# Order conditions and efficiency of multirate Runge-Kutta methods for one-dimensional atmosphere-ocean models

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## General overview

Coupling of components of earth system models (e.g. atmosphere, ocean, land etc.) is usually carried out by a coupler. Advantages of this strategy are the independent development and easy including / excluding of components. Furthermore, the time steps for the integration can be controlled by the coupler.
Disadvantages are the interpolation between the different grid structures of the components and that the coupling is utilised with a data exchange at specific time points.
A different approach is the combination of all components at once and to conduct the time integration with a multirate method. This gives the opportunity to apply individual time steps for each part. Moreover, the coupling is utilised within the set of equations. Additionally, no interpolation is required with an appropriate discretization.
Therefore, the amount of required computing resources is reduced due to less data exchange processes. On the other hand, the right choice of a multirate integration method requires the consideration of all coupling properties.
The following gives an overview about multirate Runge-Kutta (RK) methods.

## Multirate Runge-Kutta methods for one-dimensional atmosphere-ocean models

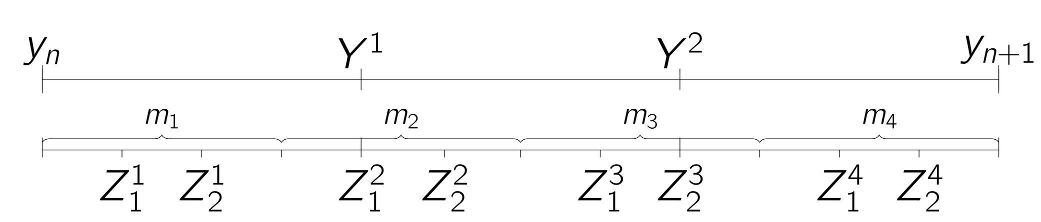
### 1D atmosphere-ocean model

- Integration of set of equation: $\dot{y} = F(y)$
- $y = y(z, t)$ - only vertical coordinates
- Splitting into two parts, i.e. atmosphere $g(y)$ and ocean $f(y)$: $\dot{y} = F(y) = f(y) + g(y)$
- No further splitting due to (non-)stiff parts of set of equation

### Multirate Generalized Additive Runge-Kutta methods (MGARK):

Günther et al., 2016, Bremicker-Trübelhorn et al., 2017

Exemplary time interval for MGARK




Extended Butcher tableau:

$$\begin{array}{ccc|c} m_1\cdot\mathbf{A}^{g,g} & & & \mathbf{A}^{g,f,1} \\ \vdots & \ddots & & \vdots \\ m_1\cdot\mathbf{b}^{g,g^T} & \dots & m_N\cdot\mathbf{A}^{g,g} & \mathbf{A}^{g,f,N} \\ \hline m_1\cdot\mathbf{A}^{f,g,1} & \dots & m_N\cdot\mathbf{A}^{f,g,N} & \mathbf{A}^{f,f} \\ \hline m_1\cdot\mathbf{b}^{g,g^T} & \dots & m_N\cdot\mathbf{b}^{g,g^T} & \mathbf{b}^{f,f^T} \end{array}$$

$$\begin{array}{c|c} \mathbf{c}^{f,f} & \mathbf{A}^{f,f} \\ \hline & \mathbf{b}^{f,f^T} \end{array} \qquad \begin{array}{c|c} \mathbf{c}^{g,g} & \mathbf{A}^{g,g} \\ \hline & \mathbf{b}^{g,g^T} \end{array}$$

$$\sum_{\lambda=1}^{N} [m_\lambda] = 1$$

Order conditions for up to order $p$:

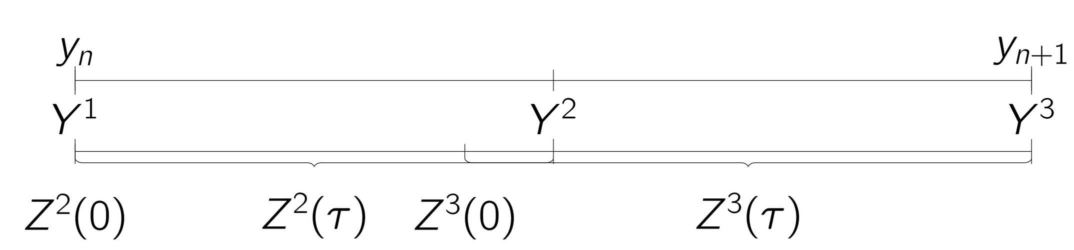
Runge-Kutta order conditions for up to order $p$ and

