (one-dimensional) description of the flow in the vasculature with a three-dimensional elastic tissue description, based on the assumption that the radii of the vessels composing the vasculature is much smaller of the typical tissue length.

In order to reduce the computational effort, the 1D and the 3D model are coupled via an immersed boundary methods, resulting in a singular term in the elasticity equation, that does not require the explicit resolution of the 1D vasculature within the finite element mesh. The use of this non-matching immersed method is particularly suitable for the efficient simulation of vascularized tissues as it allows freedom in the choice of the mesh dimension with respect to the finest scale structures. We model the fluid vasculature using a one-dimensional model, numerically solved via a high order finite volume method for the cross-sectional area, the average cross sectional pressure, and the flow rate. The numerical scheme has an independent discretization of each vessel (one-dimensional segment) and uses a local time stepping technique. The 3D solid phase is treated as a linear elastic material whose dynamics is governed by a Poisson equation for the stress tensor. The right hand side is the singular term that encodes vessel position, pressure, and their deformation, and it has support on a one-dimensional manifold defined by the vascular network.

We show a detailed validation of the convergence properties of the multiscale coupled scheme in simple cases, discussing the role of mechanical and numerical parameters. Next, we investigate and analyze different approaches to couple the 1D and the 3D formulations, discussing their accuracy and stability properties considering a synthetic vascular tissue sample with a randomly generated microvasculature. Finally, we discuss possible strategies to derive an effective tissue description to solve multiscale inverse problems in the context of MRE, in which the scale of relevant quantities (e.g., the microvasculature structure or pressure) is below available data resolution.ext, we discuss possible strategies for upscaling the derived model towards an effective description of the vascularized tissue.

Vorticity-stabilized virtual elements for the Oseen equation

Dassi, Franco
University of Milano-Bicocca, Italy

In this talk we consider Oseen problems in a convection-dominated regime. To solve this kind of partial differential equation, we combine the virtual element divergence-free discretisation shown in Beirão da Veiga et al. 2018 and the stabilisation approach proposed in N. Ahmed et al. 2021. Such a stabilisation is based on SUPG-like terms of the vorticity equation and internal jump terms for the velocity gradients and it was properly adapted for the virtual element framework. We will provide some theoretical convergence results that underline both the robustness of the scheme in different regimes and the low influence of pressure on the velocity error. Finally, we will show some numerical results that validate the theory also from a numerical point of view.

A new perspective on time-stepping schemes: Beyond strong stability

Ern, Alexandre
École nationale des ponts et chaussées & INRIA, France

We introduce a technique that makes every explicit Runge–Kutta time stepping method invariant-domain preserving when applied to high-order discretizations of the Cauchy problem associated with nonlinear conservation equations. The main advantage over the popular strong stability preserving (SSP) paradigm is more flexibility in the choice of the ERK scheme, thus allowing for a less stringent restriction on the time step and circumventing order barriers. The technique is agnostic to the space discretization. In a second step, we extend the technique to implicit-explicit (IMEX) time-stepping schemes for the Cauchy problem where the evolution operator comprises a hyperbolic part and a parabolic part with diffusion and stiff relaxation terms. Numerical experiments are presented to illustrate the theory.
This is joint work with J.-L. Guermond (Texas A&M).

Optimal adaptive algorithms for indefinite and time-dependent problems

Feischl, Michael
Technische Universität Wien, Austria

In the recent work [Feischl, Math. Comp., 2022], we prove new optimality results for adaptive mesh refinement algorithms for non-symmetric, indefinite, and time-dependent problems by proposing a generalization of quasi-orthogonality which follows directly from the inf-sup stability of the underlying problem. This completely removes a central technical difficulty in modern proofs of optimal convergence of adaptive mesh refinement algorithms and leads to simple optimality proofs for the Taylor–Hood discretization of the Stokes problem, a finite-element/boundary-element discretization of an unbounded transmission problem, and an adaptive time-stepping scheme for parabolic equations. The main technical tools are new stability bounds for the LU-factorization of matrices together with a recently established connection between quasi-orthogonality and matrix factorization.

Super-localized orthogonal decomposition for convection-dominated diffusion problems

Freese, Philip
Universität Augsburg, Germany

In this talk, we present a multi-scale method for convection-dominated diffusion problems in the regime of large Péclet numbers. The application of the solution operator to piece-wise constant right-hand sides on some arbitrary coarse mesh defines a finite-dimensional coarse ansatz space with favorable approximation properties. For some relevant error measures, including the L^2-norm, the Galerkin projection onto this generalized finite element space even yields ε-independent error bounds.

