Weierstrass Institute for
Applied Analysis and Stochastics

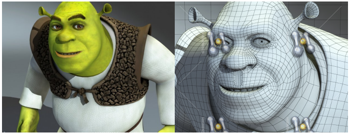
Leibniz Association

Anisotropic Finite Element Mesh Adaptation through High-Dimensional Embeddings

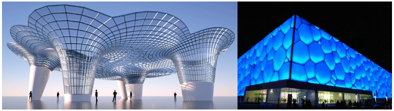
Franco Dassi, Hang Si, Simona Perotto and Timo Streckenbach

- Meshes are backbone in 3d applications.

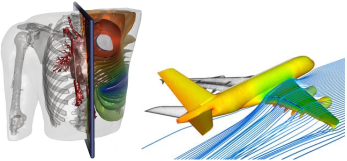
Courtesy: David Gu, Uni. Stonybrook
Computer Graphics



Courtesy: DreamWorks Pictures
Computer Animation



Courtesy: (L) Uni Stanford (R) Water Cube Stadium in Beijing
Architecture Design

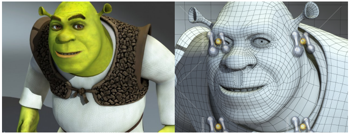


Courtesy: (L) Uni Utah (R) www.dlr.de
Numerical Simulation

Motivation

- Meshes are backbone in 3d applications.
- **The Problem**: How to generate a "good"mesh?

Courtesy: David Gu, Uni. Stonybrook
Computer Graphics



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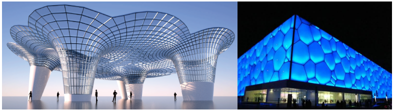

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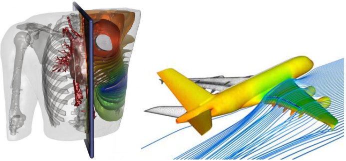


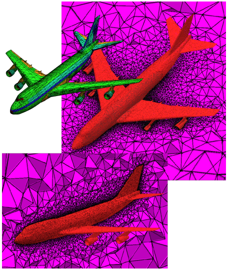
Courtesy: (L) Uni Utah (R) `www.dlr.de`
Numerical Simulation

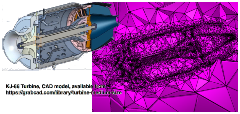
- Meshes are backbone in 3d applications.
- **The Problem**: How to generate a "good"mesh?

In 2001, H. Edelsbrunner wrote:

> *... Mesh generation is a topic in which a meaningful combination of different approaches to problem solving is inevitable.*

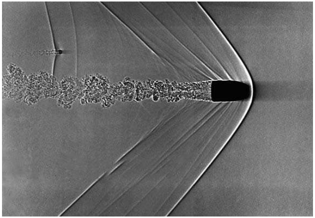
- A research project of WIAS since 2002.
- The goal is two-fold:
  - to study the underlying mathematical problems; and
  - to develop robust and efficient algorithms and softwares.
- It is freely available at `http://www.tetgen.org`.
  - latest version 1.5 (released in Nov. 2013).
  - about 10,000 downloads (Nov. 2013 - now).
  - about 20+ commercial licenses.



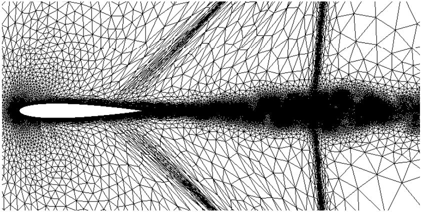


KJ-66 Turbine, CAD model, available from
https://grabcad.com/library/turbine-medium-size

- **Anisotropic meshes** are very important in many numerical simulations to capture the physical behavior of a complex phenomenon at a reasonable computational cost.
- It is a very complex and challenging problem.