$p = 1:$ 1 add. condition
$p = 2:$ 2 add. conditions
$p = 3:$ 10 add. conditions

### Multirate Infinitesimal Step methods ($MIS_{\alpha\gamma}$):

$MIS_{10}$: Knoth et al., 1998, $MIS_{\alpha\gamma}$: Wensch et al., 2009, Knoth et al., 2014

Exemplary time interval for $MIS_{\alpha\gamma}n$




Integration scheme:

$$\begin{aligned} Y^1 &= y_n, \\ r_i &= \frac{1}{h}\cdot\sum_{j=1}^{i-1}\left[\gamma_{i,j}\cdot\left(Y^j - y_n\right)\right] + \sum_{j=1}^{i-1}\left[\beta_{i,j}\cdot f\left(Y^j\right)\right] \\ Z^i(0) &= y_n + \sum_{j=1}^{i-1}\left[\alpha_{i,j}\cdot\left(Y^j - y_n\right)\right] \\ \frac{dZ^i}{d\tau} &= \frac{1}{d_i}\cdot r_i + g\left(Z^i\right), \quad \tau\in[0, d_i\cdot h], \qquad (1) \\ Y^i &= Z^i\left(d_i\cdot h\right), \qquad i = 2, \dots, s^E + 1 \\ y_{n+1} &= Y^{s^E+1}. \end{aligned}$$

- $\begin{array}{c|c} \mathbf{c}^E & \mathbf{A}^E \\ \hline & \mathbf{b}^{E^T} \end{array}$, (explicit)
- $\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathbb{R}^{s^E+1, s^E+1}$ with $\alpha_{i,j} = \gamma_{i,j} = \beta_{i,j} = 0$ for all $j \geq i$,
- $\begin{bmatrix} \mathbf{A}^E & \mathbf{0} \\ \mathbf{b}^{E^T} & 0 \end{bmatrix} = (\mathbf{I} - \boldsymbol{\alpha} - \boldsymbol{\gamma})^{-1}\cdot\boldsymbol{\beta}$,
- $d_i = \sum_{j=1}^{i-1}[\beta_{i,j}]$.

Order conditions for up to order $p$:

Runge-Kutta order conditions for up to order $p$ and

$p = 1:$ 0 add. condition
$p = 2:$ 1 add. condition
$p = 3:$ 4 add. conditions

for explicit Runge-Kutta method

### Partitioned Runge-Kutta methods (PRK):

Jackiewicz et al., 2000

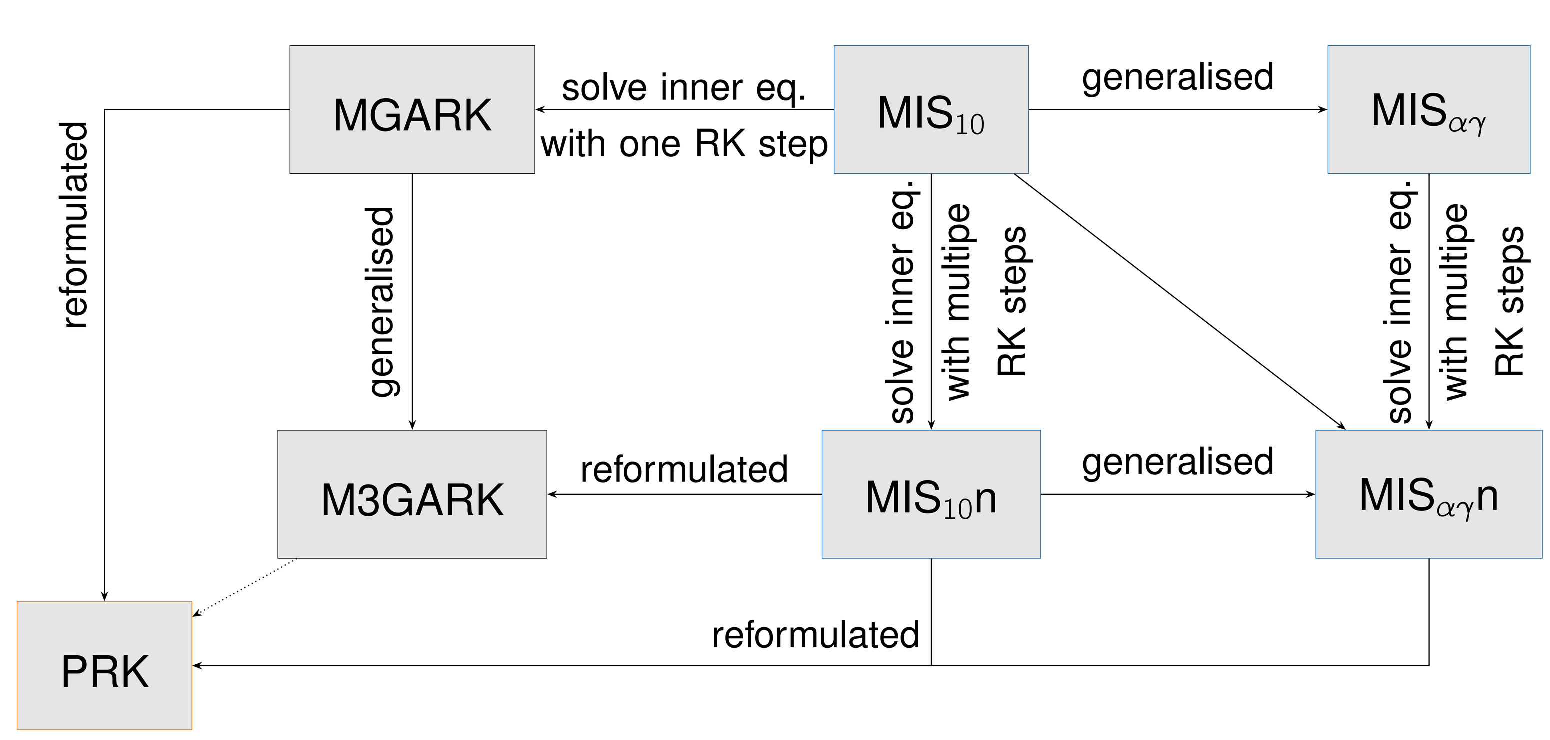
Integration scheme:

$$Y^i = y_n + h\cdot\sum_{j=1}^{s}\left[a_{i,j}^f\cdot f\left(Y^j\right) + a_{i,j}^g\cdot g\left(Y^j\right)\right]$$

$$y_{n+1} = y_n + h\cdot\sum_{i=1}^{s}\left[b_i^f\cdot f\left(Y^i\right) + b_i^g\cdot g\left(Y^i\right)\right]$$

Order conditions for up to order $p$:

$p = 1: \# = 2$
$p = 2: \# = 4$
$p = 3: \# = 14$




### $MIS_{\alpha\gamma}$: Solving of equation (1):

**Analytic approach**:

$d_i = 0$: $Y^i = Z^i(0) + h\cdot r_i$

$d_i \neq 0$: Analytic solving of $\frac{dZ^i}{d\tau} = \frac{1}{d_i}\cdot r_i + g\left(Z^i\right)$

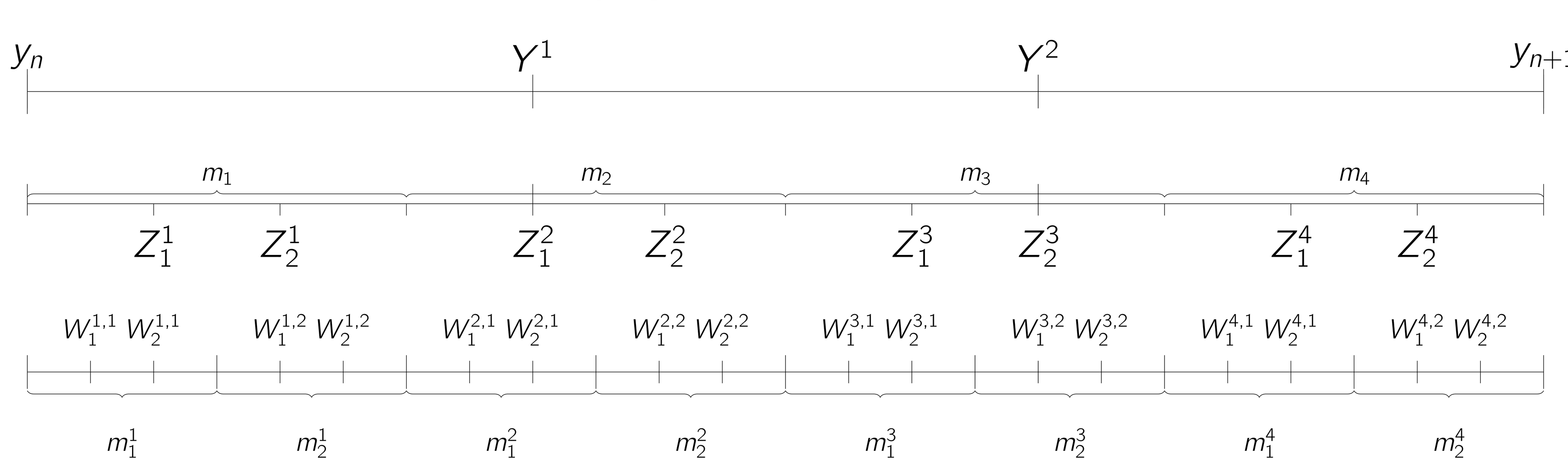
**Numerical approach, i.e. ($MIS_{\alpha\gamma}n$)**:

- Applying (new) RK method with $n$ time steps for interval $[0, d_i\cdot h]$
- No additional order conditions, i.e. only standard order conditions for RK method
- Only $MIS_{10}n$ transferable into M3GARK

**Note**: Derivation of high-order explicit RK methods for $MIS_{\alpha\gamma}n$ can be achieved by use of order conditions from PRK methods.