We construct an approximate local basis that turns the approach into a novel multi-scale method in the spirit of the Super-Localized Orthogonal Decomposition (SLOD). The error caused by basis localization, which we conjecture to decay super-exponentially, can be estimated in an a posteriori way. Numerical experiments indicate ε-independent convergence without preasymptotic effects, even in the under-resolved regime of large mesh Péclet numbers.

On reducing spurious oscillations in discontinuous Galerkin methods for convection-diffusion equations

Frerichs-Mihov, Derk
Weierstraß-Institut, Germany

Approximating the solution of convection-diffusion equations in the convection-dominated regime by standard methods on affordable grids usually lead to unphysical values, so called spurious oscillations. Standard discontinuous Galerkin methods are known on the one hand to produce sharp layers but on the other hand are also not able to prevent the pollution of the solution. This talk introduces a post-processing method that uses so called slope limiter to automatically detect regions where the solution is polluted and to correct the solution in these regions. This talk introduces some state of art slope limiting techniques and shows numerical results when they are applied on standard benchmark problems.

Inverse problems on non-parametric domains. Flow reconstruction from medical data using non linear dimensionality reduction

Galarce, Felipe
Weierstraß-Institut, Germany

Co-authors:
D. Lombardi (Centre de Recherche INRIA de Paris & Laboratoire Jacques-Louis Lions, France)
O. Mula (Eindhoven Univerity of Technology)

Solving in real time inverse problems for biomedical applications requires learning techniques that involve simulations and databases from different patients which inevitably involve anatomical variations.
We present a state estimation method which allows to take this variability into account without needing any a priori knowledge on a parametrization of the anatomical differences. We rely on morphometric techniques involving Multidimensional Scaling and couple them with reconstruction techniques that make use of reduced modeling [1].
We prove the potential of the method on a simple application inspired from the reconstruction of blood flows and quantities of medical interest with Doppler ultrasound imaging [2, 3, 4].

References:

1] N. Saeed, H. Nam, M. Imtiaz Ul Haq and D. Muhammad Saqib Bhatti, *A Survey on Multidimensional Scalin*, ACM Comput., (2018).
[2] F. Galarce, J.F. Gerbeau, D. Lombardi and O. Mula, *State estimation with nonlinear reduced models*, Application to the reconstruction of blood flows with Doppler ultrasound images. SIAM Journal on Scientific Computing, (2021). arXiv:1904.13367.
[3] F. Galarce, D. Lombardi, O. Mula, *Reconstructing Haemodynamics Quantities of Interest from Doppler Ultrasound Imaging*, International Journal for Numerical Methods in Biomedical Engineering, (2020),
arXiv:2006.04174.
[4] F. Galarce, D. Lombardi, O. Mula, *Fast reconstruction of 3D blood flows from Doppler ultrasound images and reduced models*, Computer Methods in Applied Mechanics and Engineering, (2021), arXiv:1904.13367.

Robust error bounds for the Navier–Stokes equations using implicit-explicit second order BDF method with variable steps

García-Archilla, Bosco
Universidad de Sevilla, Spain

We present the analysis and numerical experimentation of a fully discrete method for the Navier-Stokes equations, using inf-sup stable elements and grad-div stabilization. For the time integration, the second order backward differentiation formula (BDF) with variable step sizes is used. We consider the cases where the nonlinear terms are treated explicitly and semi implicitly. Grad-div stabilization allows to obtain error bounds where the constants involved do not depend on the Reynolds number. When the nonlinear terms are treated explicitly, the resulting method suffers from a CFL-type restriction which can be checked in practical examples.

Nonlinear iterative approximation of steady flow of chemically reacting fluids

Gazca Orozco, Pablo Alexei
Charles University, Czech Republic

In this talk, I will present some recent results obtained in collaboration with P. Heid and E. Süli on an iterative scheme for computing the solutions of a system describing an incompressible fluid with power-law-like rheology, with a power-law exponent depending on a scalar variable that solves an advection-diffusion PDE; in particular, this exponent varies in space. Under small data assumptions, we prove that a Zarantonello-type iteration converges to the (unique) solution of the problem. The proposed iteration scheme is remarkably simple and it amounts to solving a linear Stokes-Laplace system at each step. I will conclude with numerical experiments and some discuss possible uses as a nonlinear preconditioner.