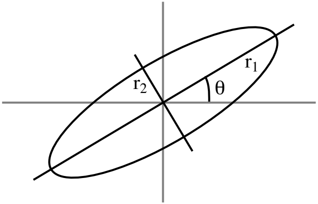


Courtesy: A. Davidhazy

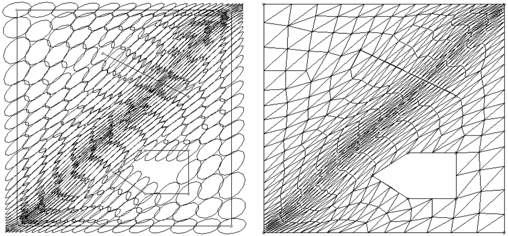


Courtesy: P. Frey

- Anisotropy is due to that the “space” is not flat, i.e., its geometry is non-Euclidean.
- Anisotropy can be described through a field $\mathcal{M}$ of metric tensors associated with a space domain $\Omega \subseteq \mathbb{R}^d$, where each metric tensor $M(\mathbf{x}) \in \mathcal{M}, \mathbf{x} \in \Omega$ is a $d \times d$ symmetric positive definite matrix.
- A metric tensor M can be geometrically represented by an oriented ellipse defined by its eigenvalues and eigenvectors.



- In the majority of works concerning anisotropic mesh generation, a (discrete) metric tensor field $\mathcal{M}$ (e.g., defined on the vertices) is used to describe the anisotropic feature of the domain. Then, a *uniform mesh* with equal (geodesic) edge length with respect to the metric tensor field $\mathcal{M}$ is sought. This will produce an anisotropic mesh of that domain.



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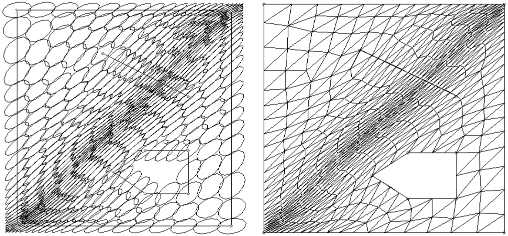
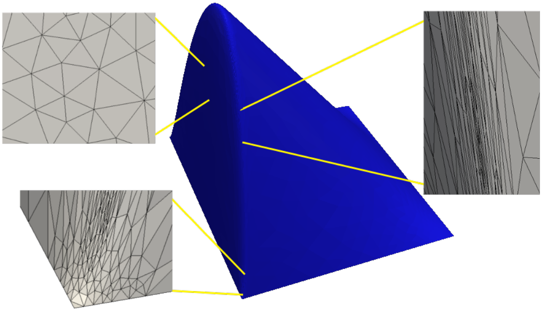
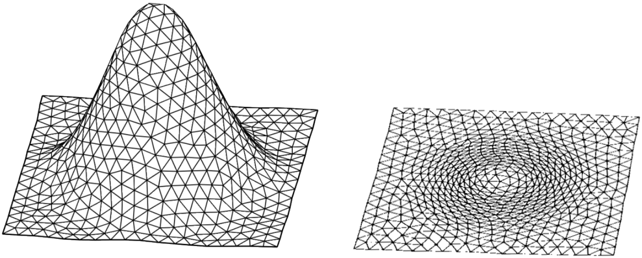


- In this work, we propose a new mesh adaptation approach, in which no metric tensor fields are involved.

Question 1: How to approximate a surface $\Gamma \subset \mathbb{R}^3$ with a small number of mesh elements.



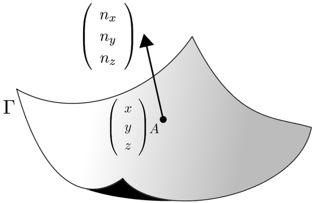
The Idea: Use additional dimensions to resolve the anisotropy.



(Courtesy of B. Lévy)

This example shows that an anisotropic mesh in $\mathbb{R}^2$ *corresponds to an isotropic mesh in* $\mathbb{R}^3$.