### Multirate Generalized Additive Runge-Kutta Method for 3 time steps (M3GARK):

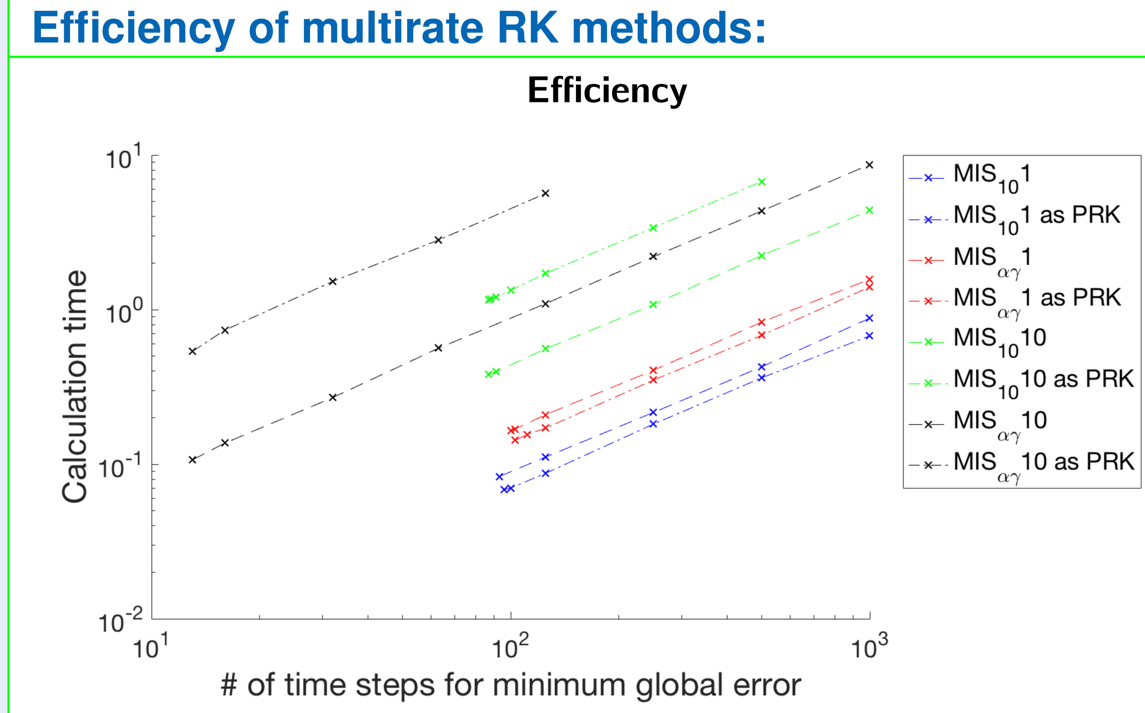
Exemplary time interval for M3GARK




Order conditions for for order $p = 3$:

$p = 1: \# = 5, \quad p = 2: \# = 9, \quad p = 3: \# = 46, \quad \Rightarrow$ 56 *additional conditions*

### Efficiency of multirate RK methods:




Example:

$$\dot{y} = -\begin{bmatrix} 0.1 & 1 \\ 1 & 0.1 \end{bmatrix}\cdot\frac{\partial y}{\partial x}$$

$t\in[0, 1], \; x\in[0, 1]$

Discretization:
MOL with $\Delta x = \frac{1}{500}$

Global error against exact solution: 0.10829

## Outlook

- Extending approach to 2D (3D) atmosphere-ocean models
- Verifying of efficiency results on more complex examples and applying most efficient strategy to idealised atmosphere-ocean models
- Splitting into more than two parts, i.e. applying multirate RK methods with more than two time steps, e.g. M3GARK
- Comparing of model results against use of multirate RK methods and coupler strategy

## References

Bremicker-Trübelhorn, Sascha and Sigrun Ortleb (2017). "On Multirate GARK Schemes with Adaptive Micro Step Sizes for Fluid-Structure Interaction: Order Conditions and Preservation of the Geometric Conservation Law". In: *Aerospace* 4.1, p. 8.
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