A multiscale hybrid-mixed model for Stokes–Brinkman–Darcy flows with divergence-compatible velocity-pressure elements

Gomes, Sonia
University of Campinas, Brazil

The current proposal extends a multiscale hybrid-mixed finite element approach, recently considered for Darcy's flows, to the whole range of Stokes and Brinkman problems. Based on divergence-compatible velocity-pressure finite elements in H(div) x L2, the formulation characterizes the approximate solution in terms of components given by well-posed global-local systems, which are subordinated to a partition of the flow domain by general subregions. Hybridization occurs by the introduction of new variables defined over the subregion boundaries (mesh skeleton): tangential traction multiplier (to weakly enforce tangential velocity continuity) and the velocity normal trace (making the H(div)-conforming inter-element connection). The finite element pair of spaces used in the subregions may have richer internal resolution than the boundary normal trace and traction multiplier. Stability of the method is ensured by the divergence compatibility property of velocity and pressure approximations and by a proper choice of the finite element space for the traction variable. Numerical results shall be presented for the verification of the main properties of the method.
This is a joint work with P.G.S. Carvalho and P.R.B. Devloo.

Adaptive grids for algebraic stabilizations of convection-diffusion-reaction equations

Jha, Abhinav
Rheinisch-Westfälische Technische Hochschule Aachen, Germany

Non-linear discretizations are necessary for convection-diffusion reaction equations for obtaining accurate solutions that satisfy the discrete maximum principle (DMP). Algebraic stabilizations belong to the very few finite element discretizations that satisfy this property.
In this talk, we consider three algebraically stabilized finite element schemes for discretizing convection-diffusion-reaction equations on adaptively refined grids. These schemes are the algebraic flux correction (AFC) scheme with Kuzmin limiter, the AFC scheme with BJK limiter **BJKR18.SeMa**, and the recently proposed Monotone Upwind-type Algebraically Stabilized (MUAS) method **JK21.arXiv**. Both, conforming closure of the refined grids and grids with hanging vertices are considered based on a residual-based a posteriori error estimator proposed in **Jha21.CAMWA**.
A non-standard algorithmic step becomes necessary before these schemes can be applied on grids with hanging vertices. The assessment of the schemes is performed with respect to the satisfaction of the global discrete maximum principle (DMP), the accuracy, e.g., smearing of layers, and the efficiency in solving the corresponding nonlinear problems.

Local pressure-corrections for incompressible flows

Kaya, Utku
Otto-von-Guericke Universität Magdeburg, Germany

Pressure-correction methods facilitate approximations of solutions to time-dependent incompressible fluid flows by decoupling the momentum equation from the continuity equation [1]. A common strategy used by several pressure-correction methods is:
(i) compute a (not necessarily divergence-free) predictor velocity field,
(ii) solve a Poisson problem for the pressure,
(iii) project the predictor velocity field onto a divergence-free one.

In cases, where an explicit time-stepping scheme is employed for the momentum equation, the Poisson problem for the pressure remains to be the most expensive step. We here present a domain decomposition method that replaces the pressure Poisson problem from step (ii) with local pressure Poisson problems on non-overlapping subregions [2,3]. No communication between the subregions is needed, thus the method is favorable for parallel computing. We illustrate the effectivity of the method via numerical results.

References
[1] J.L. Guermond, P. Minev, J. Shen, *An overview of projection methods for incompressible flows*, Comput. Methods Appl. Mech. Engrg., **195**:44-47 (2006), 6011–6045.
[2] M. Braack, U. Kaya, *Local pressure-correction for the Stokes system*, J. Comput. Math., **38**:1 (2020), 125–141.
[3] U. Kaya, R. Becker, M. Braack, *Local pressure-correction for the Navier–Stokes equations*, Internat. J. Numer. Methods Fluids., **93**:4 (2021), 1199–1212.

Anisotropic Brezzi-Douglas-Marini interpolation with applications to incompressible flows

Kempf, Volker
Universität der Bundeswehr München, Germany

Recently there has been increased interest in pressure-robust incompressible flow discretizations, where the velocity error does not depend on the pressure. In this context, the Brezzi-Douglas-Marini finite element is of note, as there are exactly divergence-free H(div)-conforming (hybrid) discontinuous Galerkin methods that use it for the velocity discretization, and the associated interpolation operator can be used as reconstruction operator for classical mixed elements to obtain a pressure-robust method. Well known challenges in the numerical solution of flow problems are boundary layer structures in the solution near walls and singularities near re-entrant edges. Both challenges can be overcome by using local anisotropic mesh grading to improve the approximation, which in turn complicates the error analysis. This talk presents recent and new results concerning anisotropic Brezzi-Douglas-Marini interpolation on simplicial and prismatic meshes and shows some numerical examples.