Let Γ be a surface in $\mathbb{R}^3$. Let $\phi : \Gamma \subset \mathbb{R}^3 \to \mathbb{R}^6$ be a map defined as,



$$\phi(A) := \begin{bmatrix} x \\ y \\ z \\ sn_x \\ sn_y \\ sn_z \end{bmatrix}$$

where A is a point in surface Γ whose coordinates are x, y and z, respectively, and n_x, n_y and n_z are the components of the normal to the surface Γ at the point p.
The constant $s \in (0, +\infty)$ is a parameter for capturing the anisotropy.

Question 2: How to generate an uniform mesh for the surface $\phi(\Gamma)$ in $\mathbb{R}^6$.

- Directly generalizing the existing algorithms in $\mathbb{R}^3$ is impractical due to the memory limitations.
- The `vorpaline` algorithm [**Lévy and Bonneel 2012**] – optimizing an d-dimensional Centroidal Voronoi Tessellation (CVT).
- ...our approach.

Define the scalar product in $\mathbb{R}^6$ to be:

$$(A,B)_{\text{6d}} = \underbrace{x_A x_B + y_A y_B + z_A z_B}_{I} + s^2 \big(\underbrace{n_x w_x + n_y w_y + n_z w_z}_{II}\big)\,.$$

This parameter will balance the contribution of the quantities I and II on whole value of $(A,B)_{\text{6d}}$. Since $I \in [-d^2, d^2]$ and $II \in [-1,1]$, where d is the measure of the diagonal of the bounding box of Γ, we need an additional constant to make I and II almost comparable. We decide to modify $(A,B)_{\text{6d}}$ in such a way

$$(A,B)_{\text{6d}} = x_A x_B + y_A y_B + z_A z_B + (h_\Gamma s)^2 \left(n_x w_x + n_y w_y + n_z w_z\right)\,.$$

where

$$h_\Gamma = \frac{d_x + d_y + d_z}{3}\,,$$

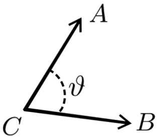
here d_x, d_y and d_z are the dimension of the bounding box of Γ.

Given two points A and B that lie on the surface Γ, we define the length of the segment l_{AB}^{6d} as

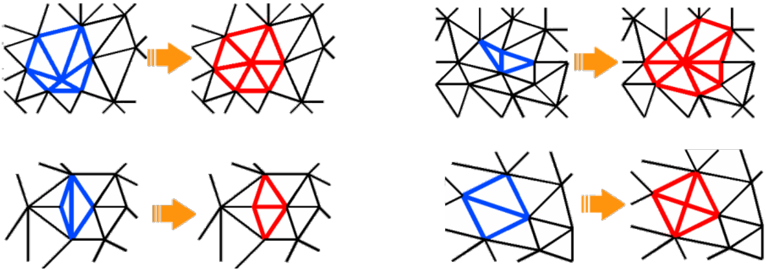
$$l_{AB}^{6d} := ||A - B||_{\text{6d}} = \sqrt{(A - B, A - B)_{\text{6d}}}\,.$$

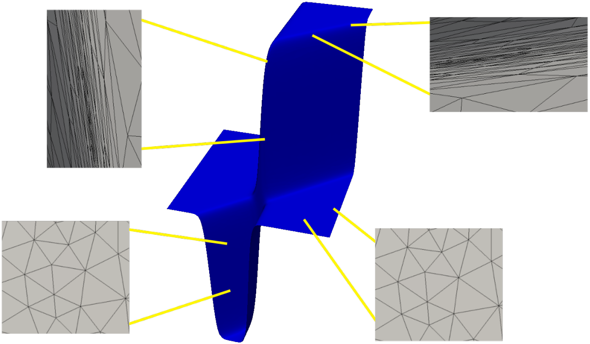
Given three points $A, B, C \in \Gamma$ we define the **6d-angle** ϑ as

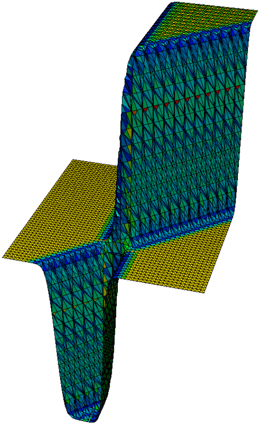
$$\cos_{\text{6d}}(\vartheta) := \frac{(A - C, B - C)_{\text{6d}}}{||A - C||_{\text{6d}}\,||B - C||_{\text{6d}}}$$

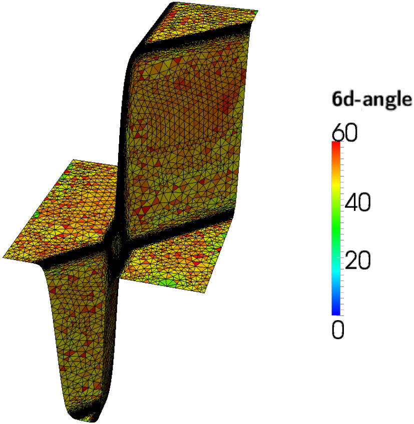


- Starting from an initial mesh of a surface $\Gamma \subset \mathbb{R}^3$
- Evaluate the lengths of the angles of the triangles in $\mathbb{R}^6$.
- Perform the standard local mesh adaptation operations to make the mesh as uniform as possible in $\mathbb{R}^6$.

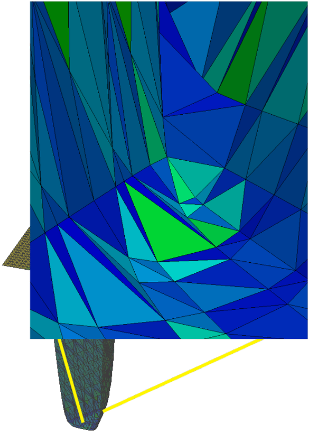


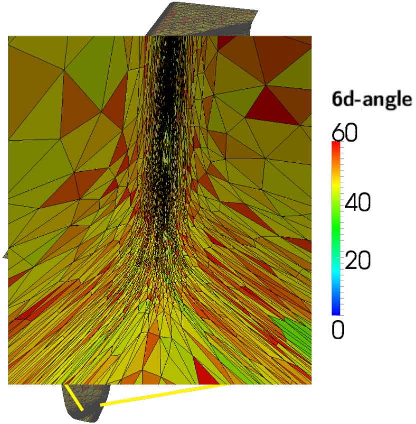






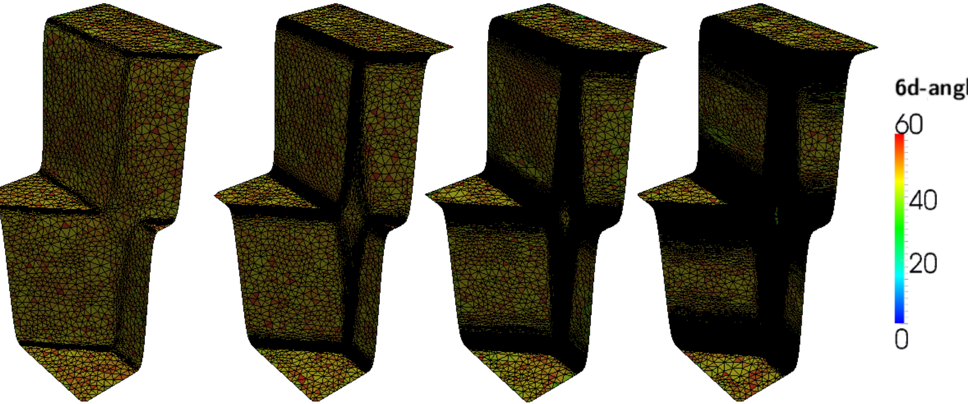
The initial mesh | A resulting mesh





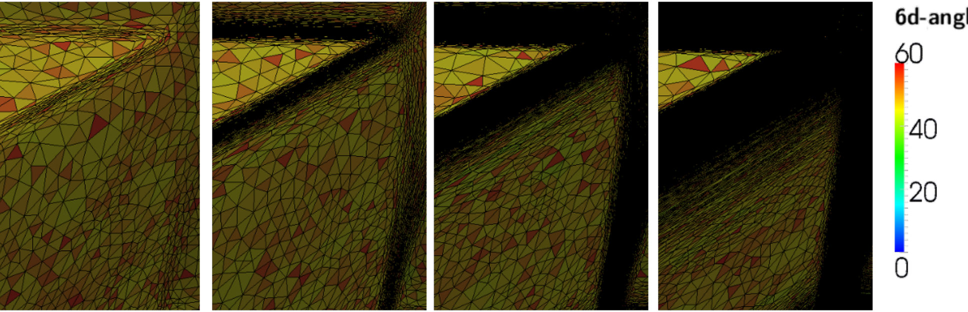
The initial mesh | A resulting mesh

The Anisotropy affected by s