Entropy-based methods for rarefied gas flows

Keßler, Torsten
Universität des Saarlandes, Germany

The Boltzmann equation is the fundamental equation for the description of rarefied gas flows that also remains valid in the case of increasing density, where its leading approximations yield the Euler and Navier-Stokes equations, respectively. In this talk, we develop an entropy-stable finite element-moment method for the Boltzmann equation. To that end, we employ a discontinuous-Galerkin method in space and time, and a spectral method in unbounded velocity space. We base our method on a converging sequence of approximations to the collision operator. We associate with each member of this sequence a normalisation map and an entropy. We show that each approximate collision operator inherits salient properties from Boltzmann's operator, such as the preservation of mass, momentum and energy, and that the linearisations near equilibrium agree. We prove that our method is entropy-stable for each member of the sequence of approximations to the collision operator. Finally, we apply our method to the Boltzmann equation with full collision operator and demonstrate the corresponding approximation properties, using benchmark test cases, in comparison to Direct Simulation Monte Carlo.
This is joint work with M. R. A. Abdelmalik, I. M. Gamba and S. Rjasanow.

A posteriori error control in the maximum norm for convection-dominated convection-diffusion equations

Kopteva, Natalia
University of Limerick, Ireland

Standard and stabilized finite element approximations are considered on shape-regular meshes for singularly perturbed convection-diffusion equations. Our initial result is that natural maximum-norm residual estimators reliably control the error in the maximum norm, assuming suitable estimates of the Green's function hold. On the other hand, residual-type estimators in the energy norm are only efficient up to a dual norm of the convective error (as shown by R. Verfürth). A main contribution of the paper is to analogously define a suitable dual seminorm of the convective error. Having defined such a dual norm, we then define the total error as the originally targeted maximum norm of the error plus the dual seminorm of the convective error plus standard data oscillation terms. Our a posteriori error estimator is then shown to be equivalent to the total error (up to a logarithmic factor). Numerical experiments illustrate the behavior and performance of our estimators in the context of uniform and adaptive mesh refinement. In particular, they show that the estimators may vastly overestimate the error in the maximum norm alone, but they closely track the total error as predicted by our theory. Adaptive refinement based on our error indicators is also shown to do an effective job at automatically resolving standard model problems whose solutions include strong layers.
This is joint work with Alan Demlow and Sebastian Franz.

An EMA-conserving, pressure-robust and Re-semi-robust reconstruction method for incompressible Navier–Stokes simulations

Li, Xu
Shandong University, China

Proper EMA-balance (E: kinetic energy; M: momentum; A: angular momentum), pressure-robustness and Re-semi-robustness (Re: Reynolds number) are three important properties of Navier–Stokes simulations with exactly divergence-free elements. This EMA-balance makes a method conserve kinetic energy, linear momentum and angular momentum under some suitable senses; pressure-robustness means that the velocity errors are independent of the continuous pressure; Re-semi-robustness means that the constants appearing in the error bounds of kinetic and dissipation energies do not explicitly depend on inverse powers of the viscosity. In this paper, based on the pressure-robust methods in [1], we propose a novel reconstruction method for a class of non-divergence-free simplicial elements which admits almost all the above properties. The only exception is the energy balance, where kinetic energy should be replaced by a properly redefined discrete energy. Our lowest order case uses the Bernardi-Raugel element on general shape-regular meshes. Some numerical comparisons with exactly divergence-free methods, pressure-robust reconstructions and the EMAC scheme are provided to confirm our theoretical results.

References:

[1] A. Linke and C. Merdon, *Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., **311** (2016), pp. 304–326.

Pressure-robust discretisations for the generalised Stokes problem on non-affine meshes

Matthies, Gunar
Technische Universität Dresden, Germany

Based on inf-sup stable finite element pairs of arbitrary order with discontinuous pressure approximation, we present a modified discretisation of the generalised Stokes problem which provides pressure robustness and maintains optimal error estimates. The approach uses a reconstruction operator mapping discretely divergence-free functions to divergence-free ones. In particular, we consider non-affine meshes and discuss the impact of the Piola transform for defining appropriate discrete function spaces.