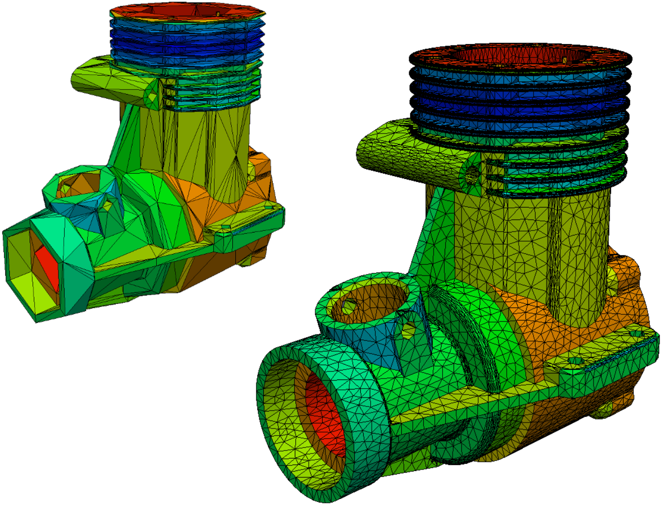
$s = 0.2$ $s = 1.$ $s = 5.$ $s = 25.$

The Anisotropy affected by s

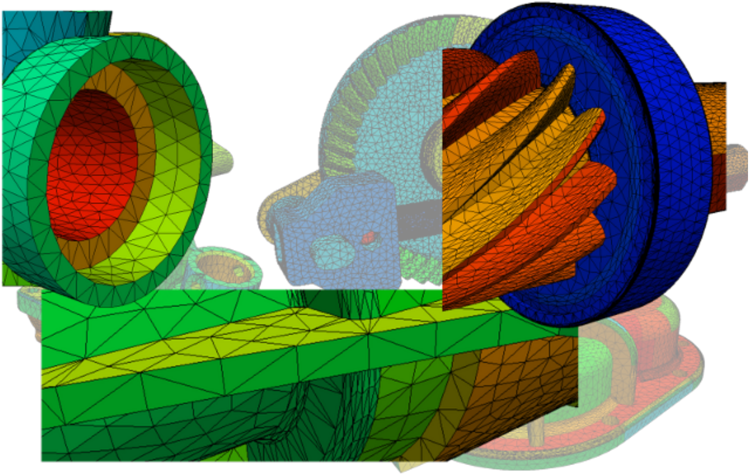


$s = 0.2$ $s = 1.$ $s = 5.$ $s = 25.$

Example: CAD Surface Remeshing



Example: CAD Surface Remeshing



Consider a flat domain Ω with a Lipschitz smooth boundary and a smooth function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. We define the embedding map $\Phi_f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^5$ as:

$$\Phi_f(\mathbf{x}) := (x,\, y,\, s\, f(x,\, y), s\, g_x(x,\, y),\, s\, g_y(x,\, y))^{\,t}\,, \qquad (1)$$

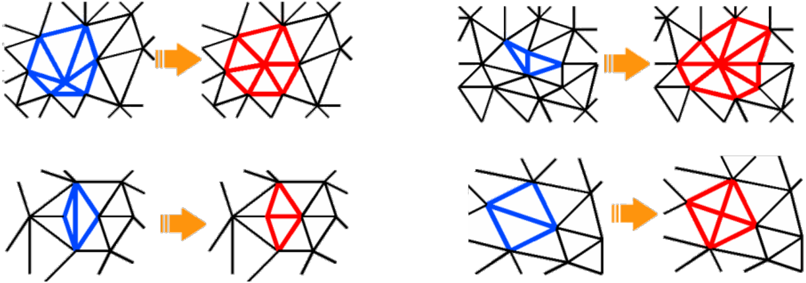
here $s \in [0, +\infty)$ is a user-specified parameter, $f(x,\, y)$, $g_x(x,\, y)$ and $g_y(x,\, y)$ are values at the point $(x,\, y)$ of the function f and its gradient components, respectively.

The Re-Meshing Procedure

an initial mesh Ω_h
a desired 5d-length, L_{5d}

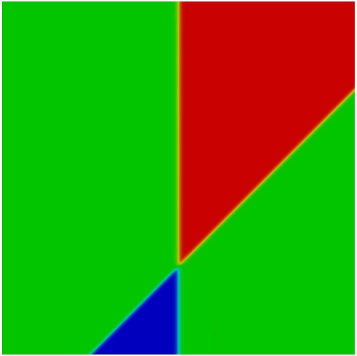
sampling
optimization

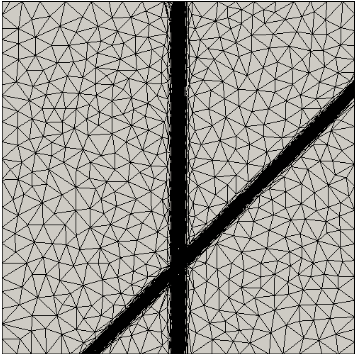
final mesh



Example

$$f_2(x,\,y) = \tanh\left(60x\right) - \tanh\left(60\left(x - y\right) - 30\right).$$





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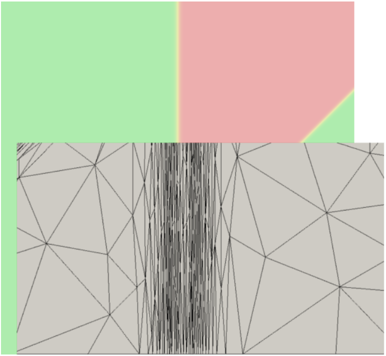
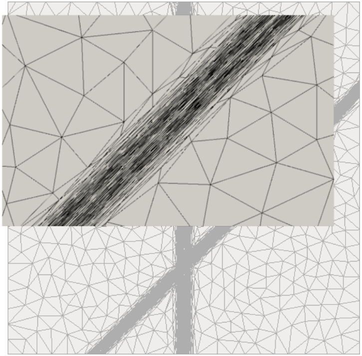
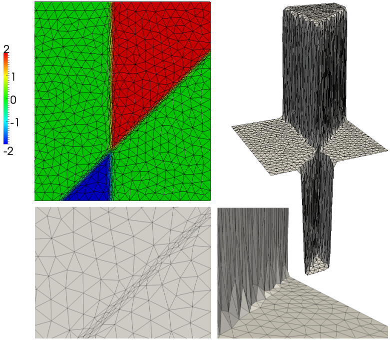


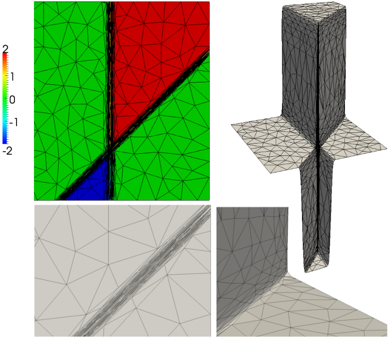


Comparison with BAMG

- Hecht, F., BAMG: Bidimensional anisotropic mesh generator. `www.ann.jussieu.fr/hecht/ftp/bamg`, `Freefem++`.
- Uses metric based mesh adaptation method.