Inf-sup stabilized Scott–Vogelius pairs on general simplicial grids by Raviart–Thomas enrichment

Merdon, Christian
Weierstraß-Institut, Germany

This talk presents a novel way to stabilize the Scott–Vogelius finite element method for the Stokes equations on arbitrary regular simplicial grids.
The key idea, inspired by [1] for $k = 1$, consists in enriching the continuous polynomials of order k of the Scott–Vogelius velocity space with appropriately chosen and explicitly given Raviart–Thomas functions. The proposed method is inf-sup stable and pressure-robust. The optimally converging H^1-conforming velocity comes with a small $H(\mathrm{div})$-conforming correction that renders the full velocity divergence-free. For $k \geq d$ in d dimensions, the method is parameter-free.
The additional degrees of freedom for the Raviart–Thomas enrichment and all non-constant pressure degrees of freedom can be condensated, which effectively results in a pressure-robust, inf-sup stable, optimally convergent $P_k \times P_0$ scheme. Some aspects are discussed and numerical studies confirm the analytic results.

References:

[1] Xu Li, Hongxing *, A low-order divergence-free H(div)-conforming finite element method for Stokes flows*, IMA Journal of Numerical Analysis, (2021).

Reduced order models for incompressible flows

Novo, Julia
Universidad Autónoma de Madrid, Spain

We consider proper orthogonal decomposition (POD) methods to approximate the incompressible Navier-Stokes equations. Our aim is to get error bounds with constants independent of inverse powers of the viscosity parameter. This type of error bounds are called robust. In the case of small viscosity coefficients and coarse grids, only robust estimates provide useful information about the behavior of a numerical method on coarse grids. To this end, we compute the snapshots with a full order stabilized method (FOM). We also add stabilization to the POD method. We study a case in which non inf-sup stable elements are used for the FOM and a case in which inf-sup stable elements are used. In the last case to approximate the pressure we use a supremizer pressure recovery method.

We show that in case we have some a priori information about the velocity, a POD data assimilation algorithm converges to the true solution exponentially fast improving the accuracy of the standard POD method.

In practical simulations one can apply some given software to compute the snapshots. It could then be the case that a different discretization for the nonlinear term is used in the FOM and the POD methods. We analyze the influence of using different discretizations for the nonlinear term. Finally, we also analyze the influence of including snapshots that approach the velocity time derivative. We study the differences between projecting onto L^2 and H^1 and prove pointwise in time error bounds in both cases.

Numerical analysis of a residual-based stabilization-motivated POD-ROM for incompressible flows

Oulghelou, Mourad
Arts et Métiers ParisTech, France

In this work, error bounds for a velocity-pressure segregated POD-ROM discretization of the Navier-Stokes equations are presented. The stability of the studied reduced order model is proven in specific norms. For velocity, in $L^\infty(L^2)$ and energy norms, with bounds that do not depend on the viscosity, and for pressure, in a semi-norm of the same asymptotic order as the L^2 norm with respect to the grid size. To check the theoretical estimates, two flow examples are studied, the flow past a cylinder and the lid-driven cavity flow. The estimates quality is then assessed in terms of their logarithmic slop with respect to the velocity POD contribution ratio. We show that the proposed error estimates allow a good approximation of the real errors slops and thus a good prediction of the their rate of convergence.

Local computation of pressure in high order finite element methods for Stokes equations

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We will suggest a new economic high-order finite element method for Stokes equations. Its main character is the local computation for the pressure. For an example, in $P^4 - P^3$ case, the method solves first the decoupled equation for a P^4-velocity. Then, a P^3-pressure p_h is calculated by local computation in the successive 5 steps. The resulting P^3-pressure is analyzed to have the optimal order $O(h^4)$ of convergence. Besides, the method overcomes the singular vertex annoying us in Scott–Vogelius finite element methods.

Reconstruction of flow domain boundaries from velocity data via multi-step optimization of distributed resistance

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We reconstruct the unknown shape of a flow domain using partially available internal velocity measurements. This inverse problem is motivated by applications in cardiovascular imaging where motion-sensitive protocols, such as phase-contrast MRI, can be used to recover three-dimensional velocity fields inside blood vessels. In this context, the information about the domain shape serves to quantify the severity of pathological conditions, such as vessel obstructions. We consider a flow modeled by a linear Brinkman problem with a fictitious resistance accounting for the presence of additional boundaries. To reconstruct these boundaries, we employ a multi-step gradient-based variational method to compute a resistance that minimizes the difference between the computed flow velocity and the available data. Afterward, we apply different post-processing steps to reconstruct the shape of the internal boundaries. We validate our method on three-dimensional examples based on synthetic velocity data and using realistic geometries obtained from cardiovascular imaging.