BAMG



HDE

Mesh Adaption via HDE

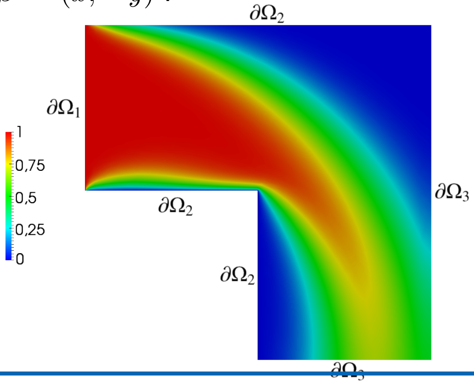
- Contrary to the classical mesh adaptation procedure, the proposed adaptation strategy in this paper does not involve both the estimation of an error and the construction of a metric field. In each iteration of the mesh adaptation, we use the following steps:

  $$\texttt{SOLVE}\rightarrow\texttt{RECOVER GRADIENT}\rightarrow\texttt{ADAPT},$$

  and this process stops when it converges or a desired maximum number of iterations is reached.

$$\begin{cases} -\mu\Delta u + \overrightarrow{\beta}\cdot\nabla u &= 0 \qquad \text{in } \Omega\,, \\ u &= 1 \qquad \text{in } \partial\Omega_1\,, \\ u &= 0 \qquad \text{in } \partial\Omega_2\,, \\ \mu\frac{\partial u}{\partial n} &= 0 \qquad \text{in } \partial\Omega_3\,, \end{cases}$$

here $\mu = 0.05$, $\overrightarrow{\beta} = (x, -y)^t$.



$$
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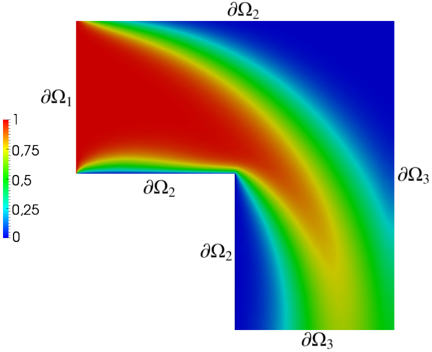
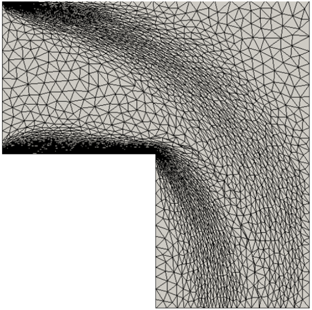
\mu \frac{\partial u}{\partial n} &= 0 \qquad \text{in } \partial\Omega_3\,,
\end{cases}
$$

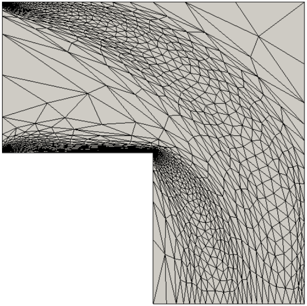
here $\mu = 0.05$, $\overrightarrow{\beta} = (x, -y)^t$.

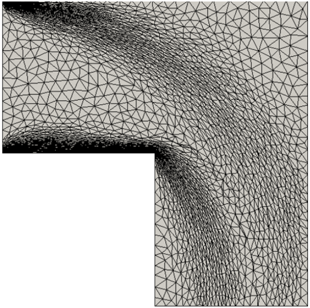


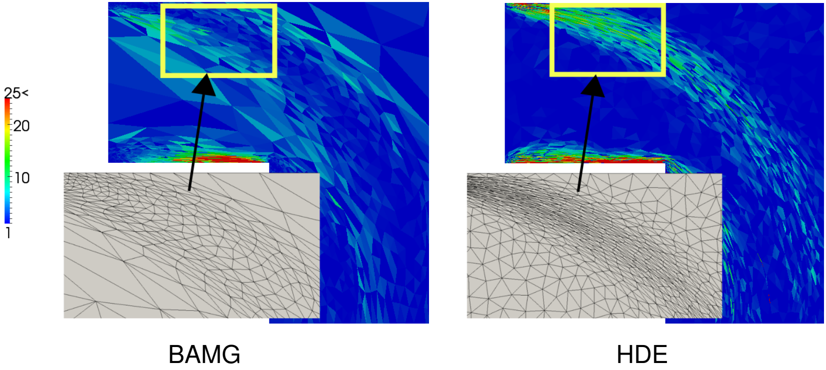


$$e_{tot} := \int_{\Omega_h} |u_h - u_{ref}|^2 \, dx \qquad \text{and} \qquad \sigma_{max} := \max_{T \in \Omega_h} \sigma_T$$

	BAMG	HDE
Ele.	4438	4337
e_{tot}	4.143e-03	6.650e-03
σ_{max}	6.880e+01	3.456e+02






25<
20
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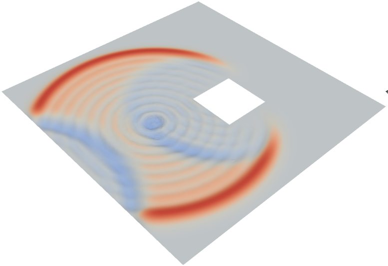
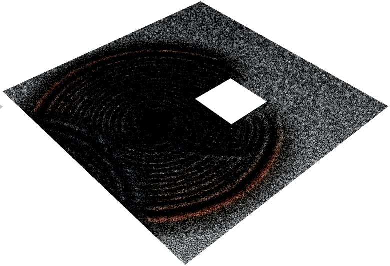
1
BAMG
HDE

Example: Wave

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \mu \Delta u &= f \qquad \text{in } \Omega\,, \\ u &= 0 \qquad \text{in } \partial\Omega\,, \end{cases}$$

here $\mu = 1.$, f discrete Dirac function.

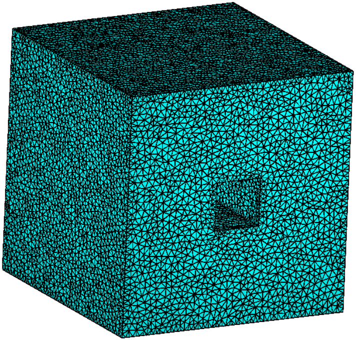




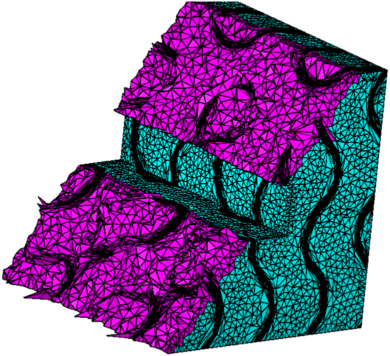
animation

Anisotropic Tetrahedral Meshes via HDE

$$f(x,y,z) = \tanh(10(\sin(2\pi x)\cos(2\pi y) + \sin(2\pi y)\cos(2\pi z) + \sin(2\pi z)\cos(2\pi x)))$$

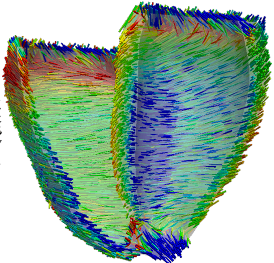


The initial mesh

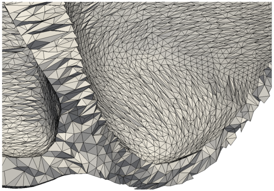


An adapted mesh

Discrete Metric Tensor Filed



A visualization of the metric tensor field
(Courtesy of B. Cristóbal)



- We have presented a method for anisotropic mesh adaptation based on higher dimensional embeddings (HDE).
- Experimental results showed that this method produced meshes are comparable those generated by metric-based mesh adaptation methods.
- HDE tends to capture anisotropy more accurately.
- However, HDE tends to over stretched in some area. The mesh gradation in HDE is less than metric-based method.
- Deep analysis is needed for HDE method.
- A very interesting question is how its relation with metric-based method.