Discretization of compressible Stokes flow using Hamiltonian and Onsager structures

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In my contribution I will motivate energy-variational formulations of compressible fluid flows in Lagrangian and Eulerian form, discuss a transformation framework that relates Hamiltonian and Onsager structures in both coordinate systems, present some simple numerical experiments and discuss the extension to viscoelastic flows.

Fluid-rigid body interactions with large motion and contact

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We consider a simple problem: a solid, e.g. a sphere, is falling towards the bottom in a container filled with a liquid.

First, we present efficient simulation methods that cope with this large-displacement problem. The fundamental problem is that the solid is moving substantially and therefore the flow domain undergoes strong changes.

Then we deal with the contact of the solid with the bottom of the container. Analytically this problem is already well studied and here the unpleasant result is that contact in finite time is not possible if the fluid will be assumed to be incompressible. In reality, however, we will find that the ball bounces off the ground and comes to rest only after a few contacts.

This situation is difficult to grasp numerically. Contact implies a topology change of the domain. We introduce different modeling and discretization approaches and present numerical benchmarks, which are also backed by experimental data.

Bathymetry reconstruction via optimal control in well-balanced finite element methods for the shallow water equations

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The reconstruction of bathymetry from known free surface elevation via numerical solution of the shallow water equations (SWE) is an ill-conditioned and generally ill–posed inverse problem. Without proper regularization, a method may fail to converge, or steady-state topography may exhibit spurious oscillations even if a smooth exact solution is known to exist. The presence of noise in the data, or a poor choice of discretization techniques aggravates such issues. To filter out perturbations caused by ill-conditioning and/or presence of unresolvable fine-scale features, a numerical method for the inverse SWE problem must be equipped with carefully designed stabilization operators [2].

In this work, we discretize the two-dimensional shallow water equations using continuous linear finite elements. The current implementation uses a new well-balanced extension of the algebraic Lax–Friedrichs method analyzed in [1] to the SWE system with non-flat topography. The steady-state bathymetry is calculated via time marching. Two approaches are used to cure the lack of well-posedness and avoid oscillations. The first one adds an artificial diffusion term to the conservation law for the water height (as in [2]). In the second approach, the regularization term consists of numerical fluxes that are constructed using a new optimal control method. An optimization problem is formulated for scalar flux potentials with the aim of minimizing the perturbation of the discretized shallow water equations and deviations from the measured free surface elevation. To suppress oscillations caused by non-smooth data, we use the total variation denoising approach developed in [3]. The first numerical results for one- and two-dimensional test problems are promising. These preliminary results include convergence studies for benchmarks with noise in the initial data and experiments with discontinuous bathymetry.

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On the inf-sup stabillity of Crouzeix–Raviart Stokes elements in 3D

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We consider non-conforming discretizations of the stationary Stokes equation in three spatial dimensions by Crouzeix–Raviart type elements. The original definition in the seminal paper by M. Crouzeix and P.-A. Raviart in 1973 is implicit and also contains substantial freedom for a concrete choice. In our talk, we introduce canonical Crouzeix-Raviart basis functions in 3D in analogy to the 2D case in a fully explicit way. We prove that this canonical Crouzeix–Raviart element for the Stokes equation is inf-sup stable for polynomial degree k = 2 (quadratic velocity approximation). We identify spurious pressure modes for the conforming (k,k-1) 3D Stokes element and show that these are eliminated by using the canonical Crouzeix–Raviart space.

Hybrid H(div) - conforming finite elements for incompressible flows

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In this talk we present discretization techniques for Stokes and Navier–Stokes equations based on H(div)-conforming finite elements for the velocity. One family of methods is based on hybridized discontinuous Galerkin (HDG) methods, the other one is a mixed formulation introducing the stress as a new variable. Since the obtained velocity approximation is exactly divergence free, we obtain stability for high Reynolds numbers. We compare energy dissipation for direct simulation (i.e. an implicit turbulence model) with various established methods for turbulence. H(div)-conformity allows the mapping of vector-fields by the Piola transform. Thus, the exact divergence free conditon is preserved under morphing of the domain. We show examples from fluid-structure interaction as well as model order reduction where this property is of advantage. The presented methods are available within the open source finite element package NGSolve.

Numerical treatment of the vectorial equations of solar oscillations

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The Galbrun's equation with additional rotational and gravitational terms model stellar oscillations. Recently, well-posedness of the continuous problem was proven, by using a suitable generalized Helmholtz decomposition in the analysis. In the discretization we aim to preserve a discrete version of the generalized Helmholtz decomposition, which is crucial for stability and helpful for the numerical analysis. The decomposition presents a strong connection between stable discretizations for the Galbrun equations and Stokes and nearly incompressible linear elasticity problems. We derive conditions on the discretization that preserve the structural properties of the continuous problem and introduce the tools needed for the numerical analysis.

A PU localization for space-time goal oriented adaptive refinement for incompressible flow problems

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In this talk, we will present a space-time extension of the variational partition of unity (PU) approach [3] for localization of dual weighted residual (DWR) type error estimators [2]. This allows for estimation and control of both spatial and temporal errors w.r.t. a given quantity of interest. For CFD, typical quantities are drag and lift coefficients. The focus will be on application of the novel PU localization [4] to the global DWR estimator for adaptive refinement. Finally, we will discuss some numerical results based on the tensor-product space-time code introduced in [1] that substantiate our developments.

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Virtual elements for a fluid-structure interaction problem

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In collaboration with L. Beirão da Veiga, C. Canuto, R. H. Nochetto

The Virtual Element Method (VEM) is a recent technology introduced in [Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, 2012, M3AS] for the discretization of partial differential equations.
The VEM can be interpreted as a novel approach that generalizes the classical Finite Element Method to arbitrary even non-convex element-geometry.
The present talk is both an introduction to the VEM for the Stokes equation, aiming at showing the main ideas of the method, and a brief look at some application to fluid-structure interaction problems.
In the first part part of the talk we will describe the basics of the divergence-free virtual elements for the Stokes equation.
In the second part, we will present a first application of VEM for a fluid structure interaction problem. We study, both theoretically and numerically, the equilibrium of a hinged rigid leaflet with an attached rotational spring, immersed in a stationary incompressible fluid within a rigid channel.

Space-time discontinuous Galerkin methods for weak solutions of hyperbolic linear symmetric Friedrichs systems

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We study weak solutions and its approximation of hyperbolic linear symmetric Friedrichs systems describing acoustic, elastic, or electro-magnetic waves. Stability and convergence estimates are provided for a discontinuous Galerkin discretization in space and time with respect to a mesh-dependent DG norm, where we also consider the case of piecewise discontinuous weak solutions. A reliable error estimator is constructed, and numerical results demonstrate the efficiency of the approach.
This is a joint work with Daniele Corallo and Willy Dörfler.

Asymptotic-preserving plasma model in 3D

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A key coefficient characterizing plasmas is the so-called Debye length λ_D, whose size indicates to what extent electric charges can deviate from the neutral case: If λ_D = *O(1)* we face the non-neutral regime, while for $\lambda_D \to 0$ the plasma becomes quasi-neutral. Plasma models have rather different properties in these two regimes due to the singular perturbation arising from the $\lambda_D \to 0$ limit. Since both regimes may coexist in some plasma phenomena, it is desired to design numerical schemes that are *robust* for arbitrary λ_D. More precisely, they should be *asymptotic-preserving* (AP), in the sense that the limit $\lambda_D \to 0$ of the scheme yields a viable discretization for the continuous limit model.
An asymptotic preserving scheme for single- and multi-fluid Euler-Maxwell system was proposed in [1] for one spatial dimension. We start from their dimensionless model and extend the scheme to three dimensions. The key ingredients are
(i) a discretization of Maxwell's equations based on primal and dual meshes in the spirit of discrete exterior calculus (DEC)/the finite integration technique (FIT),
(ii) finite volume method (FVM) for the fluid equations on the dual mesh, and
(iii) mixed implicit-explicit timestepping.

This scheme turns out to be AP for $\lambda_D \to 0$ both in terms of structure and empirically in numerical test. Special care is necessary for the boundary conditions which must be valid in both regimes. Additionally, if the electromagnetic fields have to be modeled in an insulating region beyond the plasma domain, additional stabilization is necessary to accommodate Gauss's law.

References:

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